Cut vertices and their renormalization: A generalization of the Wilson expansion

A. H. Mueller
Department of Physics, Columbia University, New York, New York 10027
(Received 13 July 1978)

Cut vertices, a generalization of matrix elements of composite operators, are introduced. Their renormalization is discussed. The Bogolubov-Parasiuk-Hepp-Zimmermann method of renormalization of cut vertices allows one to obtain a generalization of the Wilson expansion where cut vertices multiplied by singular functions appear rather than local operators times singular functions. A Callan-Symanzik equation for the moments of the structure function in $e^+ + e^- \to$ hadron (p) + anything is derived. This equation is valid to all orders of perturbation theory in both gauge and nongauge theories. Examples of renormalization through the two-loop level are given.

I. INTRODUCTION

The parton model has had enormous success in dealing with processes having a large off-shell current. However, only the total annihilation cross section for $e^+ + e^- \to$ hadrons along with deeply inelastic electron and neutrino scattering have been given a firm basis in field theory. In particular, asymptotically free gauge theories[1,2] make precise predictions for these two types of processes. These predictions seem to be in good agreement with experiment. Processes such as large-mass μ-pair production, inclusive e^+e^- annihilation, and inclusive high-$p_\perp$ hadron production have been discussed a great deal in terms of the parton model. However, these discussions have not been too convincing in the context of a gauge theory of the strong interactions.

Interesting perturbation calculations have been performed which suggest that in the case of inclusive hadron production in e^+e^- collisions[3,4] and for μ-pair production in hadron-hadron collisions[4-7] something like the renormalization group is at work. The most extensive calculations have been done by Gribov and Lipatov[3] for single-particle inclusive hadron production in e^+e^- annihilation in an Abelian gauge theory. Gribov and Lipatov sum all terms of the form $(g^2 \ln q^2)^n$ where q^2 is the photon mass squared. The results look exactly like those which would arise from a Callan-Symanzik[8,9] equation. However, the Gribov-Lipatov calculation also relates deeply inelastic electron to inclusive annihilation by the replacement $\omega \to 1/\omega$, and it is clear that such a relation will not be true in nonleading logarithms. It is, perhaps, not clear whether the nonleading logarithms will obey a Callan-Symanzik equation.

Recent calculations[4-7] for μ-pair production in both Abelian and non-Abelian gauge theories have also suggested that the $\ln q^2$ terms arrange themselves much as in the case of deeply inelastic electron scattering. However, here the calculations have been much less extensive, only one factor of $g^2 \ln q^2$ for the non-Abelian theory, and so the suggestion of a renormalization-group behavior is much less convincing.

The danger of generalizing from a calculation of a single logarithm, or even a set of leading logarithms, is probably evident. A famous example is the form-factor calculation first carried out by Sudakov.[10] The leading logarithmic series gives a behavior $e^{-cg^2 \ln^2 q^2}$, but after 20 years there is still no compelling evidence that such a form will maintain itself in nonleading logarithms.

Inclusive annihilation of a massive scalar particle $\phi(q)$ into an on-shell scalar particle $\phi(p)$ along with an arbitrary number of unidentified particles has been discussed[11,12] in a ϕ^4 theory. It was found that a Callan-Symanzik equation does hold for the moments of the structure function much as in deeply inelastic electron scattering. Recently, Ellis, Georgi, Machacek, Politzer, and Ross[13] have made the exciting discovery that the method of Ref. 11 will also work for gauge theories, at least for leading logarithms, if one works in an axial gauge, $n \cdot A = 0$.

Stimulated by the work of Ref. 13, a generalization of the Wilson expansion has been developed. This generalization makes use of cut vertices and applies to gauge theories as well as nongauge theories. It should be applfcable in any gauge although we have examined only the Feynman gauge in detail. In this method a vertex, called a bare cut vertex, is introduced. [See Eq. (12) and Fig. 4.] The simplest bare vertex, $\Gamma_\sigma^{(0)}(p) = \gamma_- p_-^{-\sigma}$, consists of two fermion lines, one above and one below a horizontal line (a cut). One adds radiative corrections by using the usual Feynman rules except that all lines above the cut have the usual $+i\epsilon$ changed to $-i\epsilon$ in their propagators. Also, any line crossing the cut has $-i/(p^2 - m^2 + i\epsilon)$ replaced by $-2\pi\delta(p^2 - m^2)$.

In the calculation of radiative corrections, ultraviolet divergences will appear. These divergences can be subtracted by a slight generalization of the standard Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ)[14-16] formalism for defining composite operators.[17] If the bare cut vertex is *analogous* to the zeroth-order term in $\bar{\psi}\gamma_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_n}\psi$, then the counterterms needed to render the cut vertex finite are analogous to $\bar{\psi}\gamma_{\mu_1}\partial_{\mu_2}\cdots A_{\mu_j}\cdots\partial_{\mu_n}\psi$ and $F_{\gamma\mu_1}\partial_{\mu_2}\cdots F_{\gamma\mu_n}$. The cut vertices are not local in general and the analogy to operators is only a convenient mnemonic.

Cut vertices and their renormalization are discussed for a ϕ^4 theory in Sec. II and for an Abelian gauge theory in Sec. III. In Sec. III additional subtractions are performed in an Abelian gauge theory for the process relevant to the calculation of single-particle inclusive e^+e^- annihilation. It is shown that the moments of the structure functions factorize, for large q^2, into a singular function depending on the q^2 of the photon, but completely independent of the particles produced, times a cut vertex which depends on the particle observed. Such a relation is a generalization of the Wilson[18] and light-cone[19,20] expansions where the cut vertices replace the matrix elements of local operators. The factorization property means that there is no need to know bound-state dynamics in order to get useful predictions for composite particle production. If one were to try to use this formalism to calculate on-mass-shell quark production, presumably the cut vertex would vanish as the quark goes on-shell. The mechanism for this vanishing is unknown as yet.

The Callan-Symanzik equation for the singular functions follows directly from the BPHZ prescription for subtraction. In the Callan-Symanzik equation there is a term which corresponds to the anomalous dimension of the renormalized cut vertex. Except for a few terms, to *lowest order* the anomalous dimensions are related to the analogous anomalous dimensions occurring in deeply inelastic electron scattering[21] by the relation $\sigma=-n+1$.[3] From this relation the Gribov-Lipatov inclusive annihilation results follow immediately. (See Ref. 26.)

In Sec. V the cut vertices necessary for the discussion of inclusive annihilation in non-Abelian gauge theories with fermions are introduced. To lowest order in g^2 the anomalous dimensions are again determined from deeply inelastic electron scattering[22,23] by the substitution $n=-\sigma+1$.

In Sec. VI applications to $e^+ + e^- \to \text{hadrons}(p)$ + anything are discussed in an asymptotically free gauge theory. The Callan-Symanzik equation for the structure functions is given explicitly. The energy-momentum conservation sum rule is seen to emerge from a zero eigenvalue in the anomalous-dimension matrix. The energy dependence of the average multiplicity, $\bar{n}$, of produced hadrons in e^+e^- annihilation is seen not to be a property of the renormalization group as the anomalous-dimension matrix has a singularity at that moment which determines $\bar{n}$. This is in contrast to nongauge theories where the energy dependence of $\bar{n}$ is determined by anomalous dimensions.[11] It is also suggested that in order to understand the q^2 dependence of particles produced having $\omega=2p\cdot q/q^2$ such that $\ln(1/\omega)\gg\ln\ln q^2$, one probably requires knowledge not obtainable from the renormalization group.

Nonperturbative tunneling effects[24] have not been included in the above analysis. Formally, there is no distinction between the application of the renormalization group discussed in this paper and the application to deeply inelastic electron scattering. What is missing here is the relation to short-distance behavior. Although deeply inelastic electron scattering is a light-cone rather than a short-distance phenomenon, analyticity connects a given moment of the structure function to a short-distance limit of local operators. One expects semiclassical effects not to modify short-distance effects severely. In the present application the moments of the inclusive annihilation struction functions are not related to any short-distance limit. Presumably, however, processes occurring close to the light cone suppress semiclassical effects just as short-distance processes do.

Finally, the main purpose of this paper is to give a general discussion of cut vertices. In a subsequent paper the limits of the applicability of the renormalization group to hadronic processes will be discussed and specific processes will be dealt with in detail.

II. ϕ^4 THEORY

In this section cut vertices will be discussed. Both spacelike and timelike cases will be considered in order to illustrate how renormalization is carried out in the most simple case. Only those vertices which actually occur in physical large-q^2 limits will be done in detail. Begin with the spacelike case.

A. Spacelike cut vertices

Consider the amplitude

$$T(p^2,p\cdot q,q^2)=\int d^4x\,d^4y\,d^4z\,e^{iqx+ip(y-z)}\langle\overline{T}(\phi(x)\phi(y))T(\phi(z)\phi(0))\rangle_c[\Delta_F'(p^2)\Delta_F'(q^2)]^{-2}, \quad (1)$$

where $\Delta_F'(p^2)$ is the full renormalized propagator for the ϕ field and $\bar{T}$ denotes an anti-time-ordered product. Now if $\tilde{T}$ is defined by

$$\tilde{T}(p^2, p\cdot q, q^2) = -i \int d^4x\, d^4y\, d^4z\, e^{iqx+ip(y-z)} \langle T(\phi(x)\phi(y)\phi(z)\phi(0))\rangle_c [\Delta_F'(p^2)\Delta_F'(q^2)]^{-2}, \qquad (2)$$

then

$$2\,\mathrm{Im}\,\tilde{T}(p^2, p\cdot q, q^2) = T(p^2, p\cdot q, q^2)$$

so long as p^2 and q^2 are below their thresholds. In the present section p and q are assumed to be spacelike vectors. When $q^2 \to -\infty$, $\tilde{T}$ has the expansion

$$\tilde{T}(p^2, \omega, q^2) \sim \sum_{n=0}^{\infty} \int d^4y\, d^4z\, e^{ip(y-z)} \langle T\phi(y) \mathcal{O}_{\alpha_1\cdots\alpha_n}(0)\phi(z)\rangle [\Delta_F'(p^2)]^{-2} \left(\frac{-2q_{\alpha_1}}{q^2}\right)\left(\frac{-2q_{\alpha_2}}{q^2}\right)\cdots\left(\frac{-2q_{\alpha_n}}{q^2}\right) E_n(q^2),$$

where $\omega = -2p\cdot q/q^2$ and $\mathcal{O}_{\alpha_1\ldots\alpha_n}(x) = i^n \phi(x)\partial_{\alpha_1}\cdots\partial_{\alpha_n}\phi(x)$. The above equation is valid so long as $|\omega| < 1$. With the normalization

$$\int d^4y\, d^4z\, e^{ip(y-z)} \langle T\phi(y)\mathcal{O}_{\alpha_1\ldots\alpha_n}(0)\phi(z)\rangle [\Delta_F'(p^2)]^{-2}\Big|_{p^2=0} = p_{\alpha_1}\cdots p_{\alpha_n} + \text{terms with } g_{\alpha_i\alpha_j},$$

one has

$$E_n(q^2) = \frac{1}{\pi}\int_1^\infty \omega^{-n-1} d\omega\, T(0, \omega, q^2)$$

for large q^2. The p^2 dependence is put back in by noting that

$$\Gamma_n(p^2) E_n(q^2) = \frac{1}{\pi}\int_1^\infty \omega^{-n-1} d\omega\, T(p^2, \omega, q^2),$$

where

$$\int d^4y\, d^4z\, e^{ip(y-z)} \langle T\phi(y)\mathcal{O}_{\alpha_1\ldots\alpha_n}(0)\phi(z)\rangle [\Delta_F'(p^2)]^{-2} = \Gamma_{\alpha_1\ldots\alpha_n}(p) = \Gamma_n(p^2) p_{\alpha_1}\cdots p_{\alpha_n} + \text{terms with } g_{\alpha_i\alpha_j}.$$

Neglecting renormalization for the moment, one can write

$$\Gamma_{\alpha_1\ldots\alpha_n}(p) = \Gamma^{(0)}_{\alpha_1\ldots\alpha_n}(p) - \frac{i}{(2\pi)^4}\int d^4k\, \tilde{T}(p,k)[\Delta_F'(k^2)]^2 \times \Gamma^{(0)}_{\alpha_1\cdots\alpha_n}(k), \qquad (3)$$

where $\Gamma^{(0)}_{\alpha_1\cdots\alpha_n}(k) = k_{\alpha_1}\cdots k_{\alpha_n}$. In order to eliminate $g_{\alpha_i\alpha_j}$ terms in Γ, it is convenient to pick a set of indices where they vanish. In particular, define $p_- = (1/\sqrt{2})(p_0 - p_3)$. Then

$$\Gamma_{-\cdots-}(p) = \Gamma_n(p^2) p_-^n = p_-^n - \frac{i}{(2\pi)^4}\int d^4k\, \tilde{T}(p,k)[\Delta_F'(k^2)]^2 k_-^n,$$

where the minus sign occurs n times in $\Gamma_{-\cdots-}(p)$. This equation is illustrated in Fig. 1. The integrals appearing in the perturbation expansion are only logarithmically divergent and may be subtracted in the standard way in which composite operators are defined.[17]

Instead of (3) consider the vertex equation

$$\Gamma_{\alpha_1\cdots\alpha_n}(p) = \Gamma^{(0)}_{\alpha_1\cdots\alpha_n}(p) + \frac{1}{(2\pi)^4}\int d^4k\, \mathcal{O}T(p,k)[\Delta_F'(k^2)]^2 \times \Gamma^{(0)}_{\alpha_1\cdots\alpha_n}(k), \qquad (4)$$

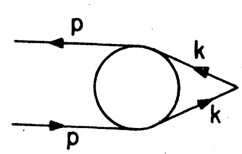

FIG. 1. A spacelike uncut vertex.

where Θ is a phase-space constraint to be specified later. The Γ's defined by Eq. (3) and Eq. (4) are in fact the same. Choosing all minus indices, one has

$$\Gamma_n(p^2)p_-^{\,n} = p_-^{\,n} + \frac{1}{(2\pi)^4}\int d^4k\,\Theta T(p,k)[\Delta_F'(k^2)]^2 k_-^{\,n}. \qquad (5)$$

This equation is shown in Fig. 2. Defining

$$p^2 = -p^2, \quad k^2 = -K^2, \quad \omega = -\frac{2p\cdot k}{K^2}$$

and using

$$d^4k = \pi K^3 dK d\omega \frac{dk_-}{p_-},$$

one has

$$\Gamma_n(p^2)p_-^{\,n} = p_-^{\,n} + \frac{1}{16\pi^3}\frac{1}{p_-}\int K^3 dK\, d\omega [\Delta_F'(K)]^2 \times T(P,\omega,K)k_-^{\,n}\Theta(k_-)dk_-, \qquad (6)$$

where the $\Theta(k_-)$ is appropriate for $p_- > 0$. The k_- integration can be done:

$$\int_0^{k_-^{\max}} dk_- k_-^{\,n} = \frac{1}{n+1}(k_-^{\max})^{n+1}.$$

$k_-^{\max}$ is most easily determined by choosing a simple coordinate system and writing

$$p = (0,0,0,-P),$$

$$k = K(\sinh\zeta, \cosh\zeta\sin\theta\cos\phi, \cosh\zeta\sin\theta\sin\phi, \cosh\zeta\cos\theta).$$

Then

$$k_- = \frac{K}{\sqrt{2}}(\sinh\zeta - \cosh\zeta\cos\theta)$$

and

$$\omega = -\frac{2P}{K}\cosh\zeta\cos\theta.$$

Now $\zeta < 0$ since $k_0 < 0$ in order that physical states have positive energy. Then

$$k_- = \frac{K}{\sqrt{2}}\left(\frac{K}{2P}\omega - \sinh|\zeta|\right).$$

The maximum value of k_- for fixed ω and K occurs when $\theta = \pi$, in which case

$$k_-^{\max} = \frac{2p_-}{\omega[1+(1-4P^2/K^2\omega^2)^{1/2}]}.$$

Thus

$$\Gamma_n(p^2) = 1 + \frac{2^n}{(2\pi)^3(n+1)}\int K^3 dK|\Delta_F'(K)|^2\int d\omega\, T(P,\omega,K)\omega^{-n-1}[1+(1-4P^2/K^2\omega^2)^{1/2}]^{-n-1}. \qquad (7)$$

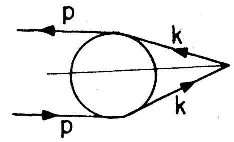

FIG. 2. A spacelike cut vertex.

After appropriate subtractions are made, this is the same as Eq. (17) of Ref. 11. The great advantage of the method just described over that of Ref. 11 is that the projections here are done automatically in terms of a bare cut vertex, $P_-^{\,-n}$. When the only relevant operators in the Wilson expansion are those involving two fields and their derivatives, this is not so important. However, the present method generalizes to gauge theories, whereas the method of Ref. 11 could only be generalized to gauge theories if one knew how to project out Lorentz quantum numbers on multiparticle states in an efficient manner. Such a projection appears to be a formidable task.

B. Timelike cut vertices

Now define

$$T(p,q) = \int d^4x\, d^4y\, d^4z\, e^{iqx - ip(y-z)} \times \langle \bar{T}(\phi(x)\phi(y))T(\phi(z)\phi(0))\rangle |\Delta_F'(p)\Delta_F'(q)|^{-2}, \qquad (8)$$

where p and q are timelike and $p_0 > 0$. This amplitude is a prototype for $e^+ + e^- \to \gamma(g) \to$ hadron(p) + anything in that Eq. (8) represents the probability for a field $\phi(q)$ to decay into a field $\phi(p)$ + anything. In analogy to (6), define

$$\Gamma_\sigma(p^2)p_-^{\,-\sigma} = p_-^{\,-\sigma} + \frac{1}{(2\pi)^4}\int d^4k\, T(p,k)|\Delta_F'(k)|^2 k_-^{\,-\sigma}, \qquad (9)$$

where σ is an arbitrary complex number. Equation (9) is illustrated in Fig. 3. Writing

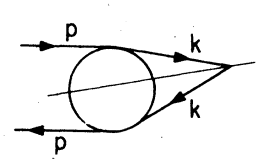

FIG. 3. A timelike cut vertex.

$$d^4k = \pi K^3\, dK\, d\omega\, \frac{dk_-}{p_-}$$

and $k^2 = K^2$, $p^2 = P^2$, $\omega = 2p\cdot k/K^2$ one obtains

$$\Gamma_\sigma(p^2)p_-^{-\sigma} = p_-^{-\sigma} + \frac{1}{16\pi^3}\, p_- \int K^3\, dK\, d\omega\, T(P,\omega,K)\,|\Delta_P'(K)|^2 \times \int_{k_-^{\min}}^{k_-^{\max}} dk_-\, k_-^{-\sigma}.$$

$k_-^{\max}$ and $k_-^{\min}$ are most easily evaluated by choosing

$$p = (P,0,0,0)\,,$$

$$k = K(\cosh\zeta, \sinh\zeta\,\sin\theta\cos\phi, \sinh\zeta\,\sin\theta\sin\phi, \sinh\zeta\,\cos\theta)\,.$$

Then $\omega = (2P/K)\cosh\zeta$ and $k_- = (K/\sqrt{2})(\cosh\zeta - \sinh\zeta\cos\theta)$, which gives

$$k_-^{\max} = \frac{K^2}{4p_-}\,\omega[1 + (1 - 4P^2/K^2\omega^2)^{1/2}]\,,$$

$$k_-^{\min} = \frac{2p_-}{\omega}\,[1 + (1 - 4P^2/K^2\omega^2)^{1/2}]^{-1}\,.$$

Thus

$$\Gamma_\sigma(P^2) = 1 + \frac{2^{-\sigma}}{(2\pi)^3(\sigma-1)} \int K^3 dK\,|\Delta_F'(K)|^2 \int d\omega\, T(P,\omega,K) \times \left\{ \omega^{\sigma-1}[1 + (1 - 4P^2/K^2\omega^2)^{1/2}]^{\sigma-1} - \left(\frac{4P^2}{K^2}\right)^{\sigma-1} \omega^{-\sigma+1}[1 + (1 - 4P^2/K^2\omega^2)^{1/2}]^{-\sigma+1} \right\}. \tag{10}$$

Except for subtractions, discussed in Appendix A, this is the same as Eq. (31) of Ref. 11. The rather simple looking equation, Eq. (9), has all of the complicated projections already included. $\Gamma_\sigma(p^2)p_-^{-\sigma}$ is what we call a cut vertex; $p_-^{-\sigma}$ is the bare cut vertex. Once one is able to define cut vertices, as in Eq. (9), and give a renormalization prescription for them, one is immediately able to write Callan-Symanzik equations for the moments of T. We postpone a detailed discussion of this point until we have dealt with gauge theories.

III. MASSIVE VECTOR-MESON THEORIES

In this section cut vertices for neutral vector-meson theories will be defined and their renormalization discussed. Yang-Mills theories will be considered in Sec. V.

(1) Statement of the problem and some definitions. The theory to be discussed in this section is defined by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}{}^2 + i\bar\psi\gamma_\mu(\partial_\mu - igA_\mu)\psi - m\bar\psi\psi - \tfrac{1}{2}m^2A_\mu A_\mu\,.$$

For simplicity we take the vector meson and the fermion to have the same renormalized mass, m. In analogy to the definitions given in Sec. II, define a bare cut vertex

$$\Gamma^{(0)}_{ab,\alpha_1\cdots\alpha_n}(p) = (\gamma_{\alpha_1})_{ab}\, p_{\alpha_2}\cdots p_{\alpha_n}\,, \tag{11}$$

where a symmetrization of the indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ is understood. a, b are the Dirac indices, and the bare cut vertex is illustrated in Fig. 4. Again specialize to all minus indices and continue n to σ by defining

$$\Gamma_\sigma^{(0)}(p) = \gamma_- p_-^{-\sigma} \tag{12}$$

and

$$\Gamma^{(0)}_{\sigma,ab}(p) = (\gamma_-)_{ab}\, p_-^{-\sigma}$$

for any complex σ. Define

$$\Gamma_{\sigma,ab}(p) = \Gamma^{(0)}_{\sigma,ab}(p) + \frac{1}{(2\pi)^4}\int d^4k\, T_{ab,cd}(p,k)\Gamma^{(0)}_{\sigma,cd}(k)\,, \tag{13}$$

where

$$T_{ab,cd}(p,h) = \int d^4x\, d^4y\, d^4z\, e^{ip(x-y)+ikx}\langle \tilde{T}(\psi_d(x)\bar\psi_{b'}(y))T(\psi_{a'}(z)\bar\psi_c(0))\rangle_c [S_F'(p)]^{-1}_{aa'}[S_F^{*\prime}(p)]^{-1}_{b'b} \tag{14}$$

with an asterisk denoting complex conjugation. In matrix notation (13) appears as

$$\Gamma_\sigma(p) = \gamma_- p_-^{-\sigma} + \frac{1}{(2\pi)^4}\int d^4k\, T(p,k)\gamma_- k_-^{-\sigma}. \tag{15}$$

Now (15) is logarithmically divergent as it stands. What is needed is a subtraction prescription which

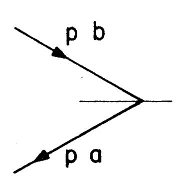

FIG. 4. A bare cut vertex for two fermions.

renders $\Gamma_\sigma(p)$ finite. At that point we shall be in a position to give the generalization of the Wilson expansion.

The major complexity in renormalizing (15), in contrast to (9), is the fact that the renormalization is not multiplicative. That is, there are other vertices which must be introduced as counterterms in order to render (15) finite. This is the same as the problem one encounters in defining the operator $\bar{\psi}\gamma_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}\psi$ in which case one is forced to consider also operators of the form

$$\bar{\psi}\gamma_{\alpha_1}\partial_{\alpha_2}\cdots A_{\alpha_i}\cdots A_{\alpha_j}\cdots\partial_{\alpha_n}\psi .$$

In that case, for the physically interesting case of deeply inelastic electron scattering, only the gauge-invariant operator

$$\bar{\psi}\gamma_{\alpha_1}(\partial_{\alpha_2}-igA_{\alpha_2})\cdots(\partial_{\alpha_n}-igA_{\alpha_n})\psi$$

occurs in the Wilson expansion. We must now introduce cut vertices involving two fermions and a photon.

Define bare cut vertices

$$\Gamma^{(0)}_{11,\sigma\alpha}(p,k)=-gg_{\alpha-}\gamma_-\frac{p_-^{-\sigma}}{k_-}, \tag{16}$$

$$\Gamma^{(0)}_{12,\sigma\alpha}(p,k)=gg_{\alpha-}\gamma_-\frac{(p+k)_-^{-\sigma}}{k_-}. \tag{17}$$

The vertices are illustrated in Fig. 5 and satisfy the Ward identities

$$k_\alpha\Gamma^{(0)}_{11,\sigma\alpha}(p,k)=-g\Gamma^{(0)}_\sigma(p), \tag{18}$$

$$k_\alpha\Gamma^{(0)}_{12,\sigma\alpha}(p,k)=g\Gamma^{(0)}_\sigma(p+k). \tag{19}$$

In fact only $\alpha=+$ will be used, so define

$$\Gamma^{(0)}_{11,\sigma}(p,k)=-g\gamma_-\frac{p_-^{-\sigma}}{k_-}, \tag{20}$$

$$\Gamma^{(0)}_{12,\sigma}(p,k)=g\gamma_-\frac{(p+k)_-^{-\sigma}}{k_-}. \tag{21}$$

In addition to the contribution to $\Gamma_\sigma(p)$ given by (15)

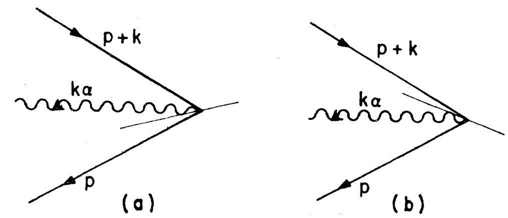

FIG. 5. (a) $\Gamma^{(0)}_{11,\sigma}(p,k)$. (b) $\Gamma^{(0)}_{12,\sigma}(p,k)$.

we must also add the contribution given by

$$\Gamma_\sigma(p)=\frac{1}{(2\pi)^8}\int d^4k\,d^4k_1\sum_{j=1}^{2}T_j(p,k,k_1)\Gamma_{1j,\sigma}(k,k_1), \tag{22}$$

where

$$T_1(p,k,k_1)=\int d^4x\,d^4y\,d^4z\,d^4w\,e^{ip(z-y)+ikx-ik_1w}\times\langle\overline{T}(\psi(x)\bar{\psi}(y)A_-(w))T(\psi(z)\bar{\psi}(0))\rangle_c \tag{23}$$

and

$$T_2(p,k,k_1)=\int d^4x\,d^4y\,d^4z\,d^4w\,e^{ip(z-y)+ikx-ik_1w}\times\langle\overline{T}(\psi(x)\bar{\psi}(y))T(\psi(z)\bar{\psi}(0)A_-(w))\rangle_c . \tag{24}$$

Amputation of propagators having momentum p is understood in (24) and fermion indices are suppressed. Equation (22) is shown in Fig. 6. Vertices with two fermions and n photons will be defined a little later on. Finally, renormalization will mix two fermion states with two photon states so that one needs to introduce the bare cut vertex for two photons

$$\Gamma^{(0)}_{\alpha\beta,\sigma}(k)=4[g_{\alpha\beta}k_-^2-k_-(k_\alpha g_{\beta-}+k_\beta g_{\alpha-})+k^2g_{\alpha-}g_{\beta-}]k_-^{-\sigma-1} \tag{25}$$

shown in Fig. 7.

(2) Some one-loop examples of renormalization. Before discussing the general problem of renormalization of cut vertices, it may be helpful to see how renormalization of cut vertices occurs at the one-loop level. Consider the graph shown in Fig. 8. This contribution to $\Gamma_\sigma(p)$ is

$$\Gamma_\sigma(p)=\frac{g^2}{(2\pi)^4}\int d^4k\frac{\gamma_\alpha[\gamma\cdot(p+k)+m]\gamma_-[\gamma\cdot(p+k)+m]\gamma_\alpha(p+k)_-^{-\sigma}(-2\pi)\delta(k^2-m^2)}{[(p+k)^2-m+i\epsilon]^2}. \tag{26}$$

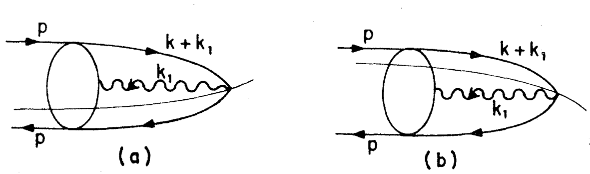

FIG. 6. An illustration of Eq. (22).

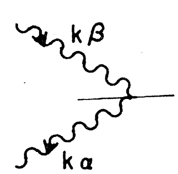

FIG. 7. A bare cut vertex for two photons.

First consider the divergent part of (26). The divergent part of Γ_σ does not depend on m so $m=0$ will be taken. Also, for the divergent part we may replace the γ matrices in the numerator by

$$\gamma_\alpha \underline{\gamma}\cdot\underline{k}\gamma_- \underline{\gamma}\cdot\underline{k}\gamma_\alpha = -2\underline{k}^2\gamma_-,$$

where $\underline{\gamma}\cdot\underline{k} = \gamma_1 k_1 + \gamma_2 k_2$. Then

$$\Gamma_\sigma(p) \sim \frac{g^2}{4\pi^3}\gamma_- \int d^4k \frac{(p+k)_-^{-\sigma}\delta(k^2)\underline{k}^2}{|(p+k)^2|^2}$$

or (27)

$$\Gamma_\sigma(p) \sim \frac{g^2}{8\pi^2}\frac{\ln\Lambda^2}{(\sigma-1)(\sigma-2)}\gamma_- p_-^{-\sigma}.$$

Thus the divergent piece of Γ is of the same form as the bare cut vertex.

Before giving a subtraction procedure for this graph, let us examine (26) further in the limit when $p \to \hat{p}$ where $\hat{\underline{p}}=0$, $\hat{p}_+=0$, $\hat{p}_-=p_-$. Then

$$\Gamma_\sigma(\hat{p}) = -\frac{g^2}{(2\pi)^3}\int dk_+ dk_- d^2\underline{k} \frac{\gamma_\alpha[\gamma_+(p+k)_- - \underline{\gamma}\cdot\underline{k}+m]\gamma_-[\gamma_+(p+k)_- - \underline{\gamma}\cdot\underline{k}+m]}{[2(p+k)_-k_+ - \underline{k}^2 - m^2]^2}\gamma_\alpha(p+k)_-^{-\sigma}\delta(k^2-m)$$

$$= \frac{g^2}{8\pi^3 p_-^2}\int k_- dk_- d^2\underline{k} \frac{[\gamma_-(\underline{k}^2+m^2) + 2\gamma_+(p+k)_-^2 - 4m(p+k)_-](p+k)_-^{-\sigma}(p+k)_-^{-\sigma}}{(\underline{k}^2+m^2)^2}$$

Thus in addition to the divergent contribution given in (27) there is a term equal to

$$+\frac{g^2}{4\pi^2 m^2}\left[\frac{\gamma_+ p_-^{-\sigma+2}}{(\sigma-3)(\sigma-4)} - \frac{2mp_-^{-\sigma+1}}{(\sigma-2)(\sigma-3)}\right]. \qquad (28)$$

In subtracting the divergent piece we must be careful not to introduce new types of vertices of the type given in (28). When Zimmermann[16,17] subtracts he uses a power series to define composite operators. This is not possible here because the p_- behavior is not polynomial. Our prescription for subtraction is to define the renormalized Γ as

$$\Gamma_\sigma(p) = \Gamma_\sigma^u(p) - \hat{\Gamma}_\sigma^u(\hat{p}), \qquad (29)$$

where we insert the superscript on the right-hand side of (29) to emphasize that those vertices are unrenormalized. $\hat{\Gamma}_\sigma^u(p)$ is that part of Γ having the form γ_- for the Dirac indices. Matrices of the form $\gamma\cdot p$, etc. are explicitly dropped in $\hat{\Gamma}$. The justification for subtracting only the $\gamma_- p_-^{-\sigma}$ term is the fact that such a structure is the only possible divergent piece of Γ by power counting. A discussion of power counting for cut vertices is given in Appendix C.

Consider now the graphs shown in Fig. 9. It is necessary to consider these graphs together as they separately have divergences at $k_-=0$. Then

$$\Gamma_\sigma(p) = -\frac{g^2}{(2\pi)^4}\int d^4k \frac{\gamma_-[\gamma\cdot(p+k)+m]\gamma_-}{(p+k)^2-m^2}\left[-2\pi\delta(k^2-m^2)\frac{(p+k)_-^{-\sigma}}{k_-} + \frac{i}{k^2-m^2}\frac{p_-^{-\sigma}}{k_-}\right]. \qquad (30)$$

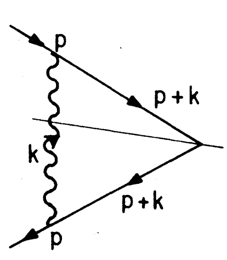

FIG. 8. A radiative correction to $\Gamma_\sigma(p)$.

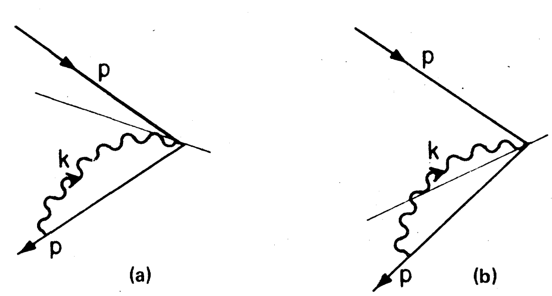

FIG. 9. Further radiative corrections to $\Gamma_\sigma(p)$.

For the divergent part

$$\Gamma_\sigma(p) \sim \frac{g^2\gamma_-}{8\pi^3 p_-}\int (p+k)_- \frac{dk_-}{k_-}\frac{d^2k}{\underline{k}^2}\{(p+k)_-^{-\sigma}\Theta(k_-) + p_-^{-\sigma}[\Theta(p_- + k_-) - \Theta(k_-)]\}$$

or

$$\Gamma_\sigma(p) \sim \frac{g^2 \ln\Lambda^2}{8\pi^2}\gamma_- p_-^{-\sigma}\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta}(1+\zeta)\{(1+\zeta)^{-\sigma}\Theta(\zeta) + [\Theta(1+\zeta) - \Theta(\zeta)]\}.$$

Thus

$$\Gamma_\sigma(p) \sim -\frac{g^2\ln\Lambda^2}{8\pi^2}\left[\sum_{l=1}^{\infty}\left(\frac{1}{l+1} - \frac{1}{\sigma+l-1}\right) - \frac{1}{\sigma-1}\right]\gamma_- p_-^{-\sigma}. \tag{31}$$

For positive integral $\sigma \geq 4$,

$$\Gamma_\sigma(p) \sim -\frac{g^2\ln\Lambda^2}{8\pi^2}\sum_{l=2}^{\sigma-2}\frac{1}{l}\gamma_- p_-^{-\sigma}.$$

The divergent part again has the form $\gamma_- p_-^{-\sigma}$. A renormalized vertex for the graphs of Fig. 9 can be defined by

$$\Gamma_\sigma(p) = \Gamma^u_\sigma(p) - \hat{\Gamma}^u_\sigma(\hat{p})$$

as before.

Finally, consider the graph shown in Fig. 10,

$$\Gamma_{\alpha\beta}(k) = -\frac{g^2}{(2\pi)^4}\int d^4p \frac{\operatorname{tr}\gamma_\alpha\{(\gamma\cdot p + m)\gamma_\beta[\gamma\cdot(p+k)+m]\gamma_-[\gamma\cdot(p+k)+m]\}(p+k)_-^{-\sigma}2\pi\delta(p^2-m^2)}{|(p+k)^2 - m^2|^2}. \tag{32}$$

One can write $\Gamma_{\alpha\beta}$ as

$$\Gamma_{\alpha\beta}(k) = g_{\alpha-}g_{\beta-}F_1 + g_{\alpha-}k_\beta F_2 + g_{\beta-}k_\alpha F_3 + g_{\alpha\beta}F_4 + k_\alpha k_\beta F_5.$$

Then it is clear that

$$F_1 = k_-^{-\sigma-1}f_1(k^2),$$
$$F_2 = k_-^{-\sigma}f_2(k^2),$$
$$F_3 = k_-^{-\sigma}f_3(k^2),$$
$$F_4 = k_-^{-\sigma+1}f_4(k^2),$$
$$F_5 = k_-^{-\sigma+1}f_5(k^2).$$

Dimensional counting says that F_1 is quadratically divergent, F_2, F_3, and F_4 are logarithmically divergent, and F_5 is convergent. This can be verified from (32). Equation (32) does not have current conservation. To obtain current conservation one must add the graphs shown in Fig. 11. Then current conservation requires that the divergent parts of F_1, F_2, F_3, and F_4 take the tensor form

$$\Gamma_{\alpha\beta}(k) = [g_{\alpha\beta}k_-^2 - k_-(g_{\alpha-}k_\beta + g_{\beta-}k_\alpha) + k^2 g_{\alpha-}g_{\beta-}]F(k), \tag{33}$$

which is the form of the bare cut vertex. The quadratic divergence is reduced to a logarithmic divergence. $\Gamma_{\alpha\beta}$ is renormalized by defining

$$\Gamma_{\alpha\beta}(k) = \Gamma^u_{\alpha\beta}(k) - \hat{\Gamma}^u_{\alpha\beta}(k),$$

where

$$\hat{\Gamma}^u_{\alpha\beta}(k) = [g_{\alpha\beta}k_-^2 - k_-(g_{\alpha-}k_\beta + g_{\beta-}k_\alpha) + k^2 g_{\alpha-}g_{\beta-}]k_-^{-\sigma-1}F(0). \tag{34}$$

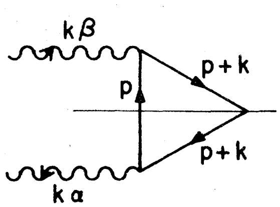

FIG. 10. A radiative correction to $\Gamma_{\alpha\beta}(k)$.

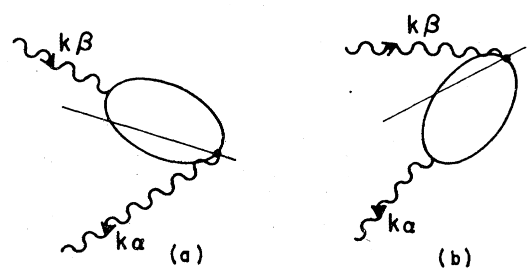

FIG. 11. Further radiative corrections to $\Gamma_{\alpha\beta}(k)$.

That is, one separates out the particular tensor form given in (25) and evaluates the coefficient of this form at $\hat{k}$. This is then subtracted from $\Gamma_{\alpha\beta}$. Dimensional counting says that the particular tensor given in (25) is the only tensor which can have a divergent coefficient.

(3). Renormalization of cut vertices. Let us begin by listing the bare vertices which will be need needed. We have previously introduced

$$\Gamma_{\sigma}^{(0)}(p)=\gamma_{-}p_{-}^{-\sigma}, \tag{12}$$

$$\Gamma_{11,\sigma}^{(0)}(p,k)=-g\gamma_{-}\frac{p_{-}^{-\sigma}}{k_{-}}, \tag{20}$$

$$\Gamma_{12,\sigma}^{(0)}(p,k)=g\gamma_{-}\frac{(p+k)_{-}^{-\sigma}}{k_{-}}, \tag{21}$$

and

$$\Gamma_{\alpha\beta,\sigma}^{(0)}(k)=4[g_{\alpha\beta}k_{-}^{2}-k_{-}(g_{\alpha-}k_{\beta}+g_{\beta-}k_{\alpha})+k^{2}g_{\alpha\beta}]k^{-\sigma-1}. \tag{25}$$

The additional vertices which are needed are

$$\Gamma_{ni,\sigma}^{(0)}(p,k_1,\ldots,k_n)=(-1)^{n+i-1}g^{n}\gamma_{-}\frac{(p+k_1+\cdots+k_{i-1})_{-}^{-\sigma}}{k_{1-}k_{2-}\cdots k_{n-}} \tag{35}$$

illustrated in Fig. 12. This vertex has n photons of Lorentz index + and the cut comes between the $i-1$ and i photons. Only + index photons are considered because it is only photons of this type which are necessitated as counterterms for divergences in the perturbation expansion of (22).

Before giving a general prescription for renormalizing cut vertices, a few definitions are necessary.[17] First, we wish to give a definition of a renormalization part. Roughly speaking a renormalization part is a proper subgraph γ which has a divergence. If the subgraph γ does not include the bare cut vertex, then a renormalization part is any subgraph which has superficial degree of divergence greater than or equal to zero, with one exception. The exception is the four-photon vertex which has a superficial degree of divergence equal to zero but does not diverge when a gauge-invariant combination of graphs is taken. Thus a subgraph γ, having only four external photon lines, is not a renormalization part. If γ contains a bare cut vertex, then it is a renormalization part if (i) it only has two external fermion lines, (ii) it has two external fermion lines and an arbitrary number of external + index photon lines, or (iii) it only has two external photon lines.

Now a forest U is a set of nonoverlapping renormalization parts γ of a graph G. U may include the null set and may include G itself. The general prescription for renormalization of a cut vertex is

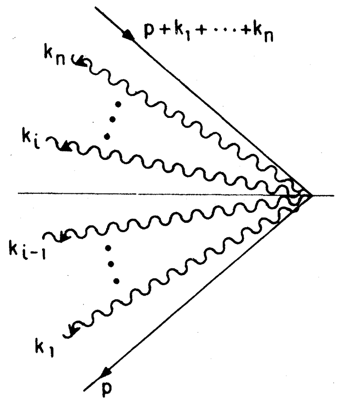

FIG. 12. A two-fermion and n-photon bare cut vertex.

$$\Gamma_{\sigma}(p)=\sum_{U}\prod_{\gamma\in U}(-t^{\gamma})\Gamma_{\sigma}^{u}(p), \tag{36}$$

where Γ_{σ}^{u} is the unrenormalized cut vertex. We view (36) in the following way. There is presumed to be a current-conserving regulation of Γ_{σ}^{u}, say by Pauli-Villars regularization. When a subtraction is done on a renormalization part at a certain order of g it is supposed that all graphs of that order of g are included. Thus we may assume the appropriate Ward identities for a given renormalization part. Then if γ does not include a bare cut vertex, t^{γ} is the usual subtraction one does in renormalizing a massive neutral vector-meson theory. If γ contains an elementary cut vertex, we do the following. If γ has only two external fermion lines, $t^{\gamma}\Gamma_{\sigma}^{\gamma}(p)=\hat{\Gamma}_{\sigma}^{\gamma}(\hat{p})$. If γ has two external fermion lines and n + index photon lines, then

$$t^{\gamma}\Gamma_{ni,\sigma}^{\gamma}(p,k_1,k_2\cdots k_n)=\hat{\Gamma}_{ni,\sigma}^{\gamma}(\hat{p},\hat{k}_1\cdots\hat{k}_n),$$

where the caret means that one takes only the γ_{-} Dirac-index term. If γ only has two external photon lines, then $t^{\gamma}\Gamma_{\alpha\beta}^{\gamma}(k)=\hat{\Gamma}_{\alpha\beta}^{\gamma}(k)$. The only complexity in (36) which is not already covered by Zimmermann's discussion of renormalization of composite operators is that here we have no rigorous power-counting theorem. Power counting is discussed in Appendix C.

IV. ADDITIONAL SUBTRACTIONS AND THE CALLAN-SYMANZIK EQUATION

In dealing with ordinary time-ordered products the Wilson expansion is derived by doing additional subtractions on Green's functions. These subtractions separate a Green's function into a singular Wilson coefficient times the renormalized Green's function involving a composite operator. We shall do much the same in this section, only differing in that cut vertices rather than local operators will occur.

A. Additional subtractions

Consider the amplitude

$$M_{\nu\mu}(p,q)=\int d^4x\, d^4y\, d^4z\, e^{ip(z-y)+iqx}\langle \overline{T}(\bar{\psi}(y)j_\nu(x))T(\psi(z)j_\mu(0))\rangle_c\,, \tag{37}$$

where amputation of the external fermion lines is understood. This process is shown in Fig. 13. $M_{\nu\mu}$ is defined by the usual subtraction procedure

$$M_{\nu\mu}(p,q)=\sum_U \prod_{\gamma\in U}(-t^\gamma)M^u_{\nu\mu}(p,q)\,. \tag{38}$$

Now take $p_0=(\vec{p}^{\,2}+m^2)^{1/2}$. Then

$$\frac{1}{4\pi}\operatorname{tr}\left(\frac{\gamma\cdot p+m}{2m}M_{\nu\mu}\right)=-\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\overline{W}_L+\frac{1}{m^2}\left[p_\mu p_\nu-\frac{p\cdot q}{q^2}(p_\mu q_\nu+p_\nu q_\mu)+\frac{g_{\mu\nu}(p\cdot q)^2}{q^2}\right]\overline{W}_2\,. \tag{39}$$

In particular,

$$\frac{1}{4\pi}\operatorname{tr}\left(\frac{\gamma\cdot p+m}{2m}M_{--}\right)=\frac{q_-^{\,2}}{q^2}\overline{W}_L-\frac{q^2p_-^{\,2}}{m^2q^2}\overline{W}_2 \tag{40}$$

when q_- and p_μ are finite with q_+ large. From (40) it is apparent that $\overline{W}_L$ and $\overline{W}_2$ can be determined from M_{--} alone. For simplicity we restrict ourselves to this component of $M_{\nu\mu}$.

For a given Feynman graph contributing to (37), break up the graph into two parts[17] as shown in Fig. 14 where a particular cut is indicated. Call the right-hand part of the graph τ and the left-hand part λ. The propagators connecting the two parts of the graph are included in λ. We require that τ be such that the lines connecting λ and τ be (i) just two fermion lines, (ii) fermion lines along with an arbitrary number of + photon lines, or (iii) two photon lines.

Define $\mathfrak{U}(\tau)$ to be the set of all normal forests, U_2, of the subgraph τ. (In a normal forest $\tau\in U_2$.) Also, define $\mathfrak{M}_\tau$ to be the set of all forests, U_1, in λ. Then, if t_2^γ denotes the usual subtraction operator in a massive vector-meson theory,

$$M_{--}=\sum_\tau\sum_{U_1\in\mathfrak{M}_\tau}\sum_{U_2\in\mathfrak{U}(\tau)}\prod_{\gamma\in U_1}(-t_1^\gamma)t_1^\tau\prod_{\gamma\in U_2}(-t_2^\gamma)M^u_{--}+\sum_U\prod_{\gamma\in U}(-t_1^\gamma)M^u_{--} \tag{41}$$

is an algebraic identity known as Zimmermann's identity. In discussing t_1^τ it is useful to consider separately the three classes of lines which can connect τ and λ.

First consider decompositions of the form shown in Fig. 15 where two fermion lines connect λ and τ. Call $M^\tau_{\nu\mu}(k,q)$ the renormalized right-hand part of Fig. 15. One wants $t_1^\gamma M^\tau_{\nu\mu}(k,q)$ to be equal to the additional divergences, involving the subgraph τ, which arise when $q^2\to\infty$ for fixed k and fixed $k\cdot q/q^2$. (We shall only consider the dominant power in q.)

We must first isolate the tensor structures which can contribute. Write

$$M_{\nu\mu}(k,q)=\sum_i t^i_{\nu\mu}(k,q)M_i(k^2,k\cdot q,q^2)\,, \tag{42}$$

where the M_i are dimensionless and the $t^i_{\nu\mu}$ are matrices in the Dirac indices. The rules for which t^i can contribute when $q^2\to\infty$ are as follows: (i) Only conserved tensors need be considered since slightly off-shell fermions will not inhibit current

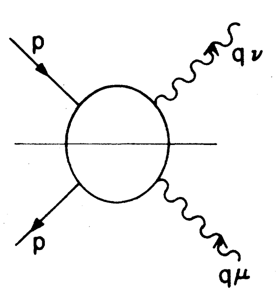

FIG. 13. The hadronic part of the amplitude, $M_{\nu\mu}$, occurring in e^+e^- inclusive annihilation.

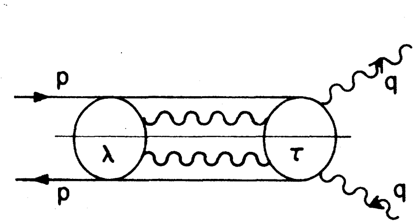

FIG. 14. A general separation of $M_{\nu\mu}$.

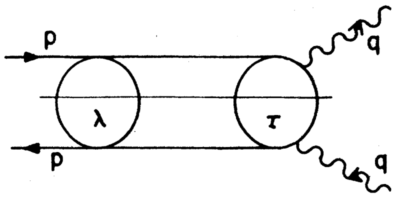

FIG. 15. A separation of $M_{\nu\mu}$ exposing two fermion lines.

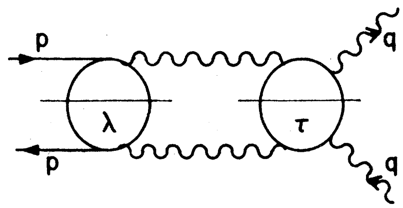

FIG. 16. A separation of $M_{\nu\mu}$ exposing two photon lines.

conservation when the momentum q flows through the whole graph. (ii) The tensors should be symmetric in ν and μ. (iii) Drop all terms with explicit factors of m or k^2. (iv) Drop all factors of $\gamma\cdot k$ compared to $\gamma\cdot q$. The allowed tensors are of the form

$$t^L_{\nu\mu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{\gamma\cdot q}{q^2} \tag{43}$$

and

$$t^2_{\nu\mu} = \tfrac{1}{2}[2g_{\mu\nu}\gamma\cdot q q\cdot k + (\gamma_\mu k_\nu + \gamma_\nu k_\tau)q^2 - (\gamma_\mu q_\nu + \gamma_\nu q_\mu)k\cdot q - (k_\mu q_\nu + k_\nu q_\mu)\gamma\cdot q]\frac{1}{(q^2)^2}\,. \tag{44}$$

Note that these two tensors correspond to the two terms in (39). Setting $\mu=\nu=-$,

$$t^L_{--} = +\frac{q_-{}^2\gamma\cdot q}{(q^2)^2} \approx +\frac{q_-{}^2 q_+\gamma_-}{(q^2)^2} \tag{45}$$

and

$$t^2_{--} = -\frac{q^2}{(q^2)^2}\gamma_- k_-\,. \tag{46}$$

Call

$$\hat{M}^\tau_{\nu\mu}(k,q) = \sum_{i=L,2} t^i_{\nu\mu}(k,q)M^\tau_i(k^2, k\cdot q, q^2)\,, \tag{47}$$

then

$$t^\tau_1 M^\tau_{--}(k,q) = \hat{M}^\tau_{--}(\hat{k},q)\,. \tag{48}$$

Graphs of the type shown in Fig. 14, where two fermions and an arbitrary number of + photons connect the two parts of the graph, are handled in the following way. Suppose $M^\tau_{\alpha_1\cdots\alpha_n,\nu\mu}(k_1\cdots k_n, kq)$ is the renormalized amplitude amplitude for the right-hand side of Fig. 14. $\alpha_1\cdots\alpha_n$ are the photon indices and the divergent part, for large q^2, has a factor $q_{\alpha_1 -}g_{\alpha_2 -}\cdots g_{\alpha_n -}$. Then

$$k_{\alpha_1}\cdots k_{\alpha_n}M^\tau_{\alpha_1\cdots\alpha_n,\nu\mu}(k_1\cdots k_n, k, q) = k_{1-}\cdots k_{n-}M^\tau_{+\cdots+,\nu\mu}(k_1\cdots k_n, k, q)$$

is determined by the usual Ward identities in terms of $M^\tau_{\nu\mu}$. The subtraction dictated by t^τ_1 for $M^\tau_{+\cdots+,\nu\mu}$ is thus determined by that for $M^\tau_{\nu\mu}$.

Consider now the decomposition shown in Fig. 16. These two photons connect the left-hand and right-hand sides of the graph. Let $M^\tau_{\alpha\beta\nu\mu}(k,q)$ be the four current amplitudes. Again we write

$$M^\tau_{\alpha\beta,\nu\mu}(k,q) = \sum_i t^i_{\alpha\beta,\nu\mu}(k,q)M^\tau_i(k^2, k\cdot q, q^2)\,, \tag{42}$$

where the t^i are the possible tensor structures and M_i are dimensionless. For large q^2 only two of the eight tensors listed in Ref. 25 give divergent coefficients in leading order of q^2. If our $\alpha, \beta, \nu, \mu, k, q$ correspond to $\alpha, \beta, \lambda, \sigma, q_2, q_1$ in Brown and Muzinich, then these tensors are their equations (A7) and (A10). They are

$$t^1_{\alpha\beta,\nu\mu}(k,q) = 2(g_{\mu\nu}q^2 - q_\mu q_\nu)\{g_{\alpha\beta}(k\cdot q)^2 - (q_\alpha k_\beta + q_\beta k_\alpha)k\cdot q + q_\alpha q_\beta k^2\}\frac{1}{(q^2)^3} \tag{49}$$

and

$$t^2_{\alpha\beta,\nu\mu}(k,q) = (g_{\mu\nu}q_\rho q_\sigma + g_{\mu\rho}g_{\nu\sigma}q^2 - g_{\mu\rho}q_\nu q_\sigma - g_{\nu\sigma}q_\mu q_\rho)(g_{\alpha\beta}k_\rho k_\sigma + g_{\alpha\rho}g_{\beta\sigma}k^2 - g_{\alpha\rho}k_\beta k_\sigma - g_{\beta\sigma}k_\alpha k_\rho)\frac{1}{(q^2)^2}\,. \tag{50}$$

When $(\mu,\nu)=(-,-)$ and $q_+ \sim q^2$ large,

$$t^1_{\alpha\beta,--} = -\frac{2q_-{}^2 q_+{}^2}{(q^2)^3}[g_{\alpha\beta}k_-{}^2 - k_-(g_{\alpha -}k_\beta + g_{\beta -}k_\alpha) + g_{\alpha -}g_{\beta -}k^2] \tag{51}$$

and

$$t^2_{\alpha\beta,--} = -\frac{q^2}{(q^2)^2}[g_{\alpha\beta}k_-{}^2 - k_-(g_{\alpha -}k_\beta + g_{\beta -}k_\alpha) + g_{\alpha -}g_{\beta -}k^2]\,. \tag{52}$$

Each of these tensors takes the form of the bare cut vertex given in (25). We define

$$t_1^\tau M_{\alpha\beta}^\tau(k,q)=\sum_{i=1}^{2} t^i_{\alpha\beta,--}(k,q)M_i^\tau(0,k\cdot q,q^2)\,. \qquad (53)$$

We have thus given a detailed description of the meaning of t_1^τ in (41). At this point we could also define t_1^γ in essentially the same manner as the product of t_2^γ and the additional subtractions just outlined for t_1^τ. We shall discuss this a little more fully after a diagonalization has been performed, at which point it will be apparent that the t_1^γ subtractions are just those we have previously discussed in the renormalization of cut vertices.

B. Diagonalization of the subtracted amplitudes

In this section we shall diagonalize Eq. (41). For the term shown in Fig. 15 one can write

$$M_{ab,--}(p,q)=\frac{1}{(2\pi)^4}\int d^4k\, T_{ab,cd}(p,k)\hat{M}^\tau_{cd,--}(\hat{k},q)\,, \qquad (54)$$

where T is given in (14). We are suppressing the $-t_1^\gamma$ terms of (41) for the moment and dropping the sum over τ. Now define, for $p^2=m^2$,

$$\frac{1}{4\pi}\mathrm{tr}\left[\frac{\gamma\cdot p+m}{2m}M_{--}(p,q)\right]=\overline{M}(p,q) \qquad (55)$$

and

$$\frac{1}{4\pi}\sum_{ab}\left(\frac{\gamma\cdot p+m}{2m}\right)_{ab}T_{ab,cd}=\overline{T}_{cd}\,. \qquad (56)$$

Then

$$\overline{M}(p,q)=\frac{1}{(2\pi)^4}\int d^4k\,\overline{T}_{cd}(p,k)\hat{M}^\tau_{cd,--}(\hat{k},q)\,. \qquad (57)$$

The two tensors in $\hat{M}^\tau_{cd,--}(k,q)$ correspond to $\overline{W}_L$ and $\overline{W}_2$ given in (39). One obtains

$$\overline{W}_2=\frac{m^2}{q^2p_-^2}\frac{1}{(2\pi)^4}\int d^4k\,\overline{T}_{cd}(p,k)(\gamma_-)_{cd}k_-M_2^\tau(0,k\cdot q,q^2) \qquad (58)$$

and

$$\overline{W}_L=\frac{1}{q^2}\frac{1}{(2\pi)^4}\int d^4k\,\overline{T}_{cd}(p,k)(\gamma_-)_{cd}\frac{1}{k_-}k\cdot qM_L^\tau(0,k\cdot q,q^2)\,. \qquad (59)$$

Define

$$\int_0^1 \nu\overline{W}_2(\nu,q^2)\omega^{\sigma-1}d\omega=C_\sigma^{(1)}E_\sigma^{(1)}(q^2) \qquad (60)$$

for large q^2. $C_\sigma^{(1)}$ is a normalization factor to be fixed later. Then

$$C_\sigma^{(1)}E_\sigma^{(1)}(q^2)=\tfrac{1}{2}mp_-^{\sigma-1}\frac{1}{(2\pi)^4}\int d^4k\,\overline{T}_{cd}(\gamma_-)_{cd}k_-^{-\sigma}\times\int_0^1 d\omega\,\omega^\sigma M_2^\tau(0,\omega,q^2)\,. \qquad (61)$$

Referring to Fig. 15 we see that, after putting the τ sum of (41) back in (61), and including a disconnected part

$$C_\sigma^{(1)}E_\sigma^{(1)}(q^2)=\lambda_\sigma^{(1,1)}\int_0^1 d\omega\,\omega^\sigma M_2(0,\omega,q^2)\,, \qquad (62)$$

where

$$\lambda_\sigma^{(1,1)}=\frac{p_-^{\sigma-1}}{4}\mathrm{tr}[(\gamma\cdot p+m)\Gamma_\sigma(p)]\Big|_{p^2=m^2} \qquad (63)$$

is the contribution of the bare cut fermion vertex and its renormalization.

From (61) it is apparent that the renormalization of Γ_σ and the $(-t_1^\gamma)$ factors in (41) are identical procedures. If we choose $C_\sigma^{(1)}$ to be equal to $\lambda_\sigma^{(1,1)}$ it is apparent that $E_\sigma^{(1)}$ is independent of the particle labeled by p, all such dependence being in Γ_σ. Equation (62) is the analog of the Wilson expansion with the cut vertices being analogous to the matrix elements of local operators. Also, if we define

$$m\int_0^1 \overline{W}_L\omega^{\sigma-2}d\omega=C_\sigma^{(1)}F_\sigma^{(1)}(q^2) \qquad (64)$$

then

$$C_\sigma^{(1)}F_\sigma^{(1)}(q^2)=\lambda_\sigma^{(1,1)}\int_0^1 d\omega\,\omega^{\sigma-1}M_L(0,\omega,q^2)\,. \qquad (65)$$

Terms with two fermions and an arbitrary number of photons are simply related to the two-fermion case by Ward identities. They give the contribution to Γ_σ, and hence to $\lambda_\sigma^{(1,1)}$ and $C_\sigma^{(1)}$, in Eq. (62) from the bare vertices (35). Finally, consider the two photon intermediate states shown in Fig. 16. One can write

$$M_{--}(p,q)=\frac{1}{(2\pi)^4}\int d^4k\,M_{\alpha\beta}(p,k)t_1^\tau M^\tau_{\alpha\beta,--}(k,q)\,. \qquad (66)$$

Taking the two tensors (51) and (52) one obtains

$$\overline{W}_2=\frac{m^2}{q^2p_-^2}\int d^4k\,\overline{M}_{\alpha\beta}(p,k)[\,g_{\alpha\beta}k_-^2-k_-(g_{\alpha-}k_\beta+g_{\beta-}k_\alpha)+g_{\alpha-}g_{\beta-}k^2]M_2^\tau(0,k\cdot q,q^2) \qquad (67)$$

and

$$\overline{W}_L = \frac{2}{(q^2)^2} \int d^4k\, \overline{M}_{\alpha\beta}(p,k)[g_{\alpha\beta}k_-^2 - k_-(g_{\alpha-}k_\beta + g_{\beta-}k_\alpha) + g_{\alpha-}g_{\beta-}k^2]\,\frac{1}{k_-^2}(k\cdot q)^2 M_1^\tau(0, k\cdot q, q^2)\,. \tag{68}$$

If these contributions to the moments are defined as

$$\int_0^1 \nu \overline{W}_2 \omega^{\sigma-1} d\omega = C_\sigma^{(2)} E_\sigma^{(2)}(q^2) \tag{69}$$

and

$$m \int_0^1 \overline{W}_L \omega^{\sigma-2} d\omega = C_\sigma^{(2)} F_\sigma^{(2)}(q^2)\,, \tag{70}$$

then

$$C_\sigma^{(2)} E_\sigma^{(2)}(q^2) = \lambda_\sigma^{(1,2)} \sum_\tau \int_0^1 \omega^\sigma d\omega\, M_2^\tau(0,\omega,q^2) \tag{71}$$

and

$$C_\sigma^{(2)} F_\sigma^{(2)}(q^2) = \lambda_\sigma^{(1,2)} \sum_\tau \int_0^1 \omega^\sigma d\omega\, M_1^\tau(0,\omega,q^2)\,. \tag{72}$$

Here

$$\lambda_\sigma^{(1,2)} = \frac{p_-^{\sigma-1}}{16}\,\mathrm{tr}(\gamma\cdot p + m)\int d^4k\, M_{\alpha\beta}(p,k)\Gamma_{\alpha\beta}^{(0)}(k) \tag{73}$$

is the contribution from the two-photon bare cut vertices.

C. The Callan-Symanzik equation

Once one has the Wilson expansion or its analog given here in (60) and (62), it is a simple matter to derive a Callan-Symanzik equation for moments of the structure functions. In doing so it is convenient to choose $C_\sigma^{(1)} = 1$ and $C_\sigma^{(2)} = 0$. We can do this by a finite renormalization of the cut vertices so that $\lambda_\sigma^{(1,1)} = 1$ and $\lambda_\sigma^{(1,2)} = 0$. Define the usual Callan-Symanzik differential operator

$$D_i = m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial g} - 2\gamma_i\,. \tag{74}$$

$i=1$ stands for external fermions and $i=2$ for external gluons. Apply D_1 to Eq. (41). One obtains, for large q^2,

$$D_1 M_{ab,--}(p,q) = \frac{1}{(2\pi)^4}\int d^4k [D_1 T_{ab,cd}(p,k)]\hat{M}_{cd}(\hat{k},q)\,, \tag{75}$$

where again the $-t_1^\tau$ factors have been suppressed. D_1 cannot act on the right-hand side of Fig. 14 because τ would no longer give a divergent contribution in the limit of large q^2. Thus one obtains

$$D_1\overline{W}_2 = \frac{m^2}{q^2 p_-^2}\frac{1}{(2\pi)^4}\int d^4k\, D_1\overline{T}_{cd}(p,k)(\gamma_-)_{cd} \times M_2(0,k\cdot q,q^2) \tag{76}$$

and

$$D_1\overline{W}_L = \frac{1}{q^2}\frac{1}{(2\pi)^4}\int d^4k\, D_1\overline{T}_{cd}(p,k)(\gamma_-)_{cd}\frac{1}{k_-}k\cdot q \times M_L(0,k\cdot q,q^2)\,. \tag{77}$$

After adding the contributions of Figs. 14 and 16 we obtain

$$D_1 E^{(1)} = a_\sigma^{(1,1)}E_\sigma^{(1)} + a_\sigma^{(1,2)}E_\sigma^{(2)}\,, \tag{78}$$

where

$$a_\sigma^{(1,1)} = \frac{p_-^{\sigma-1}}{4}\frac{(\gamma\cdot p+m)_{ba}}{(2\pi)^4} \times \int d^4k\, D_1 T_{ab,cd}(p,k)(\gamma_-)_{cd}k_-^{-\sigma} + \cdots\,. \tag{79}$$

The terms not written explicitly are the contributions of the bare cut vertices involving two fermions and n photons. This equation is illustrated in Fig. 17. To complete the equations one must add the contributions obtained from considering four current processes. From these one obtains

$$D_2 E_\sigma^{(2)} = a_\sigma^{(2,1)}E_\sigma^{(1)} + a_\sigma^{(2,2)}E_\sigma^{(2)}\,, \tag{80}$$

where $a_\sigma^{(2,1)}$ and $a_\sigma^{(2,2)}$ are illustrated in Fig. 18. The set of equations (78) and (80) are exactly as

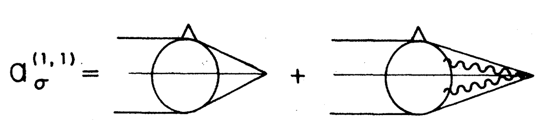

FIG. 17. Graphs contributing to $a_\sigma^{(1,1)}$. The caret on the left-hand side of the graph is a mass insertion.

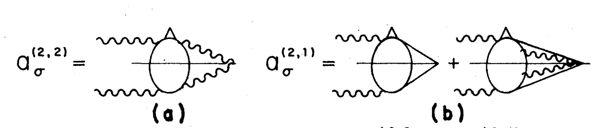

FIG. 18. Graphs contributing to $a_\sigma^{(2,2)}$ and $a_\sigma^{(2,1)}$. The carets on the left-hand parts of the graphs are mass insertions.

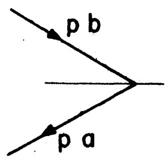

FIG. 19. A bare fermion cut vertex. The a and b represent internal-symmetry (color) labels.

found in Ref. 21. For completeness we list the values of the $a_\sigma^{(i,j)}$ to order g^2:

$$a_\sigma^{(1,1)} = -\frac{g^2}{8\pi^2}\frac{1}{(\sigma-1)(\sigma-2)} + \frac{g^2}{4\pi^2}\left[\sum_{l=1}^{\infty}\left(\frac{1}{l+1} - \frac{1}{\sigma+l-1}\right) - \frac{1}{\sigma-1}\right], \quad (81)$$

$$a_\sigma^{(1,2)} = -\frac{g^2}{8\pi^2}\frac{\sigma^2-3\sigma+4}{\sigma(\sigma-1)(\sigma-2)}, \quad (82)$$

$$a_\sigma^{(2,1)} = -\frac{g^2}{4\pi^2}\frac{\sigma^2-3\sigma+4}{(\sigma-1)(\sigma-2)(\sigma-3)}, \quad (83)$$

$$a_\sigma^{(2,2)} = 0. \quad (84)$$

V. NON-ABELIAN GAUGE THEORIES

In this section we shall give the cut vertices necessary to deal with Yang-Mills gauge theories. We shall deal with a general gauge group, G, having structure functions C_{ijk} with the fermions (quarks) in a representation R. The Lagrangian under consideration is

$$\mathcal{L} = -\tfrac{1}{4}F^i_{\mu\nu}F^i_{\mu\nu} + i\bar{\psi}_a\gamma_\mu(\partial_\mu\delta_{ab} - igT^i_{ab}A^i_\mu)\psi_b - m\bar{\psi}\psi$$

with

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + gC_{ijk}A^j_\mu A^k_\nu .$$

The bare cut vertex involving two fermions is

$$\Gamma_\sigma^{ab}(p) = \gamma_- p_-^{-\sigma}\delta_{ab}, \quad (85)$$

where the a, b indices refer to the representation R. These indices will often be suppressed. This vertex is shown in Fig. 19. The vertices involving two fermions and a single gauge field are shown in Fig. 20. They are

$$\Gamma_\sigma^{ai,b}(p,k) = +\frac{g\gamma_-}{k_-}T^i_{ab}(p+k)_-^{-\sigma} \quad (86)$$

and

$$\Gamma_\sigma^{a,bi}(p,k) = -\frac{g\gamma_-}{k_-}T^i_{ab}(p)_-^{-\sigma}. \quad (87)$$

As in the Abelian case, these vertices obey Ward identities which in fact determine their form. Thus

$$k_-\Gamma_\sigma^{ai,b}(p,k) = +gT^i_{aa'}\Gamma_\sigma^{a'b}(p+k) \quad (88)$$

and

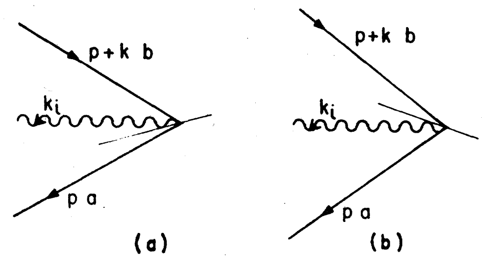

FIG. 20. (a) $\Gamma^{a,bi}(p,k)$. (b) $\Gamma_\sigma^{ai,b}(p,k)$.

$$k_-\Gamma_\sigma^{a,bi}(p,k) = -g\Gamma_\sigma^{ab'}(p)T^i_{b'b}. \quad (89)$$

Vertices with more than one gauge field become rather complicated. We shall list, for reference purposes, those vertices having two fermions and two gauge fields. They are illustrated in Fig. 21. The vertices are

$$\Gamma_\sigma^{aij,b}(p,k_1,k_2) = \frac{g^2\gamma_-(p+k_1+k_2)_-^{-\sigma}}{k_{1-}+k_{2-}} \times \left(\frac{T^iT^j}{k_{2-}} + \frac{T^jT^i}{k_{1-}}\right)_{ab}, \quad (90)$$

$$\Gamma_\sigma^{ai,bj}(p,k_1,k_2) = -\frac{g^2\gamma_-(p+k_1)_-^{-\sigma}}{k_{1-}k_{2-}}(T^iT^j)_{ab}, \quad (91)$$

$$\Gamma_\sigma^{a,bij}(p,k_1,k_2) = \frac{g^2\gamma_-}{k_{1-}+k_{2-}}p_-^{-\sigma}\left(\frac{T^iT^j}{k_{1-}} + \frac{T^jT^i}{k_{2-}}\right)_{ab}. \quad (92)$$

These vertices obey the Ward identities

$$k_{1-}k_{2-}\Gamma_\sigma^{ai,bj}(p,k_1,k_2) = (gT^i_{aa'})\Gamma_\sigma^{a'b'}(p+k_1)(-gT^j_{b'b}) \quad (93)$$

and

$$k_{1-}k_{2-}\Gamma_\sigma^{aij,b}(p,k_1,k_2) = (gT^i_{aa''})(gT^j_{a''a'})\Gamma_\sigma^{a'b}(p+k_1+k_2) + k_{1-}igC_{jil}\Gamma_\sigma^{al,b}(p,k_1+k_2). \quad (94)$$

Finally we need cut vertices corresponding to the two photon vertices of (25). However, two gauge particle vertices are not gauge invariant for the same reason that two fermion vertices are not

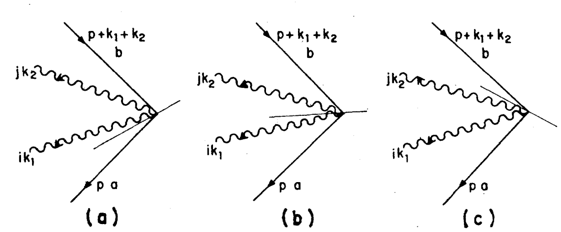

FIG. 21. (a) $\Gamma_\sigma^{a,bij}(p,k_1,k_2)$. (b) $\Gamma_\sigma^{ai,bj}(p,k_1,k_2)$. (c) $\Gamma_\sigma^{aij,b}(p,k_1,k_2)$.

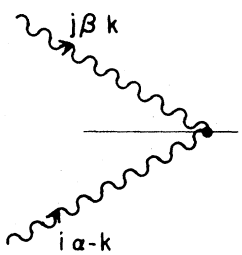

FIG. 22. $\Gamma^{ij}_{\alpha\beta}(k)$.

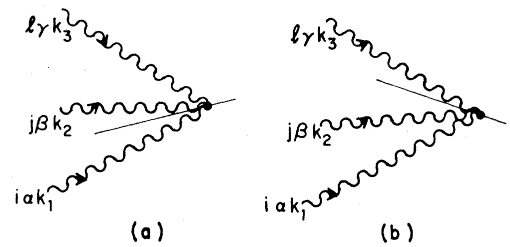

FIG. 23. (a) $\Gamma^{i,jl}_{\alpha,\beta\gamma}(k_1,k_2,k_3)$. (b) $\Gamma^{ij,l}_{\alpha\beta,\gamma}(k_1,k_2,k_3)$.

gauge invariant. One must include vertices involving up to four gauge particles along with an arbitrary number of + Lorentz index gauge fields. We shall list only the two and three gauge field vertices. The higher vertices are straightforward to construct, but they are somewhat cumbersome. The two line vertex is shown in Fig. 22 and is given by

$$\Gamma^{ij}_{\alpha\beta}(k)=4\delta_{ij}[g_{\alpha\beta}k_-^{\,2}-k_-(g_{\alpha-}k_\beta+g_{\beta-}k_\alpha)+g_{\alpha-}g_{\beta-}k^2](k_-)^{-\sigma-1}\,. \tag{95}$$

The three line vertices are shown in Fig. 23. They are given by

$$\Gamma^{ij,l}_{\alpha\beta,\gamma}(k_1,k_2,k_3)=\frac{2igC_{ijl}}{k_{2-}}\,g_{\beta-}(g_{\alpha\gamma}k_{1-}k_{3-}-k_{1-}g_{\gamma-}k_{3\alpha}-k_{3-}g_{\alpha-}k_{1\gamma}+g_{\alpha-}g_{\gamma-}k_1\cdot k_3)(k_{3-})^{-\sigma-1}$$
$$+2igC_{ijl}[k_{3-}(g_{\beta-}g_{\gamma\alpha}-g_{\alpha-}g_{\gamma\beta})+g_{\gamma-}(k_{3\beta}g_{\alpha-}-k_{3\alpha}g_{\beta-})](k_{3-})^{-\sigma-1}+\text{terms where }(i,\alpha,k_1)\leftrightarrow(j,\beta,k_2)\,. \tag{96}$$

$\Gamma^{i,jl}_{\alpha,\beta\gamma}(k_1,k_2,k_3)$ is identical to $\Gamma^{ij,l}_{\alpha\beta,\gamma}(k_1,k_2,k_3)$ except for a factor of (-1) along with the change of $(k_{3-})^{-\sigma-1}$ to $(-k_{1-})^{-\sigma-1}$.

The vertices (96) obey the Ward identities

$$k_{2\beta}\Gamma^{ij,l}_{\alpha\beta,\gamma}(k_1,k_2,k_3)=iC_{jir}g\Gamma^{rl}_{\alpha\gamma}(k_3) \tag{97}$$

and

$$k_{2\beta}\Gamma^{i,jl}_{\alpha,\beta\gamma}(k_1,k_2,k_3)=iC_{jlr}g\Gamma^{ir}_{\alpha\gamma}(-k_1)\,. \tag{98}$$

The Callan-Symanzik equations for the amplitudes occurring in inclusive e^+e^- annihilation are identical to (78) and (80). For convenience we write this equation as

$$\left[\left(m^2\frac{\partial}{\partial m^2}+\beta\frac{\partial}{\partial g}\right)\delta_{ij}-\gamma^\sigma_{ij}\right]E^{(j)}_\sigma(q^2)=0\,, \tag{99}$$

where the $\gamma^\sigma_{ij}=2\gamma_i\delta_{ij}+a^{(i,j)}_\sigma$ and the a's are calculated according to Eq. (79). To lowest order in g^2, in fact $2\gamma^\sigma_{ij}$ is the same as given by Gross and Wilczek and Georgi and Politzer with the replacement $n=-\sigma+1$ (Ref. 26):

$$\gamma^\sigma_{11}=\frac{g^2}{16\pi^2}C_2(R)\left[\left(1-\frac{2}{(\sigma-1)(\sigma-2)}\right)+4\sum_{l=1}^{\infty}\left(\frac{\sigma-2}{(l+1)(l+\sigma-1)}\right)-\frac{4}{\sigma-1}\right]\,, \tag{100}$$

$$\gamma^\sigma_{12}=-\frac{g^2}{16\pi^2}\frac{2(\sigma^2-3\sigma+4)}{\sigma(\sigma-1)(\sigma-2)}C_2(R)\,, \tag{101}$$

$$\gamma^\sigma_{21}=-\frac{g^2}{16\pi^2}\frac{4(\sigma^2-3\sigma+4)}{(\sigma-1)(\sigma-2)(\sigma-3)}T(R)\,, \tag{102}$$

$$\gamma^\sigma_{22}=\frac{g^2}{16\pi^2}\left[C_2(G)\left(\frac{1}{3}-\frac{4}{\sigma(\sigma-1)}-\frac{4}{(\sigma-2)(\sigma-3)}+4\sum_{l=1}^{\infty}\frac{\sigma-2}{(l+1)(l+\sigma-1)}-\frac{4}{\sigma-1}\right)+\tfrac{4}{3}T(R)\right]\,. \tag{103}$$

For $\sigma-2$ an integer ≥ 2, one has the identity

$$\sum_{l=1}^{\infty}\frac{\sigma-2}{(l+1)(l+\sigma-1)}-\frac{1}{\sigma-1}=\sum_{l=2}^{\sigma-2}\frac{1}{l}\,. \tag{104}$$

In the above $C_2(G)\delta_{ij}=\sum_{lk}C_{ilk}C_{jlk}$ and $\mathrm{tr}(T^iT^j)=T(R)\delta_{ij}$.

VI. APPLICATION TO $e^+ + e^- \to$ HADRON (p) + ANYTHING

It is not our purpose in this paper to dwell on applications; however, it may be helpful to discuss the sorts of predictions one obtains from the renormalization group applied to $e^+ + e^-$ annihilation. The cross section for $e^+ + e^- \to \pi^+(p)$ + anything is given by

$$\omega_p \frac{d\sigma}{d^3p} = \frac{\alpha^2}{2\pi} \frac{1}{(q^2)^3} (k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} k\cdot k') M_{\nu\mu}(p,q) , \tag{105}$$

where the process is shown in Fig. 24. k is the momentum of the electron, k' the momentum of the positron, $q = k + k'$, and p is the momentum of the π^+. In Eq. (105) the mass of the electron is neglected. $M_{\nu\mu}$ is given by

$$M_{\nu\mu}(p,q) = \int d^4x\, d^4y\, d^4z \langle \overline{T}(\phi^*(x) j_\nu(y)) T(\phi(z) j_\mu(0))\rangle e^{iqx - ip(y-z)} \left| \Delta'_F(p) \right|^{-2} , \tag{106}$$

where ϕ is a field for the π^+. Now write

$$-\frac{1}{8\pi m} M_{\nu\mu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \overline{W}_L + \frac{1}{m^2} \left[p_\mu p_\nu - \frac{p\cdot q}{q^2}(p_\mu q_\nu + p_\nu q_\mu) + \frac{g_{\mu\nu}(p\cdot q)^2}{q^2} \right] \overline{W}_2 . \tag{107}$$

Then

$$\omega_p \frac{d\sigma}{d^3p} = \frac{2m\alpha^2}{(q^2)^2} \left[2\overline{W}_L - \frac{\omega}{2m}\left(1 + \cos^2\theta + \frac{4m^2}{q^2\omega^2}\sin^2\theta\right) \nu \overline{W}_2 \right] , \tag{108}$$

where m is the mass of the π^+ and the electron mass has been neglected. If the π mass is dropped compared to $q^2\omega^2$, then

$$\omega_p \frac{d\sigma}{d^3p} = \frac{2m\alpha^2}{(q^2)^2} \left(2\overline{W}_L - \frac{\omega}{m} \frac{1+\cos^2\theta}{2} \nu \overline{W}_2 \right) . \tag{109}$$

We have normalizations such that exactly the same formula, Eq. (108), holds for the production of fermions after spin sums have been performed. Now take a moment of (109),

$$\int \omega_p \frac{d\sigma}{d^3p} \omega^\sigma d\omega = \frac{2\alpha^2}{(q^2)^2} \sum_{i=1}^{2} C_{\sigma+2}^{(i)} \left[2F_{\sigma+2}^{(i)}(q^2) - E_{\sigma+2}^{(i)}(q^2) \frac{1+\cos^2\theta}{2} \right] , \tag{110}$$

where, as before,

$$\int_0^1 \nu \overline{W}_2 \omega^{\sigma-1} d\omega = \sum_{i=1}^{2} C_\sigma^{(i)} E_\sigma^{(i)} , \tag{111}$$

$$m \int_0^1 \overline{W}_L \omega^{\sigma-2} d\omega = \sum_{i=1}^{2} C_\sigma^{(i)} F_\sigma^{(i)} . \tag{112}$$

The E and F obey[6]

$$\left[\left(-q^2 \frac{\partial}{\partial q^2} + \beta \frac{\partial}{\partial q} \right) \delta_{ij} - \gamma_{ij}^\sigma \right] E_\sigma^{(j)}(q^2) = 0 \tag{113}$$

along with a similar equation where $E \leftrightarrow F$. The moments of the structure functions then take the asymptotic form[6,22,23]

$$\int_0^1 \nu \overline{W}_2 \omega^{\sigma-1} d\omega = d_{2\sigma} (\ln q^2)^{-A_\sigma} , \tag{114}$$

$$m \int_0^1 \overline{W}_L \omega^{\sigma-2} d\omega = d_{L\sigma} (\ln q^2)^{-A_\sigma} , \tag{115}$$

where[22]

$$A_\sigma = \frac{1}{g^2 b_0} \{ \gamma_{11}^\sigma + \gamma_{22}^\sigma - [(\gamma_{11}^\sigma - \gamma_{22}^\sigma)^2 + 4\gamma_{12}^\sigma \gamma_{21}^\sigma]^{1/2} \} , \tag{116}$$

b_0 is given by

$$b_0 = \frac{1}{8\pi^2} \left[\tfrac{11}{3} C_2(G) - \tfrac{4}{3} T(R) \right] , \tag{117}$$

and $2\gamma_{ij}^\sigma$ is obtained, to lowest order, from the equations of Ref. 22 by using $\sigma = -n + 1$.[26]

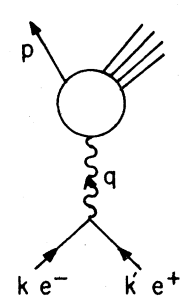

FIG. 24. An illustration of $e^+ + e^- \to$ hadron (p) + anything.

In particular, consider the integral

$$E_{\pi^+}=\frac{1}{\sigma}\int \omega_p \frac{d}{d^3p}\, d^3p\,, \qquad (118)$$

which gives the amount of energy carried off by the production of π^+'s in the e^+e^- collision. σ is the total cross section for $e^+ + e^- \to$ hadrons. Since $p \lesssim m_\pi$ contributes little to this integral we may write, always in the center-of-mass system of the e^+e^-,

$$E_{\pi^+}=\frac{2\pi}{\sigma}(\tfrac{1}{4}q^2)^{3/2}\int \omega_p \frac{d\sigma}{d^3p}\, d\cos\theta\, \omega^2 d\omega\,. \qquad (119)$$

Referring to (110) one finds

$$E_{\pi^+}=\frac{2\pi\alpha^2}{\sqrt{q^2}\sigma}\sum_{i=1}^{2} C_4^{(i)}[F_4^{(i)}(q^2)-\tfrac{1}{3}E_4^{(i)}(q^2)]\,. \qquad (120)$$

The reader can easily check that γ_{ij}^4 has a zero eigenvalue so that it is possible for π^+'s to carry off a finite fraction of the energy.

Another quantity of interest is the average multiplicity (of π^+'s in our case) of particles produced. The formula is

$$\bar{n}=\frac{1}{\sigma}\int \omega_p \frac{d\sigma}{d^3p}\frac{d^3p}{\omega_p}\,. \qquad (121)$$

Assuming that $p \gg m$ one can write

$$\bar{n}=\frac{\pi q^2}{2\sigma}\int \omega_p \frac{d\sigma}{d^3p}\,\omega\, d\omega\,. \qquad (122)$$

Again referring to (110) one finds

$$\bar{n}=\frac{4\pi\alpha^2}{\sigma}\sum_{i=1}^{2} C_3^{(i)}(F_3^{(i)}-\tfrac{1}{3}E_3^{(i)})\,. \qquad (123)$$

Unfortunately some of the γ_{ij}^σ have poles at $\sigma=3$ so that the above formula is meaningless in an asymptotically free theory. This reflects the fact that gauge theories have copious production of gauge particles which then can change into other particles. In gauge theories, in contrast to non-gauge theories, the average multiplicity of produced particles is not determined by the renormalization group alone.

We can see the above a little more clearly in a slightly different way. The equation

$$\int_0^1 \nu W_2 \omega^{\sigma-1} d\omega=\sum_{i=1}^{2} C_\sigma^{(i)}E_\sigma^{(i)} \qquad (111)$$

can be inverted to give

$$\nu \overline{W}_2(\omega,q^2)=\frac{1}{2\pi i}\sum_{i=1}^{2}\int_{L-i\infty}^{L+i\infty} d\sigma\, \omega^{-\sigma} C_\sigma^{(i)}E_\sigma^{(i)}\,, \qquad (124)$$

where L must be chosen to the right of all singularities in the σ plane. For the full cross section we may write

$$\omega_p\frac{d\sigma}{d^3p}=\frac{\alpha^2}{\pi i(q^2)^2}\sum_{i=1}^{2}\int_{L-i\infty}^{L+i\infty} d\sigma\, \omega^{-\sigma}\left(2F_{\sigma+1}^{(i)}-\frac{1+\cos^2\theta}{2}E_{\sigma+1}^{(i)}\right)C_{\sigma+1}^{(i)}\,. \qquad (125)$$

For fixed ω and large q^2, Eq. (125) should be correct. As ω becomes smaller it becomes efficient to distort the σ contour to the left. How far to the left one should distort the σ contour, however, depends crucially on the interplay between the σ and q^2 dependence in $E_\sigma^{(i)}$ and even on the σ dependence of $C_\sigma^{(i)}$. To see this, use (114) and (115) to write

$$\omega_p\frac{d\sigma}{d^3p}=\frac{\alpha^2}{\pi i(q^2)^2}\int_{L-i\infty}^{L+i\infty} d\sigma\, e^{\sigma \ln 1/\omega}\left(2d_{L\sigma+1}-d_{2\sigma+1}\frac{1+\cos^2\theta}{2}\right)e^{-A_{\sigma+1}\ln\ln q^2}\,. \qquad (126)$$

Now as soon as $\ln 1/\omega \gg \ln\ln q^2$ it should be efficient to distort the σ contour to the left. However, there are singularities in both A_σ and d_σ. In perturbation theory the rightmost singularity in A_σ is at $\sigma=3$. Near $\sigma=3$ one easily calculates that $A_\sigma \sim -C_2(G)/2\pi^2 b_0(\sigma-3)$. Keeping only this term

$$\omega_p\frac{d\sigma}{d^3p}=\frac{\alpha^2}{\pi i(q^2)^2}\int_{L-i\infty}^{L+i\infty}\exp\left[(\sigma-1)\ln 1/\omega+\frac{C_2(G)}{2\pi^2 b_0(\sigma-3)}\ln\ln q^2\right]\left(2d_{L\sigma}-\frac{1+\cos^2\theta}{2}d_{2\sigma}\right). \qquad (127)$$

If only the terms shown in the exponent were important and if $d_{L\sigma}$ had no singularities to the right of $\sigma=3$, then the effective value of σ would be

$$\sigma \approx 3+\left(\frac{C_2}{2\pi^2 b_0}\frac{\ln\ln q^2}{\ln 1/\omega}\right)^{1/2}. \qquad (128)$$

However, in perturbation theory there are also $[g^2/(\sigma-3)]^n$ type terms in the exponent so that the estimate is not to be believed. In order to determine the $\ln 1/\omega \gg \ln\ln q^2$ behavior we need to take into account all the singularities near $\sigma=3$ and asymptotic freedom is not much use here. Thus this most interesting region of $\ln 1/\omega \gg \ln\ln q^2$, which includes the region necessary to determine the average multiplicity of particles produced, remains to be understood by a method outside that of the renormalization group.

ACKNOWLEDGMENTS

I have benefited greatly from a number of discussions with David Politzer. I also gratefully acknowledge the support of the A. P. Sloan Foundation. This work was also supported in part by the U. S. Department of Energy.

APPENDIX A

In this appendix we shall indicate how a BPHZ method of subtraction of Eq. (10) leads to the amplitude of Eq. (31) of Ref. 11. In ϕ^4 theory a subgraph including the bare cut vertex is a renormalization part if and only if it has two external lines. Graphs not including the bare cut vertex have dimensions as determined in the usual way. The BPHZ formula for subtracting Γ, a particular graph, is

$$\Gamma_\sigma(p)=\sum_U \prod_{\gamma\in U}(-t^\gamma)\Gamma_\sigma^u(p)\,, \tag{A1}$$

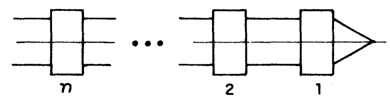

FIG. 25. A separation of a cut vertex into two-particle irreducible parts.

where U is the set of all forests of that graph and $\gamma\in U$ is the set of all nonoverlapping renormalization parts in U. Consider a typical graph as shown in Fig. 25 where two-particle irreducible parts are separated into squares. The renormalization parts including the bare cut vertex are always of the nonoverlapping type. We may view the subtractions in Fig. 25 as being done in two steps. First make all the subtractions not involving the cut vertex. Then do the subtractions involving the cut vertex. The final set of subtractions is the nonoverlapping variety and for them the BPHZ prescription is the same as the usual prescription for subtracting divergences which do not overlap. This immediately yields the formula Eq. (31) of Ref. 11.[27] The only point to be careful of here is that subtractions are done at $p=\hat{p}$ where $\hat{p}_+=0=\underline{\hat{p}}$ and $\hat{p}_-=p_-$.

APPENDIX B

In this appendix we shall examine the divergences and renormalization of all the nontrivial two-loop graphs having two external fermions and involving the bare cut fermion vertex. Consider the graphs shown in Fig. 26. The divergence in k for a fixed k_1 is a usual vertex renormalization and does not concern us here. Let us examine the divergence in k_1 for fixed k to see the nature of the counterterm (subtraction term) necessary to remove this divergence. The relevant part of the graph is shown in Fig. 27 and has the expression

$$\Gamma_{11,\sigma\alpha}(p,k)=-\frac{g^3}{(2\pi)^4}\int d^4k_1\,\frac{\gamma_\beta[\gamma\cdot(p+k_1)+m]\gamma_-[\gamma\cdot(p+k_1)+m]\gamma_\alpha[\gamma\cdot(p+k_1+k)+m]\gamma_\beta}{|(p+k_1)^2-m^2|^2[(p+k+k_1)^2-m^2-i\epsilon]}(-2\pi)\delta(k_1{}^2-m^2)(p+k_1)_-{}^{-\sigma}\,. \tag{B1}$$

The divergent part of Eq. (B1) is given by

$$\Gamma_{11,\sigma\alpha}(p,k)=\frac{g^3}{(2\pi)^3}\int d^4k_1\,\frac{\gamma_\beta\gamma\cdot k_1\gamma_-\gamma\cdot k_1\gamma_\alpha\gamma\cdot k_1\gamma_\beta(p+k_1)_-{}^{-\sigma}}{|(k_1+p)^2|^2[(p+k_1+k)^2-i\epsilon]}\,\delta(k_1{}^2)\,.$$

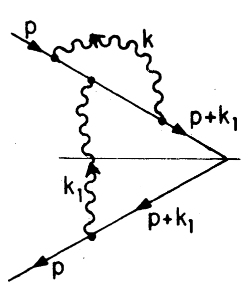

FIG. 26. A radiative correction to $\Gamma_\sigma(p)$.

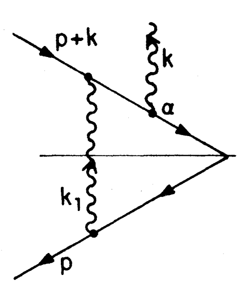

FIG. 27. A divergent subgraph of Fig. 26.

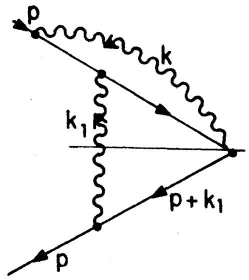

FIG. 28. A radiative correction to $\Gamma_\sigma(p)$.

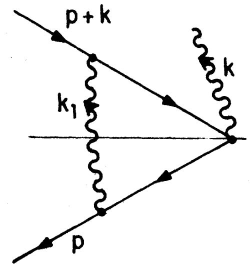

FIG. 29. A divergent subgraph of Fig. 28.

A divergence only occurs in case $\alpha=+$ in which case

$$\gamma_\beta\gamma\cdot k_1\gamma_-\gamma\cdot k_1\gamma_+\gamma\cdot k_1\gamma_\beta=\gamma_\beta\underline{\gamma}\cdot\underline{k}_1\gamma_-\underline{\gamma}\cdot\underline{k}_1\gamma_+\gamma_-\gamma_\beta k_{1+}=-4k_{1+}\underline{k}_1^{\,2}\gamma_-$$

and

$$\Gamma_{11,\sigma}(p,k)\sim-\frac{g^3}{8\pi^2}\frac{\ln\Lambda^2}{(\sigma-1)(\sigma-2)}\frac{\gamma_-p_-^{\,-\sigma}}{(p+k)_-}\,. \tag{B2}$$

Equation (B1) is not of the form of a bare cut vertex. The divergence of a form of a bare cut vertex only comes about when one takes all graphs of a given order. To see this, consider the graph shown in Fig. 28. The part involving the k_1 divergence is shown in Fig. 29. The divergent part of Fig. 29 is given by

$$\Gamma_{11,\sigma}(p,k)=-\frac{g^3}{(2\pi)^4}\int d^4k_1\frac{\gamma_\beta\gamma\cdot k_1\gamma_-\gamma\cdot k_1\gamma_\beta[-2\pi\delta(k_1^{\,2})]}{[(p+k_1)^2+i\epsilon][(p+k+k_1)^2-i\epsilon]}\frac{(p+k_1)_-^{\,-\sigma}}{k_-}\,.$$

Thus

$$\Gamma_{11,\sigma}(p,k)\sim\frac{g^3}{4\pi^3}\frac{\gamma_-}{k_-}\int d^4k_1\frac{\delta(k_1^{\,2})(p+k_1)_-^{\,-\sigma}}{[(p+k_1)^2+i\epsilon][(p+k+k_1)^2-i\epsilon]} \tag{B3}$$

and

$$\Gamma_{11,\sigma}(p,k)\sim\frac{g^3}{8\pi^2}\frac{\ln\Lambda^2}{(\sigma-1)(\sigma-2)}\frac{\gamma_-p_-^{\,-\sigma+1}}{k_-(p+k)_-}\,.$$

Equation (B3) is not the form of a bare cut vertex. However, if we add (B2) and (B3) we get

$$\Gamma_{11,\sigma}(p)\sim\frac{g^3}{8\pi^2}\frac{\ln\Lambda^2}{(\sigma-1)(\sigma-2)}\frac{\gamma_-p_-^{\,-\sigma}}{k_-}\,. \tag{B4}$$

Equation (B4) is the form of the cut vertex.

Consider now the graph shown in Fig. 30. In particular, consider the k_1 divergence due to the subgraph shown in Fig. 31. The divergent part of Fig. 31 is

$$\Gamma_{12,\sigma}(p,k)=-\frac{g^3}{(2\pi)^3k_-}\int d^4k_1\frac{\gamma_\alpha\gamma\cdot k_1\gamma_-\gamma\cdot k_1\gamma_-(p+k_1+k)_-^{\,-\sigma}\delta(k_1^{\,2})}{[(p+k_1)^2+i\epsilon][(p+k+k_1)^2-i\epsilon]}\;;$$

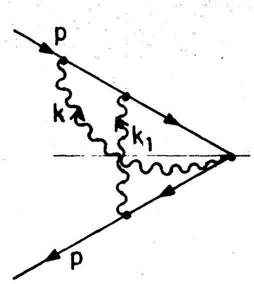

FIG. 30. A radiative correction to $\Gamma_\sigma(p)$.

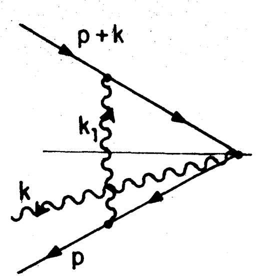

FIG. 31. A divergent subgraph of Fig. 30.

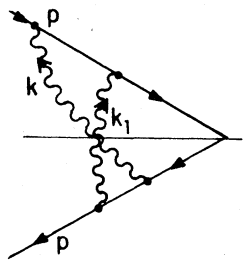

FIG. 32. A radiative correction of $\Gamma_\sigma(p)$.

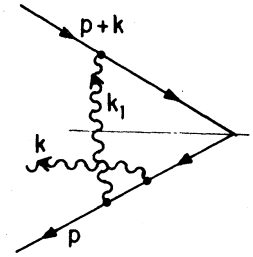

FIG. 33. A divergent subgraph of Fig. 32.

one finds

$$\Gamma_{12,\sigma}(p,k) \sim \frac{g^3}{8\pi^2}\frac{\ln\Lambda^2}{(\sigma-1)(\sigma-2)}\frac{\gamma_-(p+k)_-^{-\sigma+1}}{k_-p_-}\,. \tag{B5}$$

The k_1 divergent part of Fig. 32, shown in Fig. 33, has the expression

$$\Gamma_{12,\sigma}(p,k) = \frac{g^3}{(2\pi)^3}\int d^4k_1 \frac{\gamma_\alpha\gamma\cdot k_1\gamma_+\gamma\cdot k_1\gamma_-\gamma\cdot k_1\gamma_\alpha(p+k+k_1)_-^{-\sigma}\delta(k_1^2)}{[(p+k_1)^2+i\epsilon]\,[(p+k+k_1)^2+i\epsilon]\,[(p+k+k_1)^2-i\epsilon]}\,.$$

The divergent part here is

$$\Gamma_{12,\sigma}(p,k) \sim -\frac{g^3}{8\pi^2}\frac{\ln\Lambda^2}{(\sigma-1)(\sigma-2)}\frac{\gamma_-(p+k)_-^{-\sigma}}{p_-}\,. \tag{B6}$$

Adding (B5) and (B6) one obtains

$$\Gamma_{12,\sigma}(p,k) \sim \frac{g^3}{8\pi^2}\frac{\ln\Lambda^2}{(\sigma-1)(\sigma-2)}\frac{\gamma_-p_-^{-\sigma}}{k_-}\,,$$

which is again of the form (21).

Finally, let us turn to the most interesting, and complicated, of the two-loop graphs. The k divergent part of Fig. 28 is illustrated in Fig. 34, after letting $k_1 \leftrightarrow k$. The divergent part of Fig. 34 is

$$\Gamma_{12,\sigma}(p,k) = -\frac{ig^3}{(2\pi)^4}\int\frac{d^4k_1}{k_{1-}}\frac{\gamma_-\gamma\cdot k_1\gamma_+\gamma\cdot k_1\gamma_-(p-k)_-^{-\sigma}}{(k_1^2-i\epsilon)[(p+k_1)^2-i\epsilon]\,[(p+k_1-k)^2-i\epsilon]}\,.$$

The $\ln\Lambda^2$ term is

$$\Gamma_{12,\sigma}(p,k) = \frac{g^3}{8\pi^2}\ln\Lambda^2\ln\left[\frac{(p-k)_-}{p_-}\right]\frac{(p-k)_-^{-\sigma}}{k_-}\,. \tag{B7}$$

We should also include the k_1 divergent part of Fig. 35, shown in Fig. 36. This contribution is

$$\Gamma_{12,\sigma}(p,k) = -\frac{g^3}{(2\pi)^3}\int\frac{d^4k_1}{k_{1-}}\frac{\gamma_-\gamma\cdot k_1\gamma_+\gamma\cdot k_1\gamma_-\delta(k_1^2)}{[(p+k_1)^2-i\epsilon]\,[(p-k+k_1)^2-i\epsilon]}\,.$$

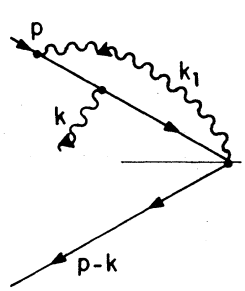

FIG. 34. A divergent subgraph of Fig. 28.

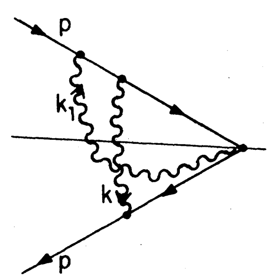

FIG. 35. A radiative correction of $\Gamma_\sigma(p)$.

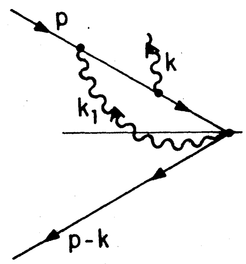

FIG. 36. A divergent subgraph of Fig. 35.

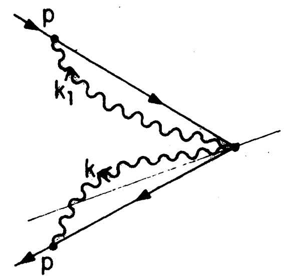

FIG. 37. A radiative correction of $\Gamma_\sigma(p)$.

Thus

$$\Gamma_{12,\sigma}(p,k) \sim -\frac{g^3}{8\pi^2}\frac{\ln\Lambda^2}{\sigma-1}\frac{\gamma_-(p-k)_-^{-\sigma}}{p_-}\,. \tag{B8}$$

Along with (B7) and (B8) we should include the k_1 divergent parts of Figs. 37 and 38. These contributions, illustrated in Figs. 39 and 40, are given by

$$\Gamma_{12,\sigma}(p,k)=\frac{g^3}{(2\pi)^4}\int\frac{d^4k_1}{k_{1-}}\frac{\gamma_-\gamma\cdot(p+k_1)\gamma_-}{[(p+k_1)^2-i\epsilon]}\left[-2\pi\delta(k_1{}^2)(p+k_1-k)_-^{-\sigma}-\frac{i}{k_1{}^2-i\epsilon}(p-k)_-^{-\sigma}\right].$$

This can be written as

$$\Gamma_{12,\sigma}(p,k)=-\frac{g^3}{8\pi^2}\frac{\ln\Lambda^2}{p_-k_-}\gamma_-\left\{\int\frac{dk_{1-}}{k_{1-}}(p-k+k_1)_-[(p+k_1-k)_-^{-\sigma}\Theta(k_{1-})+(p-k)_-^{-\sigma}(\Theta(k_{1-}+p_-)-\Theta(k_{1-}))]\right.$$
$$\left.+k_-\int\frac{dk_{1-}}{k_{1-}}[(p+k_1-k)_-^{-\sigma}\Theta(k_{1-})+(p-k)_-^{-\sigma}(\Theta(k_{1-}+p_-)-\Theta(k_{1-}))]\right\}.$$

Finally,

$$\Gamma_{12,\sigma}(p,k)=\frac{g^2}{8\pi^2}\ln\Lambda^2\left\{\frac{\gamma_-(p-k)_-^{-\sigma}}{k_-}\left[\sum_{l=1}^{\infty}\left(\frac{1}{l+1}-\frac{1}{l+\sigma-1}\right)-\frac{1}{\sigma-1}\right]+\frac{\gamma_-(p-k)_-^{-\sigma}}{p_-}\frac{1}{\sigma-1}-\frac{\gamma_-(p-k)_-^{-\sigma}}{k_-}\ln\left[\frac{(p-k)_-}{p_-}\right]\right\}. \tag{B9}$$

Adding (B7), (B8), and (B9) one obtains

$$\Gamma_{12,\sigma}(p,k)=\frac{g^3}{8\pi^2}\ln\Lambda^2\frac{\gamma_-(p-k)_-^{-\sigma}}{k_-}\left[\sum_{l=1}^{\infty}\left(\frac{1}{l+1}-\frac{1}{l+\sigma-1}\right)-\frac{1}{\sigma-1}\right], \tag{B10}$$

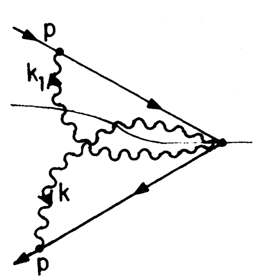

FIG. 38. A radiative correction of $\Gamma_\sigma(p)$.

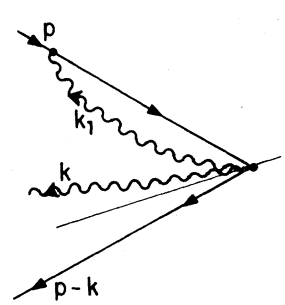

FIG. 39. A divergent subgraph of Fig. 37.

which is of the form (21). It may be noted that

$$\sum_{l=1}^{\infty}\left(\frac{1}{l+1}-\frac{1}{l+\sigma-1}\right)-\frac{1}{\sigma-1}=\sum_{l=2}^{\sigma-2}\frac{1}{l}$$

when σ is an integer. The other two-loop graphs are either identical to those given above or very similar in detail.

APPENDIX C

In this appendix a qualitative discussion of power counting for cut vertices and the relation to BPHZ renormalization will be given in an Abelian gauge theory. We are not claiming to give any rigorous demonstrations here, but rather to illustrate a few points, not discussed in the body of the paper, in the application of power counting for cut vertices.

In a usual (not cut) unrenormalized vertex, a divergence of $(\ln\Lambda^2)^n$ arises when n nonoverlapping loop momenta become large. For this divergence those loops which are disjoint have independent loop momenta. Those loops which are nonoverlapping but not disjoint have ordered loop momenta, the interior momenta being the larger. The subtraction of the $(\ln\Lambda^2)^2$ term is done in BPHZ by the forest U having renormalization parts $\gamma_1, \gamma_2, \ldots, \gamma_n$ corresponding to the n divergent loops. As a simple example consider the graph Γ, shown in Fig. 41 where all the vertices are the usual γ_μ vertices of QED. This graph has two renormalization parts consisting of Γ itself and the subgraph γ which has the lines p_2 and $p_1 - p_2$. The forests are $\{0\}$, $\{\gamma\}$, $\{\Gamma\}$, and $\{\Gamma,\gamma\}$. The forest $\{\gamma\}$ corresponds to the divergence $\ln\Lambda^2$ when $p_2\to\infty$ for fixed p_1. The forest $\{\Gamma\}$ corresponds to the $\ln\Lambda^2$ divergence when $p_1, p_2\to\infty$ together. The forest $\{\Gamma,\gamma\}$ corresponds to the $(\ln\Lambda^2)^2$ divergence when $p_2 \gg p_1 \to\infty$. The magnificent property of BPHZ is that all divergences, even the notorious overlapping divergences, are classified and subtracted in this simple way.[14-16]

The additional complication in cut vertices is that we cannot go to Euclidean space to use Weinberg's theorem and neither can we let $i\epsilon \to i\epsilon(\vec{p}^{\,2}+m^2)$ so as to use Zimmermann's Minkowski version of the power-counting theorem. However, this appears to be a nicety rather than an essential point of physics. Consider, for example, the cut graph similar to Fig. 41

$$\Gamma_\sigma(p)=\frac{g^2}{(2\pi)^8}\int d^4p_1 d^4p_2\,\frac{\gamma_\alpha(\gamma\cdot p_1+m)\gamma_\beta(\gamma\cdot p_2+m)\gamma_-(\gamma\cdot p_2+m)\gamma_\beta(\gamma\cdot p_1+m)\gamma_\alpha}{|p_1^{\,2}-m^2+i\epsilon|^2\,|p_2^{\,2}-m^2+i\epsilon|^2}$$
$$\times p_{2-}^{\;-\sigma}(-2\pi)\delta((p-p_1)^2-m^2)(-2\pi)\delta((p_1-p_2)^2-m^2).$$

For fixed p_1 there is a p_2 divergence corresponding to $p_{2+}\approx \underline{p}_2\to\infty$ with p_{2-} fixed. Thus a p_{2+} counts two units of dimensions while $\underline{p}_2$ counts one unit. The d^4p_2 counts four units while $p_2^{\,2}$ and $(p_2-p_1)^2$ count two units apiece. The one apparent danger to usual power counting is the presence of p_{2+} terms in the numerators of the fermion propagators. However, the γ_- at the cut vertex annihilates these terms. The $p_1\approx p_2\to\infty$ and $p_2\gg p_1\to\infty$ divergences are similarly analyzed by counting + components of the momentum as two units and transverse components as one unit.

The new element, the possibility of + components of loop momenta in numerators, does create a new kind of divergence when a photon line, whose momentum is fixed, enters a loop whose momentum is large. If the photon line ends on a γ_-, then the photon creates a denominator of dimension -2 while the numerator of the new fermion propagator can have a p_+ term of dimension $+2$. Thus it is possible to insert an arbitrary number of + photons into a divergent loop without losing the diver-

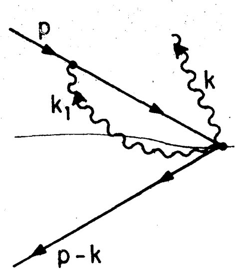

FIG. 40. A divergent subgraph of Fig. 38.

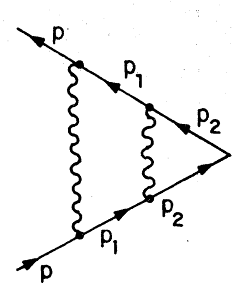

FIG. 41. An example of a graph having four forests.

gence. As we have seen in Sec. III these divergences are related by gauge invariance to the divergences without the photon lines. Photon lines which begin and end on the same, or comparable, large momentum line can give an additional p_+ only at the expense of creating an additional p_- so that usual counting prevails.

In summary, when a large loop momentum crosses a cut, + components of that momentum count two units, − components count no units, and transverse components count one unit. If a loop momentum does not cross a cut, counting is as usual.

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[1]H. D. Politzer, Phys. Rep. 14C, 129 (1974).
[2]D. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973).
[3]V. N. Gribov and L. Lipatov, Yad. Fiz. 15, 781 (1972) [Sov. J. Nucl. Phys. 15, 438 (1972)].
[4]C. Sachrajda, Phys. Lett. 73B, 185 (1978); CERN Report No. TH 2492, 1978 (unpublished).
[5]I. Halliday, Nucl. Phys. B103, 343 (1976).
[6]H. D. Politzer, Nucl. Phys. B129, 301 (1977); H. Georgi and H. D. Politzer (unpublished).
[7]G. Altarelli, G. Parisi, and R. Petronzio, Phys. Lett. 76B, 351 (1978); 76B, 356 (1978).
[8]C. Callan, Phys. Rev. D 2, 1541 (1972).
[9]K. Symanzik, Commun. Math. Phys. 18, 227 (1970).
[10]V. Sudakov, Zh. Eksp. Teor. Fiz. 30, 87 (1956) [Sov. Phys.—JETP 3, 65 (1956)].
[11]A. Mueller, Phys. Rev. D 9, 963 (1974).
[12]C. Callan and M. Goldberger, Phys. Rev. Γ 11, 1553 (1975).
[13]R. K. Ellis, H. Georgi, M. Machacek, H. D. Politzer, and G. Ross (unpublished).
[14]N. Bogoliubov and D. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959).
[15]K. Hepp, Commun. Math. Phys. 2, 301 (1966).
[16]W. Zimmermann, Commun. Math. Phys. 15, 208 (1969).
[17]W. Zimmermann, *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1971).
[18]K. Wilson, Phys. Rev. 179, 1499 (1969).
[19]R. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971).
[20]Y. Frishman, Phys. Rev. Lett. 25, 966 (1970).
[21]N. Christ, B. Hasslacher, and A. Mueller, Phys. Rev. D 6, 3543 (1972).
[22]D. Gross and F. Wilczek, Phys. Rev. D 9, 980 (1974).
[23]H. Georgi and H. D. Politzer, Phys. Rev. D 9, 416 (1974).
[24]A. Polyakov, Phys. Lett. 59B, 82 (1975).
[25]R. Brown and I. Muzinich, Phys. Rev. D 4, 1496 (1972).
[26]If $\tilde{\gamma}^{n}_{ij}$ are the γ's given by Gross and Wilczek (Ref. 22), then the precise relationship is $2\gamma^{\sigma}_{12}=-\tilde{\gamma}_{12}^{-\sigma+1}$, $2\gamma^{\sigma}_{21}=-\tilde{\gamma}_{21}^{-\sigma+1}$, $2\gamma^{\sigma}_{22}=\tilde{\gamma}_{22}^{-\sigma+1}-[g^2C_2(G)/(2\pi^2)]\,\pi\cot\pi\sigma$ and $2\gamma^{\sigma}_{11}=\tilde{\gamma}_{11}^{-\sigma+1}-[g^2C_2(R)/2\pi_2]\,\pi\cot\pi\sigma$. The continuation from integral σ is made by the relation

$$\sum_{l=2}^{\sigma}\frac{1}{l}=\left[\frac{\Gamma'(\sigma+1)}{\Gamma(\sigma+1)}-\Gamma'(1)\right]-1.$$

It is also true that $2\gamma^{\sigma}_{ii}=\tilde{\gamma}_{ii}^{\sigma-2}$ and $4\gamma^{\sigma}_{12}\gamma^{\sigma}_{21}=\tilde{\gamma}_{12}^{\sigma-2}\tilde{\gamma}_{21}^{\sigma-2}$. These relations constitute the Gribov-Lipatov symmetries for both elementary spinor and elementary vector targets and final-state particles.

[27]The b_σ given in Ref. 11 is in fact zero. This makes the Callan-Symanzik equation simpler than claimed there.