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Fermion zero modes and level crossing

Joe Kiskis
The Institute for Advanced Study, Princeton, New Jersey 08540
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The Dirac Hamiltonian in the presence of instanton and meron fields is studied. Some three-dimensional, zero-energy, normalizable solutions are found. They are related to the four-dimensional zero modes which are known and to the level-crossing picture. A geometrical interpretation of the simplicity of the results is given.

I. INTRODUCTION

Zero modes of the massless Euclidean Dirac operator in the presence of instanton fields have been given explicitly.[1] The existence of the zero modes is a consequence[2] of the triangle anomaly[3] or equivalently of the Atiyah-Singer index theorem.[4]

A more intuitive discussion of the physics involved has been given by Callan, Dashen, and Gross[5]: work in the $A_4=0$ gauge and focus upon the Euclidean time parameter x_4. As x_4 changes from $-\infty$ to $+\infty$, the one-instanton field evolves from one pure gauge configuration to another topologically distinct pure gauge. The massless three-dimensional Dirac Hamiltonian in the presence of such a field is

$$H(x_4) = -i\alpha_i(\partial + A)_i \,. \tag{1.1}$$

This operator depends parametrically on x_4 through $A_i(x_4, \vec{x})$. Knowledge of the spectrum of $H(x_4)$ as a function of x_4,

$$H(x_4)\Psi_E(x_4, \vec{x}) = E(x_4)\Psi_E(x_4, \vec{x}) \,, \tag{1.2}$$

gives information about the behavior of the quantized Dirac field in the adiabatic approximation. Callan, Dashen, and Gross have proposed that there is a positive-chirality mode with an energy that begins below zero and ends above zero as x_4 runs from $-\infty$ to $+\infty$ and that there is a negative-chirality mode with the opposite behavior. An understanding of the relationship among the zero modes, tunneling suppression, and the triangle anomaly results.[5]

If solutions to (1.2) with such properties exist, then continuity and chiral symmetry imply that there will be some time τ_0 at which the levels cross zero:

$$H(\tau_0)\Psi_\pm(\tau_0, \vec{x}) = 0 \tag{1.3}$$

with

$$\gamma_5\Psi_\pm = \pm\Psi_\pm \,. \tag{1.4}$$

After some preliminary gauge transformations have been worked out in Sec. II, (1.3) is studied in Sec. III. For an instanton located at the origin, we find *normalizable* solutions to (1.3) for $\tau_0=0$. The solutions are very simple.

Now consider a two-meron field with the two merons at

$$x_4 = \pm y \,, \quad \vec{x} = 0 \,. \tag{1.5}$$

This field can be thought of as a deformation of an instanton field.[5] In this case, we find normalizable solutions to (1.3) for

$$-y \le \tau_0 \le y \,. \tag{1.6}$$

While the existence of these solutions is related to the concept of spectral flow and to the index theorem with boundary,[6] the extreme simplicity of the solutions must have some more specific source. In Sec. IV, it is shown that the gauge fields under consideration are, in a certain way, conformally flat. Further, there is a conformally flat manifold on which the curvature of the gauge field cancels[7] the curvature of the tangent bundle in a special channel. Projected onto this manifold, the solutions to (1.3) that we found are simply constants. This result contributes to a geometrical understanding of meron field configurations.

II. A SIMPLE CHOICE OF GAUGE

In this section, we will work through gauge transformations which bring the two-meron, the one-meron, and the $x_4=0$ instanton fields into very simple forms.

Consider the starting points: The one-instanton solution is

$$A_\mu = \frac{2\Sigma_{\mu\nu}x^\nu}{x^2+\lambda^2} \,. \tag{2.1}$$

The one-antimeron solution is

$$A_\mu = \frac{\Sigma_{\mu\nu}x^\nu}{x^2} \,. \tag{2.2}$$

The two-antimeron solution that we consider has the merons on the x_4 axis:

$$A_\mu = \frac{\Sigma_{\mu\nu}(x-y)^\nu}{(x-y)^2} + \frac{\Sigma_{\mu\nu}(x+y)^\nu}{(x+y)^2} , \quad y_i = 0, \quad y_4 = y > 0 . \tag{2.3}$$

Also,

$$\Sigma_{\mu\nu} = -\Sigma_{\nu\mu} , \quad \Sigma^\dagger_{\mu\nu} = -\Sigma_{\mu\nu} , \quad \Sigma_{ij} = -\tfrac{1}{2} i\epsilon_{ijk}\sigma^k , \quad \Sigma_{i4} = -\tfrac{1}{2} i\sigma_i . \tag{2.4}$$

Since no three-vector other than $\vec{x}$ appears, we must have in each case

$$A_4(x_4, \vec{x}) = -\tfrac{1}{2} i D(x_4, \vec{x}^2)\vec{x}\cdot\vec{\sigma} , \tag{2.5}$$

$$A_i(x_4, \vec{x}) = -\tfrac{1}{2} i A(x_4, \vec{x}^2)\epsilon_{ijk}x^j\sigma^k - \tfrac{1}{2} i|\vec{x}|B(x_4, \vec{x}^2)\sigma_i - \tfrac{1}{2} i \frac{1}{|\vec{x}|} C(x_4, \vec{x}^2)x_i\vec{x}\cdot\vec{\sigma} . \tag{2.6}$$

This is the general spherically symmetric ansatz. (It happens that C is zero in each case for the fields given above.)

To get to the $A_4 = 0$ gauge, transform to

$$A'_\mu = g^{-1}(\partial + A)_\mu g$$

by

$$g(x_4, \vec{x}) = \mathrm{P}\exp\left[-\int_0^{x_4} d\tau A_4(\tau, \vec{x})\right] . \tag{2.7}$$

With (2.5), there is a major simplification, and

$$g = e^{i\hat{x}\cdot\vec{\sigma}\theta} , \tag{2.8a}$$

$$\theta(x_4, \vec{x}^2) = \tfrac{1}{2}|\vec{x}|\int_0^{x_4} d\tau\, D(\tau, \vec{x}^2) . \tag{2.8b}$$

For the two-antimeron field, all calculations can be carried out and we obtain the result that the field (2.3) is gauge equivalent to

$$A_4 = 0 , \quad A_i = -\tfrac{1}{2} i\epsilon_{ijk}\partial_j\rho\sigma_k , \tag{2.9}$$

$$\rho = \ln\{\vec{x}^2 + y^2 - x_4^2 + [\vec{x}^2 + (x_4 - y)^2]^{1/2} \times [\vec{x}^2 + (x_4 + y)^2]^{1/2}\} . \tag{2.10}$$

Although a result of the form of (2.6) with A, B, and C all nonvanishing is all that one might expect, (2.9) is actually much simpler. A full discussion of this appears in Sec. IV.

If the single-antimeron field is transformed as in (2.8) and if this is followed by another rotation with $\theta = -\pi/4$, then we obtain

$$A_4 = 0 , \quad A_i = -\tfrac{1}{2} i\epsilon_{ijk}\partial_j \ln[(\vec{x}^2 + x_4^2)^{1/2} - x_4]\sigma_k . \tag{2.11}$$

The one-instanton field is not as simple. If x_4 is not zero, then no gauge transformation of the form (2.8a) with

$$\theta = \theta(x_4, \vec{x}^2) \tag{2.12}$$

brings the field to a form with B and C equal to zero. However, at $x_4 = 0$ such a form is very easily obtained and it is

$$A_4 = 0 , \quad A_i = -\tfrac{1}{2} i\epsilon_{ijk}\partial_j\rho\sigma_k , \quad \rho = \ln(\vec{x}^2 + \lambda^2) . \tag{2.13}$$

Our final observation of this section is that at $x_4 = 0$ the instanton field and the two-antimeron field have the same form.

III. ZERO-ENERGY SOLUTIONS

This section presents some zero-energy normalizable solutions to the massless Dirac equation in the presence of the fields given in the preceding section. The "time" x_4 appears as a parameter. The motivation for this approach was given in the Introduction.

The Hamiltonian operator is

$$H = -i\alpha_i(\partial + A)_i . \tag{3.1}$$

The familiar choice of matrices is

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} . \tag{3.2}$$

Thus, solutions

$$H\Psi = 0 \tag{3.3}$$

will always come in pairs which can be chosen to be plus and minus chiral eigenstates. The interpretation of this was discussed in the Introduction. It is sufficient, then, to study

$$h\psi = 0 \tag{3.4}$$

with

$$h = -i\sigma_i(\partial + A)_i . \tag{3.5}$$

Remark. A normalizable solution to

$$H\Psi = E\Psi \tag{3.6}$$

must have $E = 0$ when F_{ij} falls faster than x^{-2} at infinity. Applying H to (3.6) gives

$$\{-(\partial + A)^2 - \tfrac{1}{4}[\alpha_i, \alpha_j]F_{ij}\}\Psi = E^2\Psi . \tag{3.7}$$

Since F falls rapidly at infinity, any x^{-1} term in A must be pure gauge. Thus, the asymptotic form of (3.7) is essentially

$$-\partial_i{}^2\Psi = E^2\Psi \ . \quad (3.8)$$

With positive E^2, (3.8) does not have solutions which are normalizable at infinity.

Return to (3.4). In (3.4) ψ carries spin and isospin indices each running from 1 to 2. Employing a device used by Jackiw and Rebbi,[8] ψ is viewed as a 2×2 matrix and is written

$$\psi = \chi\sigma_2 \ . \quad (3.9)$$

Equation (3.4) becomes

$$\sigma_i\partial_i\chi - \sigma_i\chi A_i = 0 \ . \quad (3.10)$$

With the ansatz that χ is proportional to the unit matrix

$$\chi = \chi_0 1 \ , \quad (3.11)$$

(3.10) becomes

$$(\sigma_i\partial_i - \sigma_i A_i)\chi_0 = 0 \ . \quad (3.12)$$

Specializing to a field of the form

$$A_i = -\tfrac{1}{2}i\epsilon_{ijk}\partial_j\rho\sigma_k \ , \quad (3.13)$$

(3.12) becomes

$$\sigma_i(\partial_i + \partial_i\rho)\chi_0 = 0 \ . \quad (3.14)$$

The solution to (3.14) is

$$\chi_0 = Ne^{-\rho} \quad (3.15)$$

with N some normalization constant.

Thus, at $x_4 = 0$ in the one-instanton field, there is a zero-energy bound-state solution of (3.4) given by

$$\psi = \frac{N\sigma_2}{\vec{x}^2 + \lambda^2} \ . \quad (3.16)$$

In the two-antimeron field, the result is

$$\psi = \frac{N\sigma_2}{\vec{x}^2 + y^2 - x_4^2 + [\vec{x}^2 + (x_4 - y)^2]^{1/2}[\vec{x}^2 + (x_4 + y)^2]^{1/2}} \ . \quad (3.17)$$

For

$$-y < x_4 < y \ , \quad (3.18)$$

this is nonsingular and normalizable. At

$$x_4 = \pm y \ ,$$
$$\psi = \frac{N\sigma_2}{\vec{x}^2 + |\vec{x}|(\vec{x}^2 + 4y^2)^{1/2}} \ . \quad (3.19)$$

This has a weak singularity at the origin which is normalizable. For

$$y < |x_4| \ , \quad (3.20)$$

ψ has a singularity at the origin which is not integrable.

The single-antimeron result is

$$\psi = \frac{N\sigma_2}{(\vec{x}^2 + x_4^2)^{1/2} - x_4} \ . \quad (3.21)$$

This is not normalizable.

In conclusion, (3.16) gives a normalizable solution to (3.4) for the $x_4 = 0$ instanton field. Equation (3.17) is a normalizable solution for the two-antimeron field when

$$|x_4| \leq y \ . \quad (3.22)$$

IV. GEOMETRY

We have found that the instanton field at $x_4 = 0$, the meron field, and the two-meron field have very simple forms (3.13). Further, it is very easy to find a solution to the Dirac equation in such a field. A geometrical interpretation of these results will now be given.

We have been working on the flat base manifold R^3. Now consider the massless Dirac operator on an orientable three-dimensional Riemannian manifold. The structure group of the tangent frame bundle is SO(3). The isospin group is SU(2). Thus, we are also considering an SU(2) principal bundle. As usual, the gauge field A is the connection of this bundle. Since SU(2) and SO(3) have the same algebras, there is a close formal relationship between the connection A of the isospin bundle and the connection Γ of the tangent bundle.

It will be shown that there is a conformally flat manifold on which the curvature of the tangent bundle compensates for the curvature of the isospin bundle. The Dirac equation then has a zero-energy solution which is simply a constant spinor. Conformal invariance of the Dirac equation allows us to pull this back to a solution on R^3. This will be the same solution that we have already found.

First recall the form of the Dirac operator on a Riemannian manifold. It involves the vierbein field which is defined by the relationship

$$E_i(\vec{x}) = V_i^j(\vec{x})\partial_j \ . \quad (4.1)$$

The E_i are an orthonormal basis at $\vec{x}$ and the ∂_i are the coordinate-induced basis. The inverse transformation is

$$\partial_i = V^{-1}{}_i{}^j E_j \quad (4.2)$$

and

$$V^{-1}{}_i{}^j = \delta^{jr}V_r{}^s g_{si} \ . \quad (4.3)$$

If $\Gamma_i{}^j{}_k$ is the usual metric-induced connection in a coordinate basis, then we can form

$$\Gamma_i \equiv -\tfrac{1}{2}V^{-1}{}_r{}^s(\partial_i\delta^r{}_t + \Gamma_i{}^r{}_t)V_u{}^t\delta^{uv}\sigma_{sv} \ . \quad (4.4)$$

The σ_{ij} are the same as the Σ_{ij} of (2.4) and satisfy

$$[\sigma_{ij}, \sigma_{kl}] = \delta_{ik}\sigma_{jl} - \delta_{il}\sigma_{jk} - \delta_{jk}\sigma_{il} + \delta_{jl}\sigma_{ik} \quad (4.5)$$

as SO(3) generators.

The curved-space version of (3.4) becomes

$$-i\sigma^i V_i{}^j(\partial_j+\Gamma_j+A_j)\psi=0 \ . \tag{4.6}$$

Γ is a matrix in the spin space, and A is a matrix in isospin space.

Now consider two three-dimensional manifolds M and M' with metrics g and g'. Let f be an embedding

$$f\colon\ M\to M' . \tag{4.7}$$

f induces a metric on M, the pull back

$$f^*g' \tag{4.8}$$

of g'. Suppose that

$$f^*g'=e^{2\lambda}g \tag{4.9}$$

with

$$\lambda\colon\ M\to R \ . \tag{4.10}$$

f is a conformal mapping.

Given a set of coordinates on M, it is convenient to use the coordinates on M' induced by f. We will do so.

Also, if we are given an A' on M', f pulls it back to M. However, with our choice of coordinates the components of A' on M' and f^*A' on M are the same. Furthermore,

$$g'_{ij}=e^{2\lambda}g_{ij} \tag{4.11}$$

and

$$V_i'^j=e^{-\lambda}V_i{}^j \ . \tag{4.12}$$

From (4.11) and (4.12), we obtain

$$\begin{aligned}\Gamma_i'&=\Gamma_i-\tfrac{1}{2}V^{-1}{}_r{}^s(\delta^r{}_i\partial_t\lambda-g_{it}g^{ru}\partial_u\lambda)V_v{}^t\delta^{vw}\sigma_{sw}\\ &=\Gamma_i+\delta\Gamma_i \ .\end{aligned} \tag{4.13}$$

A short calculation gives

$$V_i{}^j\delta\Gamma_j=-\delta^{rs}V_r{}^t\partial_t\lambda\sigma_{is}$$

and (4.14)

$$\sigma^i V_i{}^j\delta\Gamma_j=\sigma^i V_i{}^j\partial_j\lambda \ .$$

There are two equations under consideration. On M'

$$-i\sigma^i V_i'^j(\partial_j+\Gamma_j'+A_j')\psi'=0 \ , \tag{4.15}$$

and on M

$$-i\sigma^i V_i{}^j(\partial_j+\Gamma_j+A_j)\psi=0 \ . \tag{4.16}$$

If we take for A the pull back of A' and use (4.14), (4.16) becomes

$$-i\sigma^i e^{\lambda}V_i'^j(\partial_j+\Gamma_j'+A_j'-\partial_j\lambda)\psi=0 \ . \tag{4.17}$$

Thus, a solution ψ' to (4.15) on M' gives a solution ψ to (4.16) on M:

$$\psi=e^{\lambda}\psi' \ . \tag{4.18}$$

Now specialize further and suppose that M is R^3 so that M' is conformally flat. Then

$$g_{ij}=\delta_{ij}$$

and (4.19)

$$V_i{}^j=\delta_i{}^j \ .$$

This gives

$$\Gamma_i'=-\sigma_{it}\partial_t\lambda=+\tfrac{1}{2}i\epsilon_{ijk}\partial_j\lambda\sigma^k \ . \tag{4.20}$$

The simple expression for A_i in (3.13) is now understood. The form is that of a conformally flat connection.

With the ansatz of (3.9) and (3.11), (4.15) becomes

$$-ie^{-\lambda}\sigma^i(\partial_i+\Gamma_i'-A_i)\chi_0'=0 \ . \tag{4.21}$$

It is now obvious that we should choose M' by taking

$$\lambda=-\rho \tag{4.22}$$

to obtain for (4.21)

$$-i\sigma^i\partial_i\chi_0'=0 \ . \tag{4.23}$$

The solution to (4.23) is

$$\chi_0'=\text{constant} \ . \tag{4.24}$$

Thus, we have found a manifold M' with metric

$$g'_{ij}=e^{-2\rho}\delta_{ij} \ . \tag{4.25}$$

On this manifold, the Dirac equation (4.15) with A' of the form (3.13) has a very simple solution,

$$\psi'=N\sigma_2 \ , \tag{4.26}$$

with N some constant. Finally, from (4.26) and (4.18) we obtain a solution to our original problem on R^3 which is

$$\psi=Ne^{-\rho}\sigma_2 \ . \tag{4.27}$$

Comparison with (3.9), (3.11), and (3.15) shows that (4.27) is exactly the solution that was obtained in Sec. III.

Let us take a closer look at M' for the two-antimeron case with

$$|x_4|<y \ . \tag{4.28}$$

The metric is

$$g'_{ij}=\left(\frac{1}{\vec{x}^2+y^2-x_4^2+[\vec{x}^2+(x_4-y)^2]^{1/2}[\vec{x}^2+(x_4+y)^2]^{1/2}}\right)^2\delta_{ij} \ . \tag{4.29}$$

Topologically this is S^3. The metric changes with x_4 and gives some nonconstant curvature. However, at $x_4=0$ (4.29) becomes

$$g'_{ij}=\left(\frac{1}{2(\vec{x}^2+y^2)}\right)^2\delta_{ij}\ , \tag{4.30}$$

and M' is a three-sphere with the usual metric.

V. DISCUSSION AND CONCLUSION

We have studied the massless Dirac Hamiltonian in fields A_i obtained from instantons and merons in the $A_4=0$ gauge. For the instanton at $x_4=0$ and for a two-antimeron pair with

$$|x_4|<y\ , \tag{5.1}$$

there are zero-energy normalizable solutions. These are related to level crossing and to four-dimensional solutions as discussed in the Introduction. A geometrical interpretation of the simplicity of the results was given in Sec. IV.

It should be clear that similar results can be obtained in anti-instanton, two-meron, and meron-antimeron fields.

In the region (5.1), as x_4 changes the field A_i changes, but the energy of the solution stays equal to zero. Another quantity which does not change is

$$\int d^3x\,\epsilon_{ijk}\,\mathrm{Tr}(A_i\partial_jA_k+\tfrac{2}{3}A_iA_jA_k)\ . \tag{5.2}$$

Although this is suggestive, there are other deformations of A_i which leave (5.2) unchanged and give nonzero-energy shifts.

Finally, we remark that our analysis leaves open the question of whether or not there are zero-energy bound solutions in other channels.

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