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Covariant two-body dynamics and the Weinberg equation

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Covariant, constrained two-body dynamics, with both particles put on their mass shells, is proposed as a generalization of the Bakamjian-Thomas-Coester relativistic quantum-mechanical scheme. The Weinberg infinite-momentum limit of the latter scheme is investigated, and a covariant formulation of the two-body problem in the light-front field theory approach is made. We find several *equivalent* versions of the three-dimensional, covariant, two-body integral equations. Among these equations we get one in which the covariant two-body propagator has the form identical with the nonrelativistic case, i.e., we have a quadratic structure in the relative momentum and the ordinary reduced mass. We discuss connections between different schemes, emphasizing the variety and the uniqueness of the off-shell extensions.

I. INTRODUCTION

There exist in the literature very many practical proposals for dealing with two-body and many-body problems in the relativistic manner.[1-8,14,17,18,21,24-29] However, some approaches are noncovariant,[2-4,18,24] and other schemes possess arbitrariness[5] in writing three-dimensional, two-body integral equations. There are special recipes such as the "minimal relativity"[6] scheme, with which one would like to correlate the findings from nonrelativistic nuclear physics with those which are necessary to explain relativistic processes.

The medium-energy facilities for studying the pion-nucleus interaction provide a large amount of data, in which relativistic effects must be taken into account. Also, the isobar nucleon modification[7] of conventional nuclear physics calls for taking into account the mesonic degrees of freedom in a relativistic framework with some effective Lagrangians. The nucleon-nucleon interaction contains some effects, which are due to relativistic features, and it is desirable to disentangle them from pure dynamical effects.[8]

The original motivation for the present work was the aim to find a relativistic scheme in which the Glauber formula could be derived as the eikonal approximation in the relativistic three-body problem.[9] For a two-body subsystem we found[9] that the Abarbanel-Itzykson[10] result for the sum of ladder and crossed ladder diagrams can be reproduced for the fully on-shell t matrix in a covariant, eikonal, potential scheme, in which we use the condition $q \cdot P = 0$, where q is the Wightman-Gårding relative four-momentum and P is the total four-momentum. In an earlier work[11] we found that the real potential correction to high-energy proton-proton scattering, in the range of laboratory momentum 20–1500 GeV/c, explained very well the phenomenon of breaking the geometrical scaling.

The present paper is organized as follows. The noncovariant, relativistic, quantum two-body dynamics, proposed by Bakamjian and Thomas[2] and largely developed by Coester,[3,4] is recapitulated in Sec. II, where we also study Weinberg's infinite-momentum limit.[12] In Sec. III we propose a covariant generalization of the scheme given in Sec. II, using constrained dynamics, similar to the scheme suggested recently by Todorov,[13] but with both particles constrained by their mass shells. A field-theoretic description, based on the light-front field theory approach, is presented in Sec. IV, where we find a covariant Weinberg equation with a propagator such as in the nonrelativistic case. In Sec. V we make a comparison of different approaches including four classes of the most popular three-dimensional, two-body schemes. Conclusions and some remarks concerning the relativistic Schrödinger equation are given in Sec. VI. Two appendices contain normalization of amplitudes and several details corresponding to the case of unequal masses.

II. NONCOVARIANT RELATIVISTIC QUANTUM MECHANICS AND THE INFINITE–MOMENTUM LIMIT

In this section we recapitulate several known facts from the scheme developed by Bakamjian and Thomas,[2] Coester,[3,4] and other,[14] and we emphasize the variety of the *equivalent* two-body integral equations.

In relativistic quantum dynamics, particles are on their mass shells, but the whole two-body system may be off the energy shell. For such a system, Bakamjian and Thomas introduced an interaction v in the invariant mass of the two-body system, as a rotationally invariant function of the c.m. relative three-momentum $\vec{k}$ and a relative coordinate $\vec{\rho}$. Thus for noninteracting and interacting cases, the respective mass operators are

$$h_0=(\vec{k}^2+m_1^2)^{1/2}+(\vec{k}^2+m_2^2)^{1/2}\,,$$
$$h=h_0+v(\vec{k}^2,\vec{\rho}^2,\vec{k}\cdot\vec{\rho})\,. \tag{2.1}$$

Consecutively, the generators of the time translation are

$$H_0=(h_0^2+\vec{P}^2)^{1/2}\,,$$
$$H=(h^2+\vec{P}^2)^{1/2}\,, \tag{2.2}$$

where $\vec{P}$ is the total three-momentum.

The Møller operators are defined as

$$\Omega_\pm=\lim_{t\to\pm\infty}\exp(iHt)\exp(-iH_0t) \tag{2.3}$$

and, by applying the Kato theorem,[15] Coester[3] rewrites them as

$$\Omega_\pm=\lim_{t\to\pm\infty}\exp(iht)\exp(-ih_0t)\,. \tag{2.4}$$

Assuming a regular v, it can be shown[3] that Ω_- satisfies

$$1=\Omega_- - i\int_0^{-\infty}[\exp(ih_0t)]v\Omega_-\exp(-ih_0t)\,. \tag{2.5}$$

The Hilbert space of the free-particle states is spanned by the eigenstates of H_0 denoted as $|\vec{P}\vec{k}\rangle$. The dependence on $\vec{P}$ follows from translational invariance, and it is sufficient to consider states denoted as $|\vec{k}\rangle$. Acting on such states we have $h\Omega_-=\Omega_-h_0$, and defining a scattering operator

$$\mathcal{T}=v\Omega_-\,, \tag{2.6}$$

we get from Eqs. (2.5) and (2.6) the two-body, half-off-shell, integral equation

$$\langle\vec{k}'|\mathcal{T}|\vec{k}\rangle=\langle\vec{k}'|v|\vec{k}\rangle - \lim_{\epsilon\to 0}\int d^3\vec{k}''\langle\vec{k}'|v|\vec{k}''\rangle\langle\vec{k}''|\mathcal{T}|\vec{k}\rangle \times[\omega(k'')-\omega(k)-i\epsilon]^{-1}\,, \tag{2.7}$$

where $\omega(k)\equiv(k^2+m_1^2)^{1/2}+(k^2+m_2^2)^{1/2}$. The energy shell is determined by k^2 in Eq. (2.7), and in the following we shall denote it as

$$k^2=k_s^2\equiv\tfrac{1}{4}s-\tfrac{1}{2}(m_1^2+m_2^2)+\tfrac{1}{4}s^{-1}(m_1^2-m_2^2)^2\,,$$

where s is the invariant mass squared equal to the c.m. energy squared. The normalization of $\mathcal{T}$ is such that

$$\frac{d\sigma}{dt}=\pi k_s^{-2}\frac{d\sigma}{d\Omega}\bigg|_{\text{c.m.}}=\pi^5k_s^{-2}s[1-(m_1^2-m_2^2)^2s^{-2}]^2 \times|\langle\vec{k}'|\mathcal{T}|\vec{k}\rangle|_{k'^2=k^2=k_s^2}|^2, \tag{2.8}$$

where t is the invariant momentum transfer squared. Some more details concerning normalization are given in Appendix A.

For simplicity, we now put $m_1=m_2=m$, and in Appendix B we collect the relevant formulas for $m_1\neq m_2$. A different two-body equation from Eq. (2.7), but equivalent to it, can be found[4] if we use again the Kato theorem, and instead of Eq. (2.4) we write

$$\Omega_\pm=\lim_{t\to\pm\infty}\exp(i\tfrac{1}{4}m^{-1}h^2t)\exp(-i\tfrac{1}{4}m^{-1}k_0^2t)\,. \tag{2.9}$$

Then, we define an interaction V and a corresponding scattering operator T as

$$V=\tfrac{1}{4}m^{-1}(h^2-h_0^2)=\tfrac{1}{4}m^{-1}(h_0v+vh_0+v^2)\,, \tag{2.10}$$

$$T=V\Omega_-=\tfrac{1}{4}m^{-1}(h_0\mathcal{T}+\mathcal{T}h_0)\,. \tag{2.11}$$

From Eqs. (2.5), (2.9), (2.10), and (2.11) we get

$$\langle\vec{k}'|T|\vec{k}\rangle=\langle\vec{k}'|V|\vec{k}\rangle - \lim_{\epsilon\to 0}\int d^3\vec{k}''\langle\vec{k}'|V|\vec{k}''\rangle\langle\vec{k}''|T|\vec{k}\rangle \times(k''^2m^{-1}-k^2m^{-1}-i\epsilon)^{-1}\,. \tag{2.12}$$

Notice, that the two-body propagator in Eq. (2.12) has in the denominator the quadratic dependence of the momenta, as in the nonrelativistic case.

The same two-body propagator as in Eq. (2.12) can also be found from the following procedure. We multiply Eq. (2.6) from the left- and right-hand sides by the operator $\tfrac{1}{2}m^{-1/2}(h_0+\omega)^{1/2}$, where ω is a parameter. Then, using Eq. (2.5), we take the matrix elements between the states $\langle\vec{k}'|$ and $|\vec{k}\rangle$, set the value of ω to be equal to $\omega(k)$, and performing the t integration in Eq. (2.5) we get the two-body equation in the form of Eq. (2.12), but instead of V and T we obtain $\overline{V}$ and $\overline{T}$, respectively. The fully off-shell matrix element of $\overline{V}$, which appears in the kernel of such an equation, is

$$\langle\vec{k}'|\overline{V}|\vec{k}''\rangle=\tfrac{1}{4}m^{-1}[\omega(k')+\omega(k)]^{1/2}\langle\vec{k}'|v|\vec{k}''\rangle \times[\omega(k'')+\omega(k)]^{1/2}\,. \tag{2.13}$$

This should be contrasted with

$$\langle\vec{k}'|V|\vec{k}''\rangle=\tfrac{1}{4}m^{-1}[\omega(k')\langle\vec{k}'|v|\vec{k}''\rangle+\omega(k'')\langle\vec{k}'|v|\vec{k}''\rangle +\langle\vec{k}'|v^2|\vec{k}''\rangle]\,. \tag{2.14}$$

The matrix elements of $\overline{T}$ and $\mathcal{T}$ are related in the same way as $\overline{V}$ and v. Again, such a relation is different from the one which holds between T and $\mathcal{T}$ and follows from Eq. (2.11).

The fully off-shell matrix elements of V and v are independent of $\omega(k)$, the energy shell. (When going fully on shell the matrix elements of V and/or v may depend on energy if they are nonlocal interactions. This should not be confused with the energy independence of the fully off-shell matrix elements of V and v.) Therefore, we call V and v potentials, while $\overline{V}$, explicitly depending on $\omega(k)$, we call a quasipotential. The two-body equations, of the form of Eq. (2.12) with either V or $\overline{V}$, describe the same dynamics but in a different way. The matrix elements of V and $\overline{V}$ are very different both off shell and on shell. Only the fully on-shell elements of T and $\overline{T}$ coincide, as it should be for measurable quantities normalized in the same way. However, the off-shell extensions of T and $\overline{T}$ are completely different, though the propagators are identical in both equations. There is no question of the uniqueness of the off-shell extension of either V and T or $\overline{V}$ and $\overline{T}$ once the matrix elements $\langle\vec{k}'|v|\vec{k}''\rangle$ are given as an input. This is transparent in Eqs. (2.13) and (2.14).

The variety of equivalent two-body integral equations can be enlarged either by using the Kato theorem, with functions different from before, or by acting with operators different from $\frac{1}{2}m^{-1/2}(h_0+\omega)^{1/2}$. In Sec. V we present some more examples of them. Now we obtain from Eq. (2.12) an equation with the form of the Weinberg equation.[12] However, there is an essential difference between this equation and the original Weinberg equation. The most important difference is in the energy dependence of the fully off-shell elements of the irreducible kernel.

Let us take the limit $|\vec{P}|\to\infty$, keeping $P_0^2-\vec{P}^2$ fixed, and let us use Weinberg's parametrization[12] of the particle's momenta. For $\eta\in(0,1)$, we have

$$\vec{p}_1=\eta\vec{P}+\vec{q}_\perp,\quad \vec{p}_2=(1-\eta)\vec{P}-\vec{q}_\perp,$$

where $\vec{P}\cdot\vec{q}_\perp=0$. For a finite $|\vec{P}|\equiv P$ we consider the Lorentz transformation to the c.m. system, and we get

$$k_\parallel=\eta P\cosh\varphi-[(\eta\vec{P}+\vec{q}_\perp)^2+m^2]^{1/2}\sinh\varphi,$$

$$\vec{k}_\perp=\vec{q}_\perp,$$

where

$$\cosh\varphi=\{(\eta^2P^2+q_\perp^2+m^2)^{1/2}+[(1-\eta)^2P^2+q_\perp^2+m^2]^{1/2}\}A^{-1},$$

$$\sinh\varphi=PA^{-1}\equiv P(\{(\eta^2P^2+q_\perp^2+m^2)^{1/2}+[(1-\eta)^2P^2+q_\perp^2+m^2]^{1/2}\}^2-P^2)^{-1/2}.$$

Taking the limit $P\to\infty$, with η and $\vec{q}_\perp$ fixed, we find

$$\lim k_\parallel=(\eta-\tfrac{1}{2})[(q_\perp^2+m^2)\eta^{-1}(1-\eta)^{-1}]^{1/2},$$
$$\lim\omega(k)=[(q_\perp^2+m^2)\eta^{-1}(1-\eta)^{-1}]^{1/2},\qquad(2.15)$$
$$\lim d^3k=\tfrac{1}{4}\eta^{-3/2}(1-\eta)^{-3/2}(q_\perp^2+m^2)^{1/2}d\eta\,d^2q_\perp.$$

Finally, defining an amplitude M by the equation

$$\langle\eta''\vec{q}_\perp''|M|\eta'\vec{q}_\perp'\rangle=(2\pi)^3 2m\,\omega^{1/2}(k'')\times\langle\eta''\vec{q}_\perp''|T|\eta'\vec{q}_\perp'\rangle\omega^{1/2}(k'),\quad(2.16)$$

and similarly defining the matrix elements of I in terms of V, we get the following two-body equation, equivalent to Eq. (2.12):

$$\langle M\rangle=\langle I\rangle-\tfrac{1}{2}(2\pi)^{-3}\int d\eta''d^2q_\perp''\,\eta''^{-1}(1-\eta'')^{-1}\times\langle I\rangle\langle M\rangle\left[\frac{q_\perp''^2m^2}{\eta''(1-\eta'')}-\frac{q_\perp^2+m^2}{\eta(1-\eta)}-i\epsilon\right]^{-1}.\quad(2.17)$$

This is the Weinberg[12] form of the two-body equation. However, here the fully off-shell elements of I are independent of $\omega(k)$, the energy shell, as can be seen from Eqs. (2.16) and (2.14). This is in contrast to the irreducible kernel in the original Weinberg equation which we disucss in Sec. IV.

III. COVARIANT, TWO-BODY DYNAMICS

It is possible to find a covariant generalization of the scheme given in the preceding section. We start by discussing a free two-body system, and denote the particle's four-momenta by p_1, p_2, the total four-momentum by $P=p_1+p_2$, and for the relative four-momentum we take the following combination of p_1 and p_2:

$$q=\mu_2p_1-\mu_1p_2,\qquad(3.1)$$

where

$$\mu_1=\tfrac{1}{2}[1+(m_1^2-m_2^2)P^{-2}],$$
$$\mu_2=\tfrac{1}{2}[1-(m_1^2-m_2^2)P^{-2}],$$

m_1 and m_2 are the rest masses, and P^2 denotes the four-dimensional scalar product with the signature $(+---)$. The four-vector q is known as the Wightman-Gårding[16] relative momentum, and for the on-mass-shell momenta it is guaranteed to be a spacelike four-vector, since then the scalar product $q \cdot P$ vanishes. The Hilbert space of the free two-body states is spanned by the fully-on-mass-shell states

$$|p_1(p_1^2=m_1^2)\, p_2(p_2^2=m_2^2)\rangle.$$

Two-mass shell constraints are equivalent to the following two constraints on the relative and total four-momenta:

$$q \cdot P=0, \tag{3.2}$$

$$P^2-[(-q^2+m_1^2)^{1/2}+(-q^2+m_2^2)^{1/2}]^2=0. \tag{3.3}$$

The two-body states

$$|q(\text{Eq. (3.2)})P(\text{Eq. (3.3)})\rangle$$

have 6 degrees of freedom, as in the nonrelativistic case. This is in accordance with the fact that the contraction of the Lorentz group to the Galilei group does not decrease the number of commuting operators.

For further development of the constrained dynamics it is useful to carry out some considerations for a classical two-body system. Such a system is described in a 16-dimensional phase space of two four-momenta p_1 and p_2 and two canonically conjugate position-space four-vectors x_1 and x_2. In the x space we introduce two four-vectors for the two-body system. One is the relative position $x=x_1-x_2$, and the second is an analog of the center-of-mass position. The second four-vector we denote as capital X, and we define it through the relation

$$x_1 \cdot dp_1+x_2 \cdot dp_2=X \cdot dP+x \cdot dq\ . \tag{3.4}$$

Substituting Eq. (3.1) into Eq. (3.4), we get

$$X=\mu_1 x_1+\mu_2 x_2-[x \cdot P(m_1^2-m_2^2)P^{-4}]P\ . \tag{3.5}$$

It is straightforward to verify that if x_1, p_1, and x_2, p_2 satisfy the standard, canonical Poisson brackets, then x, q and X, P also obey them as should be because of Eq. (3.4).

In the 16-dimensional phase space we restrict ourselves to a timelike four-vector P, and we define a 12-dimensional subspace by using two constraints given by Eqs. (3.2) and (3.3) and adding to them two extra constraints for defining a parametrization in the constrained space. Our four constraints are

$$\begin{aligned}
&\varphi_1 \equiv q \cdot P=0,\\
&\chi_1 \equiv x \cdot P=0,\\
&\varphi_2 \equiv P^2-[(-q^2+m_1^2)^{1/2}+(-q^2+m_2^2)^{1/2}]^2=0,\\
&\chi_2 \equiv X_0-t=0,
\end{aligned} \tag{3.6}$$

where t is the time in the frame of reference, in which the components of the four-vector P are P_0, P_x, P_y, and P_z. It is useful, but not necessary, to use the notation of the Dirac-Poisson brackets in the constrained space. We do it in two steps, first using $\varphi_1=\chi_1=0$, then $\varphi_2=\chi_2=0$. The Dirac-Poisson brackets are denoted by a single asterisk and a double asterisk respectively, and for two quantities f and g they are defined in terms of the ordinary Poisson brackets as

$$\{fg\}^*=\{fg\}+[\{f\varphi_1\}\{\chi_1 g\}-\{f\chi_1\}\{\varphi_1 g\}](\{\varphi_1\chi_1\})^{-1},$$

$$\{fg\}^{**}=\{fg\}^* +[\{f\varphi_2\}^*\{\chi_2 g\}^*-\{f\chi_2\}^*\{\varphi_2 g\}^*](\{\varphi_2\chi_2\}^*)^{-1}.$$

For any pair of $\varphi_1, \chi_1, \varphi_2, \chi_2$ the double-asterisk brackets are equal to zero. However, the double-asterisk brackets between the components of the four-vectors x, q, X, and P are no longer of the canonical form. Nevertheless, it is possible to define a new set of variables which have the canonical form of the double-asterisk brackets needed for the quantum program.

To define the new variables we introduce a vierbein which we denote as $P\kappa l m$. It contains a unit timelike four-vector $P(P^2)^{-1/2}$, and three orthonormal spacelike four-vectors κ, l, and m. Thus, we have $P\cdot\kappa=P\cdot l=P\cdot m=\kappa\cdot l=\kappa\cdot m=l\cdot m=0$, and $\kappa^2=l^2=m^2=-1$. Such a vierbein was used by the present author in connection with the eikonal approximation[11] and the relativistic Glauber formula.[9] Its explicit construction in terms of P_0, P_x, P_y, and P_z was given by Faustov.[17] Denoting $(P^2)^{1/2}$ as M_0, and P_0+M_0 as N_0, we have the following components of the vierbein four-vectors[17]:

$$\begin{aligned}
&P(P^2)^{-1/2}=(P_0M_0^{-1}, P_xM_0^{-1}, P_yM_0^{-1}, P_zM_0^{-1}),\\
&\kappa=(P_zM_0^{-1}, P_xP_zM_0^{-1}N_0^{-1}, P_yP_zM_0^{-1}N_0^{-1}, 1+P_z^2M_0^{-1}N_0^{-1}),\\
&l=(P_xM_0^{-1}, 1+P_x^2M_0^{-1}N_0^{-1}, P_xP_yM_0^{-1}N_0^{-1}, P_xP_zM_0^{-1}N_0^{-1}),\\
&m=(P_yM_0^{-1}, P_xP_yM_0^{-1}N_0^{-1}, 1+P_y^2M_0^{-1}N_0^{-1}, P_yP_zM_0^{-1}N_0^{-1}).
\end{aligned} \tag{3.7}$$

Any four-vector can be projected on the $P\kappa lm$ vierbein. For example, instead of q and x we can store the information about these four-vectors in the following scalar products:

$$q_P \equiv q\cdot PM_0^{-1},\quad q_\kappa \equiv -q\cdot\kappa,\quad q_l \equiv -q\cdot l,\quad q_m \equiv -q\cdot m,$$

$$x_P \equiv x\cdot PM_0^{-1},\quad x_\kappa \equiv x\cdot\kappa,\quad x_l \equiv x\cdot l,\quad x_m \equiv x\cdot m.$$

The scalar products $q_{\kappa l m}$ and $x_{\kappa l m}$ are the new variables which have the following double-asterisk brackets:

$$\{x_\alpha q^\beta\}^{**} = \delta_\alpha{}^\beta,\quad \text{where } \alpha \text{ or } \beta = \kappa, l, m$$
$$\{x_\alpha \vec{P}\}^{**} = 0,\quad \{q_\alpha P^\mu\}^{**} = 0,\quad \text{where } \mu = 0,1,2,3. \tag{3.8}$$

Knowing projections on the vierbein $P\kappa lm$, and the vierbein four-vectors, we can reconstruct a given four-vector. For example, we have

$$q = PM_0^{-1}q_P + \kappa q_\kappa + l q_l + m q_m.$$

If we are given $q_{P\kappa lm}$, $x_{P\kappa lm}$, X^μ, and P^μ, then we can write ten Poincaré generators,

$$H_0 = (\vec{p}_1{}^2 + m_1{}^2)^{1/2} + (\vec{p}_2{}^2 + m_2{}^2)^{1/2},$$
$$\vec{K}_0 = \vec{x}_1(\vec{p}_1{}^2 + m_1{}^2)^{1/2} + \vec{x}_2(\vec{p}_2{}^2 + m_2{}^2)^{1/2},$$
$$\vec{P}_0 = \vec{p}_1 + \vec{p}_2, \tag{3.9}$$
$$\vec{J}_0 = \vec{x}_1\times\vec{p}_1 + \vec{x}_2\times\vec{p}_2,$$

where the index 0 on the left-hand side denotes the noninteracting case. In the constrained space we have $q_P = x_P = 0$, and

$$M_0 = (q_\kappa{}^2 + q_l{}^2 + q_m{}^2 + m_1{}^2)^2 + (q_\kappa{}^2 + q_l{}^2 + q_m{}^2 + m_2{}^2)^{1/2}.$$

The vanishing double-asterisk brackets $\{q_\alpha P^\mu\}^{**}$ and $\{q_\alpha, q_\beta\}^{**}$ enable us to formulate the quantum program with the use of the states $|q_{\kappa lm} P(P^2 = M_0{}^2)\rangle$. They are eigenstates of the free Hamiltonian

$$H_0 = \{\vec{P}^2 + [(q_\kappa{}^2 + q_l{}^2 + q_m{}^2 + m_1{}^2)^{1/2} + (q_\kappa{}^2 + q_l{}^2 + q_m{}^2 + m_2{}^2)^{1/2}]^2\}^{1/2}. \tag{3.10}$$

To introduce an interaction we follow the basic idea of Bakamjian and Thomas (many details of our work and that of Bakamjian and Thomas are different) and change the mass operator of the two-body system. For the noninteracting case we have

$$M_0 = p_{1P} + p_{2P}, \tag{3.11}$$

where p_1 and p_2 are expressed in terms of q and P as

$$p_{1,2} = \pm q + \mu_{1,2} P,$$

with

$$\mu_{1,2} = \tfrac{1}{2}[1 \pm (m_1{}^2 - m_2{}^2)M_0{}^{-2}]. \tag{3.12}$$

In the presence of interaction we keep the condition $q_P = 0$ as a condition on the state vectors, where in the $P\kappa lm$ vierbein field we put $P^2 = M_0{}^2$, so we have

$$\mu_2 p_{1P} - \mu_1 p_{2P} = 0, \tag{3.13}$$

but we modify the operator Eq. (3.11) into the operator

$$M = p_{1P} + p_{2P} + \varphi, \tag{3.14}$$

where φ is a function of $q_{\kappa lm}$ and $x_{\kappa lm}$, about which we shall tell more later on. The change from Eq. (3.11) to Eq. (3.14) is the same as from h_0 to $h = h_0 + v$ in the preceding section. [The change from (3.11) to (3.14) is also similar to the one which appears when passing from the evaluation of the propagator to the evaluation of the irreducible kernel in the Weinberg equation. In particular, in the lowest-order approximation to the irreducible kernel, when we evaluate the contribution to the energy of the system from the exchange of one leg, we must add to the energies of two propagating particles also a contribution from the exchanged leg.] In the presence of interaction our two particles remain on their mass shells, but the whole system goes off the energy shell to the nonrelativistic case. Indeed, using the vierbein $P\kappa lm$ with $P^2 = M_0{}^2$, we have as always $p_{1\kappa lm} = -p_{2\kappa lm} = q_{\kappa lm}$, and from Eqs. (3.13) and (3.14) we get $p_{1,2P} = \mu_{1,2} M_0$, leading to two mass-shell constraints $p_{1,2P}{}^2 - q_\kappa{}^2 - q_l{}^2 - q_m{}^2 = m_{1,2}{}^2$.

The function φ has to satisfy some regularity conditions, such as continuity, boundedness, monotonicity in domains, and a proper behavior at infinities. We shall assume all of them, and for more details we refer to the papers by Kato[15] and Schierholz.[18] Here, we only consider the variables on which φ depends, and the form of this dependence. We can verify that if φ is a function of the following combination of $q_{\kappa lm}$ and $x_{\kappa lm}$,

$$\varphi = \varphi(q_\kappa{}^2 + q_l{}^2 + q_m{}^2, x_\kappa{}^2 + x_l{}^2 + x_m{}^2, (q_\kappa x_\kappa + q_l x_l + q_m x_m)_+),$$

where the subscript + means a symmetrized product, then the double-asterisk brackets between any pair of constraints remain the same if we replace the constraint $\varphi_2 = 0$ by

$$\varphi_2' \equiv P^2 - [(-q^2 + m_1^2)^{1/2} + (-q^2 + m_2^2)^{1/2} + \varphi]^2 = 0. \tag{3.15}$$

Solving this constraint for the time component of P we get the following total Hamiltonian H, in the presence of interaction:

$$H = \{\vec{P}^2 + [(q_\kappa^2 + q_l^2 + q_m^2 + m_1^2)^{1/2} + (q_\kappa^2 + q_l^2 + q_m^2 + m_2^2)^{1/2} + \varphi]^2\}^{1/2}. \tag{3.16}$$

The Møller operators

$$\Omega_\pm = \lim_{t\to\pm\infty} \exp(iHt)\exp(-iH_0 t)$$

can be rewritten in two different forms using the Kato theorem with the functions

$$f_1(H) = (H^2 - \vec{P}^2)^{1/2} = M,$$

and

$$f_2(M) = \tfrac{1}{8}\mu^{-1}[M^2 + M^{-2}(m_1^2 - m_2^2)^2],$$

where $\mu = m_1 m_2 (m_1 + m_2)^{-1}$ is the ordinary reduced mass. We get

$$\Omega_\pm = \lim_{t\to\pm\infty} \exp(iMt)\exp(-iM_0 t) = \lim_{t\to\pm\infty} \exp[if_2(M)t]\exp[-if_2(M_0)t]. \tag{3.17}$$

For the interacting case we build the Poincaré generators as

$$H, \vec{P}, \vec{K}, \vec{J} = \Omega_-(H_0, \vec{P}_0, \vec{K}_0, \vec{J}_0)\Omega_-^\dagger,$$

where H_0, $\vec{P}_0$, $\vec{K}_0$, and $\vec{J}_0$ are given by Eq. (3.9). (When the two-body system has a bound state, then we have to add to the above generators an extra term corresponding to the projection on the space of the bound states. For more details concerning this modification we refer to Fong and Sucher.[14]) The generators of the space translations $\vec{P}$ are the same for the noninteracting and the interacting case because $\vec{P}$ commutes with Ω_-.

The commutation of $\vec{P}$ with H can be also used for rewriting $\Omega_\pm$ in the form containing the proper time of the two-body system $\mathcal{T} = t(1 + \vec{P}^2 s^{-1})^{-1/2}$, where s is the invariant energy squared, an eigenvalue of M_0^2. We can follow Jordan, Macfarlane, and Sudarshan,[14] and use the Heisenberg picture to represent transformation to the c.m. system.

Taking the Lorentz transformation along P_z, setting $P_x = P_y = 0$, and denoting $\cosh\alpha \equiv (1 + \vec{P}^2 s^{-1})^{1/2}$, we get for the interacting and free systems the following transformations of H and H_0:

$$H \to H\cosh\alpha - P_z\sinh\alpha, \quad H_0 \to H_0\cosh\alpha - P_z\sinh\alpha.$$

These transformations inserted in the definition of the Møller operators give the following final result, after using the commutation of H and $\vec{P}$ and the Kato theorem with $f(H) = M(\cosh\alpha)^{-2}$:

$$\Omega_\pm = \lim_{t\to\pm\infty} \exp(iHt\cosh\alpha)\exp(-iH_0 t\cosh\alpha) = \lim_{\mathcal{T}\to\pm\infty} \exp(iM\mathcal{T})\exp(-iM_0\mathcal{T}).$$

In correspondence with v and V, defined in Sec. II, we introduce, besides $\varphi = M - M_0$, also

$$\phi = f_2(M) - f_2(M_0). \tag{3.18}$$

The two-body scattering amplitude corresponding to ϕ is defined as $T = \phi\Omega_-$, and its matrix elements have the property

$$\langle q^2_{\kappa' l' m'} P'(P'^2 = M_0'^2) | T | q_{\kappa l m} P(P^2 = M_0^2)\rangle = \delta^{(3)}(\vec{P}' - \vec{P})\langle q'_{\kappa' l' m'} | T | q_{\kappa l m}\rangle, \tag{3.19}$$

which follows from translational invariance. (The dependence of a t matrix on $\vec{P}$ could be also found from the following procedure: We define the t matrix by the equation $t = (H - H_0)\Omega_-$, and derive an integral equation for it as in Sec. II. In such an equation, $\vec{P}$ appears explicitly because of Eqs. (3.10) and (3.16). However, when doing this, it is important to notice that both H and H_0 are $\vec{P}$ dependent, and therefore also the difference $H - H_0$ is $\vec{P}$ dependent. An attempt along these lines was made by L. Heller, G. E. Bohannon and F. Tabakin [Phys. Rev. C 13, 742 (1976)]; however, the $\vec{P}$ dependence was only taken into account in the propagator and was not considered in $H - H_0$ which is complicated.) Three δ functions in Eq. (3.19) show the trivial dependence of T on the total three-momentum. The nontrivial dependence of T on the fourth component of P^μ can be simply taken into account if we use the notion of the projections on the $P\kappa l m$ vierbein for the four-vector P^μ. Then, instead of writing P^μ, we put

$$P_P = M_0, \quad P_\kappa = 0, \quad P_l = 0, \quad P_m = 0.$$

In terms of the components in the frame of the $P\kappa l m$ vierbein, the four-vectors PM_0^{-1}, κ, l, and m are, of course, (1, 0, 0, 0), (0, 0, 0, 1), (0, 1, 0 0), (0, 0, 1, 0). The primed total four-momentum P' has in the unprimed vierbein frame the components $P'_{P\kappa l m}$. However, because of the δ functions in Eq. (3.19), we get

$$P'_{\kappa l m}=P_{\kappa l m}=0. \tag{3.20}$$

The component P'_P is equal to M'_0, since $P'^2=M_0'^2$. Using Eqs. (3.7) and (3.20) we find that in the $P\kappa l m$ vierbein frame, the four-vectors $P'M_0'^{-1}$, κ', l', and m' are exactly the same as PM_0^{-1}, κ, l and m. Therefore, when working with the components $P_P=M_0, P_{\kappa l m}=0$ and $P'_P=M'_0, P'_{\kappa l m}=0$, we can simplify Eq. (3.19), using only one vierbein $P\kappa l m$, and we get

$$\langle q'_{\kappa l m}P'(P'^2=M_0'^2)\,|\,T\,|\,q_{\kappa l m}P(P^2=M_0^2)\rangle=\delta^{(3)}(P'_{\kappa l m})\langle q'_{\kappa l m}\,|\,T\,|\,q_{\kappa l m}\rangle. \tag{3.21}$$

We also find that $q'_P=0$ since $q'\circ P'=0$ and $P'_{\kappa l m}=0$. Thus, the T matrix is evaluated for the following values of the four-vectors q, P, q', and P':

$$\langle q'_P=0, q'_{\kappa l m}, P'_P=M'_0, P'_{\kappa l m}=0\,|\,T\,|\,q_P=0, q_{\kappa l m}, P_P=M_0, P_{\kappa l m}=0\rangle. \tag{3.22}$$

The matrix elements $\langle q'_{\kappa l m}|T|_{\kappa l m}\rangle$ satisfy the following two-body integral equation, which can be derived here in the same way as in Sec. II,

$$\langle T\rangle=\langle\phi\rangle-2\mu\lim_{\epsilon\to 0}\int d^3q''_{\kappa l m}\langle\phi\rangle\langle T\rangle(q_\kappa''^2+q_l''^2+q_m''^2-q_\kappa^2-q_l^2-q_m^2-i\epsilon)^{-1}. \tag{3.23}$$

It is also evident that for the t matrices $\mathcal{T}$ and $\overline{T}$ we can find the corresponding covariant analogs. However, we cannot work with the projections on the $P\kappa l m$ vierbein fields and take the limit $|\vec{P}|\to\infty$, with $P_0^2-\vec{P}^2$ fixed, since then two of the vierbein four-vectors would be undetermined, having two infinite components.

IV. COVARIANT WEINBERG EQUATION WITH A QUADRATIC PROPAGATOR

In field theory the basic two-body equation is the Bethe-Salpeter equation.[19] It is an exact equation, as shown by Gell-Mann and Low[20] if the irreducible kernel contains an infinite series and the particle propagators are the full two-point Green's functions. (The Bethe-Salpeter equation for the two-body Feynman kernel can be viewed as a definition of the irreducible kernel. Such a kernel should contain all couplings to the many-body states, while all disconnected diagrams should be taken care of in the propagators corresponding to the physical masses of particles.) However, in practice one is forced to take a finite sum of diagrams for the irreducible kernel, and then the Bethe-Salpeter equation is only an approximate equation. In general, the irreducible kernel of the Bethe-Salpeter equation is nonlocal and energy dependent, but in some cases, like the ladder approximation in the scalar theory, it is local and energy independent.

In parallel to the standard Feynman perturbation theory one may also write the "old fashioned" perturbation rules for the S-matrix elements. Then, all particles in the intermediate states are on their mass shells, and instead of the Feynman propagators for each particle we have energy denominators for each state. Kadyshevsky[21] developed such a scheme and found a two-body integral equation which plays the same role as the Bethe-Salpeter equation in the standard theory. Again, if the kernel contains all irreducible diagrams and the propagator is the full one, then such a two-body equation is exact by definition of the irreducible kernel. However, in contrast to the Bethe-Salpeter case, this irreducible kernel is nonlocal and energy dependent even in the lowest-order approximation. Weinberg[12] considered the infinite-momentum limit of the old-fashioned perturbation rules and wrote a Bethe-Salpeter-type equation, with the ladder approximation for the irreducible kernel.

The Weinberg infinite-momentum rules can also be found in the light-front field theory (LFFT) approach. Such schemes were developed by many authors,[22] but most useful for us is the series of papers by Yan *et al.*[23] There, all popular renormalizable field theories were studied and a formal proof of the equivalence of the LFFT with the standard field theory is given. The S-matrix operator is defined by the Dyson formula,

$$S=T^+\exp\left(-i\int d^4x\,H_I(x)\right),$$

with T^+ denoting the $x_+=x_0+x_z$ ordered product, which can be expanded in the perturbation series analogous to the old-fashioned time-ordered perturbation expansion. Then, instead of the energy denominators we have the $p_-=p_0-p_z$ denominators, and the conservation of the $p_+=p_0+p_z$ and $\vec{p}_\perp$ components. The on-shell particle momenta are determined from the simple relation $p_+p_- -p_\perp^2=m^2$, which is *linear* in p_-.

For our purposes the main object of interest is the two-body matrix element of the S-matrix operator. Taking the S-matrix operator between the on-mass-shell states, we have a well-defined total four-momentum P in the initial state. It is a timelike four-vector and it can be used for defining the $P\kappa lm$ vierbein, introduced in the preceding section. This vierbein can be taken for defining the following combination of variables which play the role of the $\pm$ and $\perp$ components: $p_{1P}\pm p_{1\kappa}, p_{1l}, p_{1m}$, and similarly for the second particle. The momenta of the individual particles are on their mass shells, thus the relative momentum q and the total momentum P obey the relations $q\cdot P=0$ and $P^2=M_0^2=4(m^2+q_\kappa^2+q_l^2+q_m^2)$, for $m_1=m_2=m$. The case $m_1\neq m_2$ is given in Appendix B. The $+$ and $\perp$ components of the total momentum are conserved; therefore, we have the following structure of the matrix element of the scattering matrix M:

$$\langle P'_P+P'_\kappa, P'_{lm}, P'_P-P'_\kappa=M_0'^2(P'_P+P'_\kappa)^{-1}, q'_P+q'_\kappa, q'_{lm};\ q'\cdot P'=0\,|M|\,P_P=M_0, P_{\kappa lm}=0, q_P=0, q_{\kappa lm}\rangle$$
$$=\delta(P'_P+P'_\kappa-P_P)\delta^{(2)}(P'_{lm})\langle q'_P+q'_\kappa, q'_{lm}|M|q_{\kappa lm}\rangle\,, \qquad (4.1)$$

where in the final state we used the same vierbein $P\kappa lm$ as in the initial state. We note, that if $M'_0\neq M_0$, then the final state is not in the c.m. system. Introducing the standard notion of the light-cone variable $\eta=(p_{1P}+p_{1\kappa})(P_P+P_\kappa)^{-1}$, we have $\eta=\frac{1}{2}+q_\kappa P_P^{-1}$, $\eta'=\frac{1}{2}+(q'_P+q'_\kappa)(P'_P+P'_\kappa)^{-1}$, and we get the Weinberg equation

$$\langle\eta' q'_{lm}|M|\eta q_{lm}\rangle=\langle\eta' q'_{lm}|I^W|\eta q_{lm}\rangle$$
$$-\tfrac{1}{2}(2\pi)^{-3}\int d\eta'' d^2q''_{lm}\eta''^{-1}(1-\eta'')^{-1}\langle\eta' q'_{lm}|I^W|\eta'' q''_{lm}\rangle\langle\eta'' q''_{lm}|M|\eta q_{lm}\rangle$$
$$\times\,[(q_l''^2+q_m''^2+m^2)\eta''^{-1}(1-\eta'')^{-1}-(q_l^2+q_m^2+m^2)\eta^{-1}(1-\eta)^{-1}-i\epsilon]^{-1}\,, \qquad (4.2)$$

where $\langle\eta' q'_{lm}|I^W|\eta'' q''_{lm}\rangle$ is the irreducible kernel, depending on the energy shell, determined in Eq. (4.2) by $(q_l^2+q_m^2+m^2)\eta^{-1}(1-\eta)^{-1}=s$. The lowest-order approximation to I^W is given below, in Eq. (4.7). If I^W would contain all irreducible diagrams, and in Eq. (4.2) there would be the full propagator, then Eq. (4.2) would be the definition of I^W, because the two-body matrix element of M is already defined. In this sense, Eq. (4.2) may be considered as an exact equation such as the Bethe-Salpeter equation with the full irreducible kernel, and the full propagator.

It is possible to make a change of variables and get the propagator in Eq. (4.2) in a quadratic form. According to Eq. (4.1), the total four-momenta in the initial and final states are, respectively,

$$P=(P_P=M_0=\sqrt{s}, P_{\kappa lm}=0)$$

and

$$P'=(P'_P=\tfrac{1}{2}M_0+\tfrac{1}{2}M_0'^2M_0^{-1}, P'_\kappa$$
$$=\tfrac{1}{2}M_0-\tfrac{1}{2}M_0'^2M_0^{-1}, P'_{lm}=0),$$

where $M_0'^2=P'^2=(P'_P+P'_\kappa)(P'_P-P'_\kappa)$. Thus, from the condition $q'\cdot P'=0$ we get $q'_PP'_P-q'_\kappa P'_\kappa=0$, and using it, together with the above relations, we find an identity

$$q_\kappa'^2-q_P'^2=(q'_P+q'_\kappa)^2(P'_P+P'_\kappa)^{-2}P'^2$$
$$=(\eta'-\tfrac{1}{2})^2M_0'^2\,. \qquad (4.3)$$

Let us now introduce two new vierbeins $P'\kappa' lm$, and $P''\kappa'' lm$, with the l and m legs unchanged, because of $P'_{lm}=P''_{lm}=0$ and Eq. (3.7). Working with new and old vierbein components, we find from $q'\cdot P'=0=q''\cdot P''$ the following relations:

$$q'_{P'}=q''_{P''}=0,$$
$$q_{\kappa'}'^2=q_\kappa'^2-q_P'^2,\quad q_{\kappa''}''^2=q_\kappa''^2-q_P''^2\,. \qquad (4.4)$$

Finally, using Eqs. (4.3) and (4.4), we get, as in Eq. (2.15),

$$q''_{\kappa''}=(\eta''-\tfrac{1}{2})(P''^2)^{1/2}\,,$$
$$P''^2=(q_l''^2+q_m''^2+m^2)\eta''^{-1}(1-\eta'')^{-1}\,,$$
$$dq''_{\kappa''}d^2q''_{lm}=\tfrac{1}{4}\eta''^{-3/2}(1-\eta'')^{-3/2}$$
$$\times(q_l''^2+q_m''^2+m^2)d\eta'' d^2q''_{lm}\,, \qquad (4.5)$$

and substituting them in Eq. (4.2), we obtain

$$\langle q'_{\kappa' lm}|M|q_{\kappa lm}\rangle = \langle q'_{\kappa' lm}|I^W|q_{\kappa lm}\rangle$$

$$-\tfrac{1}{2}(2\pi)^{-3}\int d^3q''_{\kappa'' lm} M_0''^{-1}\langle q'_{\kappa' lm}|I^W|q''_{\kappa'' lm}\rangle\langle q''_{\kappa'' lm}|M|q_{\kappa lm}\rangle (q''^2_{\kappa''}+q''^2_l+q''^2_m-k_s^2-i\epsilon)^{-1}, \qquad (4.6)$$

where $k_s^2 \equiv \frac{1}{4}s - m^2$. For illustration, we write the ladder approximation to I^W. It is the Weinberg expression in terms of our variables

$$\langle q'_{\kappa' lm}|I^W_{\text{ladder}}|q''_{\kappa'' lm}\rangle = -g^2\{m^2+(q'_l-q''_l)^2+(q'_m-q''_m)^2+(x'-x'')(x'M_0'^2-x''M_0''^2)+|x'-x''|[\tfrac{1}{2}(M_0'^2+M_0''^2)-s]\}^{-1}, \qquad (4.7)$$

where

$$x' = q'_{\kappa'} M_0'^{-1}, \quad x'' = q''_{\kappa''} M_0''^{-1}.$$

The following features of I^W_{ladder} should be noticed: (1) For $M_0' = M_0'' = M_0$ we get the standard Yukawa interaction, if in the conventional theory we take $P^2 = P'^2 = M_0^2$, and $q \circ P = q' \circ P = 0$. (2) For $M_0' \neq M_0$ or $M_0'' \neq M_0$ we get an explicit dependence on M_0 which plays the role of the energy shell. Thus I^W_{ladder} is energy-dependent. (3) For either $M_0' \neq M_0$ or $M_0'' \neq M_0$ the matrix element $\langle q'_{\kappa' lm}|I^W_{\text{ladder}}|q_{\kappa lm}\rangle$ does not depend on the differences between the variables in the bra and ket, but separately on each of them. In this sense it is a nonlocal interaction.

Equation (4.6) may be rewritten for the amplitude T instead of M. The fully off-shell connection between these amplitudes is

$$\langle q'_{\kappa' lm}|T|q_{\kappa lm}\rangle = \tfrac{1}{2}(2\pi)^{-3} m^{-1} M_0'^{-1/2}\langle q'_{\kappa' lm}|M|q_{\kappa lm}\rangle M_0^{-1/2}. \qquad (4.8)$$

Inserting it in Eq. (4.6) we get for the matrix elements $\langle q'_{\kappa' lm}|T|q_{\kappa lm}\rangle$ the following equation:

$$\langle T\rangle = \langle V^W\rangle - \int d^3q''_{\kappa'' lm}\langle V^W\rangle\langle T\rangle[m^{-1}(q''^2_{\kappa''}+q''^2_l+q''^2_m) - m^{-1}k_s^2 - i\epsilon]^{-1}, \qquad (4.9)$$

where V^W is connected to I^W in the same way as T and M in Eq. (4.8). The kernel V^W is energy dependent and nonlocal, so we may call it a quasipotential. The propagator in Eq. (4.9) has the required structure with the quadratic denominator, as in Eq. (3.23), but the energy dependence of V^W precludes the identification of it with ϕ.

V. COMPARISON OF THREE-DIMENSIONAL FORMALISMS

In Secs. III and IV we presented the exact three-dimensional formalisms in the framework of relativistic quantum mechanics and field theory, respectively. In the literature there are many more three-dimensional, two-body formalisms, and we would like to find some connections between the presented schemes and the schemes taken from the literature.

The necessary requirement for two exact schemes to be physically correct is to give the same fully on-shell t-matrix elements. Thus, the on-shell matrix elements of either the T amplitudes, evaluated from Eqs. (3.23) and (4.9), or the M amplitudes, evaluated from Eqs. (2.17) and (4.2), must coincide for any physical values of s and the invariant momentum transfer squared t. We notice that if $P'^2 = P^2 = s$, then the $P'\kappa lm$ and $P\kappa lm$ vierbeins coincide. For the irreducible kernels, the condition is only a sum rule in the form of the right-hand sides of the above equations taken fully on-shell. Therefore, there is no unique relation between the irreducible kernels from Sec. III and the corresponding kernels from Sec. IV, although in each section separately, the off-shell extensions of the irreducible kernels are uniquely defined if the operators ϕ or φ are known, and the Lagrangian and a set of diagrams for the irreducible kernel are given, respectively. It would be completely incorrect to identify the irreducible kernels from Sec. III with these from Sec. IV both off-shell and on-shell, because the kernels in Sec. IV are energy dependent, and the off-shell extension is different in Sec. IV than in Sec. III. The example of the kernels V and $\overline{V}$, given in Sec. II, shows that it is insufficient to have the same propagators to identify the kernels either on-shell or off-shell. In connection with this we would like to make some comments concerning the so-called "minimal relativity" scheme introduced in Ref. 6. The minimal relativity factors $M_0'^{-1/2}$, $M_0^{-1/2}$ appear in Eqs. (2.16) and (4.8), but we can not identify the kernels. To emphasize this point let us make an incorrect, in general, assumption that the fully on-shell elements of I and I^W coincide. Then, from the necessary requirements of the on-shell equality of the t-matrix elements of M, we would get the following equality:

$$\left\{ \mathrm{P} \int d^3 q''_{\kappa l m} \langle I \rangle \langle M \rangle [\tfrac{1}{2}(2\pi)^3 M_0''(M_0''^2 - s - i\epsilon)]^{-1} \right\} \Big|_{M_0'^2 = M_0^2 = s}$$

$$= \left\{ \mathrm{P} \int d^3 q''_{\kappa'' l m} \langle I^W \rangle \langle M \rangle [\tfrac{1}{2}(2\pi)^3 M_0''(M_0''^2 - s - i\epsilon)]^{-1} \right\} \Big|_{M_0'^2 = M_0^2 = s}, \quad (5.1)$$

where P denotes the principal value. However, in general this equality will be violated, because the off-shell extension of I and M on the left-hand side is different than the extension of I^W and M on the right-hand side, and the principal-value integral crucially depends on the off-shell extension. It is also inappropriate to consider[6] that the potential ϕ in Eq. (3.23) is related to the nonrelativistic potential (like the Reid potential) through the minimal relativity factors. This would correspond to identifying the Reid potential with I, while the correct nonrelativistic limit of ϕ corresponds simply to taking ϕ for small momenta, with no extra factors.

To get a field-theoretic model of the interaction defined in Sec. III we should first choose a Lagrangian and a set of irreducible diagrams for I^W. Then, we solve either Eq. (4.6) or (4.9) and take the fully on-shell elements of this solution. Next, we choose an ansatz for ϕ or φ and solve an appropriate equation from Sec. III, fitting some parameters in the ansatz to the on-shell value obtained from the field-theoretic equation. The ansatz cannot be uniquely determined on the level of the two-body theory. We have to test it further in the many-body calculations.

Now, we shall consider four classes of three-dimensional schemes which originate from different arguments than the one presented in this paper. The first class is the most popular one and was established by Logunov and Tavkhelidze[24] as a reformulation of the Bethe-Salpeter equation, and independently by Blankenbecler and Sugar[25] as the two-body unitary reduction of the Bethe-Salpeter equation. The reduction procedure is nonunique, and for different possibilities see Ref. 5. The second class was developed independently by Kadyshevsky,[21] on the basis of the old-fashioned perturbation theory, and by Schierholz,[18] within the framework of relativistic quantum dynamics. The third class was proposed by Fronsdal and Lundberg[26] in a scheme related to quantum mechanics, and by Gross[27] on the basis of the Bethe-Salpeter equation. The fourth class was developed by Todorov,[28] using equal off-mass-shell continuation. Recently, it was substantiated[13] in a formalism based on constrained dynamics. We used, after Todorov, a similar approach in Sec. III, but we put both particles on their mass shells.

Let us denote the four classes by the letters LT-BS, K-S, FL-G, and T. The two-body propagators in these formalisms, for $m_1 = m_2 = m$, are

$$-[\tfrac{1}{2}(2\pi)^3 M_0''(M_0''^2 - s - i\epsilon)]^{-1}, \quad \text{LT-BS} \quad (5.2a)$$

$$[(2\pi)^3 M_0''^2 (M_0'' - \sqrt{s} - i\epsilon)]^{-1}, \quad \text{K-S} \quad (5.2b)$$

$$[(2\pi)^3 \sqrt{s}\, M_0'' (M_0'' - \sqrt{s} - i\epsilon)]^{-1}, \quad \text{FL-G} \quad (5.2c)$$

$$[\tfrac{1}{2} s^{-1/2} (M_0''^2 - s - i\epsilon)]^{-1}. \quad \text{T} \quad (5.2d)$$

The amplitudes are normalized as the M amplitude in the first three classes, and as the $\mathcal{T}$ amplitude in the fourth class. For each formalism we can find two analog equations within the scheme of Sec. III, with the same propagators as above, but with different off-shell continuations. We denote the kernels in these equations as I and $\bar{I}$ for the first three classes, and as φ^{T} and $\bar{\varphi}^{\mathrm{T}}$ for the fourth class. They are linear in ϕ and φ, respectively. We write these kernels in the way they appear in the appropriate integral equations,

$$\langle I^{\text{LT-BS}} \rangle = -2m(2\pi)^3 M_0'^{1/2} \langle q'_{\kappa l m} | \phi | q''_{\kappa l m} \rangle M_0''^{1/2}, \quad (5.3a)$$

$$\langle I^{\text{K-S}} \rangle = 4m(2\pi)^3 (M_0' + \sqrt{s})^{-1/2} M_0' \langle \phi \rangle M_0'' \times (M_0'' + \sqrt{s})^{-1/2}, \quad (5.3b)$$

$$\langle I^{\text{FL-G}} \rangle = 4m(2\pi)^3 \sqrt{s}\, (M_0' + \sqrt{s})^{-1/2} \times M_0'^{1/2} \langle \phi \rangle M_0''^{1/2} (M_0'' + \sqrt{s})^{-1/2}, \quad (5.3c)$$

$$\langle \varphi^{\mathrm{T}} \rangle = 2m s^{-1/2} \langle \phi \rangle, \quad (5.3d)$$

$$\langle \bar{I}^{\text{LT-BS}} \rangle = -\tfrac{1}{2}(2\pi)^3 (M_0' + \sqrt{s})^{1/2} \times M_0'^{1/2} \langle \varphi \rangle M_0''^{1/2} (M_0'' + \sqrt{s})^{1/2}, \quad (5.4a)$$

$$\langle \bar{I}^{\text{K-S}} \rangle = (2\pi)^3 M_0' \langle \varphi \rangle M_0'' , \tag{5.4b}$$

$$\langle \bar{I}^{\text{FL-G}} \rangle = (2\pi)^3 \sqrt{s}\, M_0'^{1/2} \langle \varphi \rangle M_0''^{1/2} , \tag{5.4c}$$

$$\langle \bar{\varphi}^{\text{T}} \rangle = \tfrac{1}{2} s^{-1/2} (M_0' + \sqrt{s})^{1/2} \langle \varphi \rangle (M_0'' + \sqrt{s})^{1/2} . \tag{5.4d}$$

Equations (5.3a) and (5.4b) have the right-hand side independent of s, so they can be called potentials. The remaining ones are quasipotentials. The numerical results found in the K-S scheme, corresponding to Eq. (5.4b), can be considered as a model of φ, since the kernel is s independent and the off-shell continuation is the same as in Sec. III. In all remaining cases if one wants to get a model of ϕ, or φ, one has to repeat the fits, taking the above kernels and the off-shell continuation as in Sec. III.

It is also possible to get the LFFT analogs corresponding to four propagators in Eq. (5.2). The kernels in these equations we denote as J for the first three classes, and as j^T for the fourth class. We get

$$\langle J^{\text{LT-BS}} \rangle = \langle q_{\kappa}' {}_{lm} | I^W | q_{\kappa}'' {}_{lm} \rangle , \tag{5.5a}$$

$$\langle J^{\text{K-S}} \rangle = 2(M_0' + \sqrt{s})^{-1/2} M_0'^{1/2} \langle I^W \rangle M_0''^{1/2} \times (M_0'' + \sqrt{s})^{-1/2} , \tag{5.5b}$$

$$\langle J^{\text{FL-G}} \rangle = 2\sqrt{s}\,(M_0' + \sqrt{s})^{-1/2} \langle I^W \rangle (M_0'' + \sqrt{s})^{-1/2} , \tag{5.5c}$$

$$\langle j^{\text{T}} \rangle = (2\pi)^{-3} s^{-1/2} M_0'^{-1/2} \langle I^W \rangle M_0''^{-1/2} . \tag{5.5d}$$

VI. CONCLUSIONS AND REMARKS

The main conclusion of our paper is that several, equivalent, exact, covariant, three-dimensional, two-body integral equations, with uniquely defined off-shell extensions, exist if the interaction operator is known or the Lagrangian and a set of diagrams are given. The two-body propagators may take different forms, but if they are the same it is yet insufficient to identify the appropriate irreducible kernels. To make the connection between different irreducible kernels we must take an ansatz for the interaction and perform separate numerical fits. The answer is not unique and must be verified in the many-body calculations.

Our work should be continued in several directions. The most natural one is the three-body covariant framework. On the two-body level, the vector-meson exchange in the nucleon-nucleon interaction should be investigated in the framework of the LFFT, using the Weinberg equation, and incorporating the results of Yan *et al.*[23] The pion-nucleus interaction should be investigated and the role of the N^* system estimated.[7] A separate group of problems is connected with QED, in particular the one-photon exchange[29] should be tested in the three-dimensional Weinberg equation. Also the bound-state problem[30] should be investigated in the LFFT, and the applications to the relativistic quark models should be encouraging because of great similarities with the nonrelativistic scheme. Now we shall make some remarks about topics related to this work.

A. Relativistic Schrödinger equation, and the limit $m_2 \to \infty, P^2 \sim m_2{}^2 \to \infty$

For the unequal-mass case the propagator in the equation for T is

$$[f_2(M_0(q'')) - f_2(M_0(k_s)) - i\epsilon]^{-1} = [\tfrac{1}{2}\mu^{-1}(q_\kappa''^2 + q_l''^2 + q_m''^2) - \tfrac{1}{2}\mu^{-1} k_s{}^2 - i\epsilon]^{-1} , \tag{6.1}$$

where

$$f_2(M) \equiv \tfrac{1}{8}\mu^{-1}[M^2 + M^{-2}(m_1{}^2 - m_2{}^2)^2] ,$$

$$k_s{}^2 \equiv \tfrac{1}{4}s - \tfrac{1}{2}(m_1{}^2 + m_2{}^2) + \tfrac{1}{4}s^{-1}(m_1{}^2 - m_2{}^2)^2 .$$

Therefore, for a local ϕ the following differential equation will hold:

$$\left[-\tfrac{1}{2}\mu^{-1}\left(\frac{\partial^2}{\partial x_\kappa{}^2} + \frac{\partial^2}{\partial x_l{}^2} + \frac{\partial^2}{\partial x_m{}^2}\right) + \phi\right]\psi(x_{\kappa lm}) = \tfrac{1}{2}\mu^{-1} k_s{}^2\, \psi(x_{\kappa lm}) . \tag{6.2}$$

For a nonlocal ϕ, Eq. (6.2) is an integro-differential equation. If ϕ is an energy-dependent quasipotential, then the nonorthogonality of eigenfunctions must be taken into account because of the appearance of the functions of s both on the left- and right-hand sides of Eq. (6.2).

In the limit $m_2 \to \infty$, $P^2 \sim m_2{}^2 \to \infty$, corresponding to an infinitely heavy particle 2 being on its mass shell, we get $\lim q = p_1$, $\lim \mu = m_1$, $\lim k_s{}^2 = E_{1\text{ab}}{}^2 - m_1{}^2$, where $E_{1\text{ab}}$ is the fully on-shell energy of particle 1 in the laboratory system, the rest system of particle 2. The $q \cdot P = 0$ condition becomes $p_1{}^2 = m_1{}^2$, so we recover the Fronsdal-Gross approach.

For $P^2 \to \infty$ the vierbein field $P\kappa lm$ does not make sense, but in the static limit, the laboratory frame is the distinguished one, and the expression $q_\kappa{}^2 + q_l{}^2 + q_m{}^2$ must be replaced by $\vec{p}_1{}^2$, which determines the first-particle energy equal to the relative energy. Therefore, Eq. (6.2) becomes

$$(\tfrac{1}{2}m_1{}^{-1}\vec{p}_1{}^2 + \phi)\,\psi = \tfrac{1}{2}m_1{}^{-1}(E_{1\text{ab}}{}^2 - m_1{}^2)\,\psi \quad \text{or}$$

$$(E_{1\text{ab}}{}^2 - \vec{p}_1{}^2 - 2m_1\phi - m_1{}^2)\,\varphi = 0 , \tag{6.3}$$

the stationary Klein-Gordon equation, corresponding to the energy E_{1ab} of the first particle.

Equation (6.2) may be useful in studying very strongly bound systems, with the binding energy comparable to the rest mass. We have in mind the cases of supercritical fields. In particular, one can study the conditions for the appearance of the pionization phenomena, if for ϕ we take the phenomenological pion-nucleus potential.

B. Symmetry $1 \rightleftharpoons 2$

The relative motion, described by the above Schrödinger equation, is written in terms of variables which are symmetric under the interchange $1 \rightleftharpoons 2$. The same symmetry also has the relative velocity

$$|\vec{v}_{21}| = [1 - m_1^2 m_2^2 (p_1 p_2)^{-2}]^{-1/2}.$$

However,

$$k_s = \mu |\vec{v}_{21}| (m_1 + m_2) s^{-1/2} (1 - v_{21}^2)^{-1/2} \neq \mu |\vec{v}_{21}| .$$

In connection with this point, see Ref. 31.

C. Variables in which the Weinberg propagator is quadratic

The particular combination of the standard light-cone variables, in which the Weinberg propagator has the quadratic structure in the denominator, is

$$q_\kappa = (\eta - \tfrac{1}{2}) (q_\perp^2 + m^2)^{1/2} \eta^{-1/2} (1 - \eta)^{-1/2} . \qquad (6.4)$$

For η we can take $\eta = \frac{1}{2} + (q_0 + q_z)(P_0 + P_z)^{-1}$, or the Weinberg η, given in Sec. II, avoiding completely the notion of the vierbein $P\kappa l m$.

For the unequal-mass case we have instead of Eq. (6.4) the following expression:

$$q_\kappa = [\eta - \tfrac{1}{2} - \tfrac{1}{2}(m_1^2 - m_2^2) M_0^{-2}] M_0 ,$$

where

$$M_0 = [(q_\perp^2 + m_1^2)\eta^{-1} + (q_\perp^2 + m_2^2)(1 - \eta)^{-1}]^{1/2} ,$$

and to get the quadratic structure of the propagator we have to multiply $(P''^2 - P^2 - i\epsilon)^{-1}$ by the factor $[1 - (m_1^2 - m_2^2)^2 M_0''^{-2} M_0^{-2}]^{-1}$.

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APPENDIX A: NORMALIZATION OF AMPLITUDES

The fully on-shell matrix elements of the S-matrix operator can be written in the following ways:

$$\begin{aligned}
&\langle P'(P'^2 = M_0'^2) q'(q'P' = 0) | S | P(P^2 = M_0^2) q(qP = 0) \rangle \\
&= \delta^{(3)}(\vec{P}' - \vec{P}) [\delta^{(3)}(q'_{\kappa l m} - q_{\kappa l m}) - 2\pi i \delta(M_0' - M_0) \langle q'_{\kappa l m} | \mathcal{T} | q_{\kappa l m} \rangle] \\
&= \delta^{(3)}(\vec{P}' - \vec{P}) [\delta^{(3)}(q'_{\kappa l m} - q_{\kappa l m}) - 2\pi i \delta(\tfrac{1}{2}\mu^{-1}\bar{q}'^2 - \tfrac{1}{2}\mu^{-1}\bar{q}^2) \langle q'_{\kappa l m} | T | q_{\kappa l m} \rangle] \\
&= \delta^{(3)}(\vec{P}' - \vec{P}) \left\{ \delta^{(3)}(q'_{\kappa l m} - q_{\kappa l m}) - (2\pi)^4 i \delta(M_0' - M_0) \prod_{i=1}^{4} [(2\pi)^3 2 w_i]^{-1/2} \langle q'_{\kappa l m} | M | q_{\kappa l m} \rangle \right\} ,
\end{aligned}$$

where

$$w_{1,2} \equiv (\bar{q}^2 + m_{1,2}^2)^{1/2}, \quad w_{3,4} \equiv (\bar{q}'^2 + m_{1,2}^2)^{1/2}, \quad (\vec{P}) = (P_{\kappa l m}) = (0,0,0), \quad P_0 = P_p = M_0$$

$$\bar{q}^2 \equiv -q^2 |_{q \cdot P = 0} = q_\kappa^2 + q_l^2 + q_m^2 = \tfrac{1}{4} M_0^2 - \tfrac{1}{2}(m_1^2 + m_2^2) + \tfrac{1}{4} M_0^{-2} (m_1^2 - m_2^2)^2 .$$

It is useful to notice the following relations:

$$M_0 = (\bar{q}^2 + m_1^2)^{1/2} + (\bar{q}^2 + m_2^2)^{1/2},$$

$$\bar{q}^2 + m_{1,2}^2 = \tfrac{1}{4} M_0^2 [1 \pm (m_1^2 - m_2^2) M_0^{-2}]^2,$$

$$d\bar{q}^2 / dM_0 = \tfrac{1}{2} M_0 [1 - (m_1^2 - m_2^2)^2 M_0^{-4}],$$

$$\begin{aligned} t &= (q' - q)^2 |_{q' \cdot P = q \cdot P = 0} \\ &= -(q'_\kappa - q_\kappa)^2 - (q'_l - q_l)^2 - (q'_m - q_m)^2 . \end{aligned}$$

The invariant differential cross section is

$$\begin{aligned} \pi^{-1} k_s^2 d\sigma / dt &= (2\pi)^4 \mu^2 |\langle \mathcal{T} \rangle|_{\bar{q}'^2 = \bar{q}^2 = k_s^2}|^2 \\ &= \tfrac{1}{16} (2\pi)^{-2} s^{-1} |\langle M \rangle|^2 \\ &= \pi^4 s [1 - (m_1^2 - m_2^2)^2 s^{-2}]^2 |\langle \mathcal{T} \rangle|^2 . \end{aligned}$$

APPENDIX B: $m_1 \neq m_2$

The operator relation between T and $\mathcal{T}$, corresponding to Eq. (2.11), is

$$T=\tfrac{1}{8}\mu^{-1}[\mathcal{T}h_0+h_0\mathcal{T} - (m_1^2-m_2^2)h_0^{-2}(\mathcal{T}h_0+h_0\mathcal{T})h_0^{-2}],$$

where $\mu \equiv m_1m_2(m_1+m_2)^{-1}$, $h_0=(k^2+m_1^2)^{1/2}+(k^2+m_2^2)^{1/2}$, the Møller operators are

$$\Omega_\pm=\lim_{t\to\pm\infty}\exp(if(h)t)\exp(-if(h_0)t),$$

with

$$f(h)\equiv\tfrac{1}{8}\mu^{-1}[h^2-h^{-2}(m_1^2-m_2^2)^2],$$

and we used,

$$V=f(h)-f(h_0),\quad T=V\Omega_-,$$

$$f(h)\Omega_-=\Omega_-f(h_0),$$

$$(h^{-2}-h_0^{-2})\Omega_-=-h_0^{-2}(\mathcal{T}h_0+h_0\mathcal{T})h_0^{-2}.$$

We note that the function f is monotonic and increasing above $|m_1^2-m_2^2|^{1/2}$.

In the limit $|\vec{P}|\to\infty$ and $\eta, \vec{q}_\perp$ fixed, we get

$$\lim k_\parallel=\tfrac{1}{2}[(2\eta-1)(q_\perp^2+m_1^2)+\eta^2(m_2^2-m_1^2)] \times[q_\perp^2+m_1^2+\eta(m_2^2-m_1^2)]^{-1/2}\eta^{-1/2}(1-\eta)^{-1/2},$$

$$\lim\omega(k)=\lim[(k^2+m_1^2)^{1/2}+(k^2+m_2^2)^{1/2}] =[(q_\perp^2+m_1^2)\eta^{-1}+(q_\perp^2+m_2^2)(1-\eta)^{-1}]^{1/2}=M_0,$$

$$\lim d^3\vec{k}=\tfrac{1}{4}\eta^{-1}(1-\eta)^{-1}M_0 \times[1-(m_1^2-m_2^2)^2M_0^{-4}]d\eta d^2q_\perp.$$

In the covariant LFFT formulation, we have

$$\eta=(p_{1\kappa}+p_{1P})M_0^{-1} =\tfrac{1}{2}+q_\kappa M_0^{-1}+\tfrac{1}{2}(m_1^2-m_2^2)M_0^{-2},$$

$$d\eta=4\eta(1-\eta)M_0^{-1}[1-(m_1^2-m_2^2)^2M_0^{-4}]^{-1}dq_\kappa,$$

$$M_0''^2-s=[f_2(M_0'')-f_2(\sqrt{s})]8\mu \times[1-(m_1^2-m_2^2)^2M_0''^{-2}s^{-1}]^{-1},$$

$$f_2(M_0'')-f_2(\sqrt{s})=\tfrac{1}{2}\mu^{-1}(q_{\kappa''}''^2+q_l''^2+q_m''^2-k_s^2).$$

The fully off-shell matrix elements of V^W and I^W are connected as follows:

$$\langle q'_{\kappa' lm}|V^W|q''_{\kappa'' lm}\rangle=\tfrac{1}{4}(2\pi)^{-3}\mu^{-1}M_0'^{-1/2}[1-(m_1^2-m_2^2)^2s^{-1}M_0'^{-2}]^{1/2} \times[1-(m_1^2-m_2^2)^2M_0'^{-4}]^{-1/2}\langle q'_{\kappa' lm}|I^W|q''_{\kappa'' lm}\rangle M_0''^{-1/2}[1-(m_1^2-m_2^2)^2s^{-1}M_0''^{-2}]^{1/2} \times[1-(m_1^2-m_2^2)^2M_0''^{-4}]^{-1/2}.$$

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Note added in proof. For any process we have a timelike four-momentum P, and two spacelike four-momenta in the initial and final states q and q', from which we can explicitly construct the vierbein four-vectors $p^\mu = Ps^{-1/2}$, $\kappa^\mu = q^\mu(-q^2)^{-1/2}$, $m^\mu = N\epsilon^{\mu\nu\rho\sigma}P_\nu q_\rho q'_\sigma$ and $l^\mu = L\epsilon^{\mu\nu\rho\sigma}P_\nu m_\rho \eta_\sigma$, where N and L are the appropriate normalization factors, such that $l^2 = m^2 = -1$.

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