Structure of the energy tensor in the classical electrodynamics of point particles

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Classical electromagnetic theory provides an energy tensor defined off the particle's world line. The definition is extended to a distribution valid "everywhere." The extended definition is essentially unique. The Lorentz-Dirac equation follows immediately without the appearance of infinities at any stage. In the distribution theory formulation momentum integrals over spacelike planes exist and are finite. The planes are not restricted to be orthogonal to the particles' world lines, and consequently a finite, conserved momentum integral exists for a system of charged particles. "Self-momentum" (the "momentum" due to the strongest singularities in the energy tensor) is conserved differentially for each particle separately, and the associated integral over a spacelike plane is zero. It may therefore be omitted. This justifies and generalizes the *ad hoc* procedure of dropping self-energy terms in electrostatics.

I. INTRODUCTION

The basic difficulty in the electrodynamics of point particles is that the (stress-) energy (-momentum) tensor θ has nonintegrable singularities. Most deductions that one would want to make from the theory require integration over either four-dimensional regions through which the world lines pass or three-dimensional surfaces cut by the world lines. If one excludes a region around each world line (characterized by a length ϵ) then the integrals have terms in ϵ^{-1} which diverge in the limit $\epsilon \to 0$.

Thus we have infinite energy for a static system of charged particles, and no more than a prescription for removing self-energy terms. Thus Dirac's derivation[1] of the Lorentz-Dirac equation

$$ma = eF_{\text{ext}} \cdot v + \tfrac{2}{3}e^2(\dot{a} - a^2 v) \tag{1}$$

required the use of an infinite mass renormalization. Thus Teitelboim's concept[2] of a charged particle's bound electromagnetic four-momentum

$$P_{\text{bound}} = \frac{e^2}{2\epsilon}v - \frac{2}{3}e^2 a$$

contains a term corresponding to infinite self-momentum. The presence here of ϵ^{-1} might be regarded only as a matter of formalism if a finite remainder were well defined. But this has been accomplished only when the integral of the momentum density is taken over the particular spacelike plane whose normal is the particle's four-velocity v. This means that in general one has had no expression for the momentum of a system of charges. A plane normal to one world line would not be normal to the others. Tabensky[3] has however shown how the infinite mass renormalization procedure can be carried out for a special class of spacelike surfaces that cut each world line orthogonally.

As in 1938, one's interest in the self-energy problems of classical electrodynamics is not only for the sake of understanding that theory better, but also to gain insights which will be useful in quantum electrodynamics. This latter theory will not, however, be considered in the present paper.

Immediately after Dirac's derivation of (1), attempts were made by Pryce[4] and Bhabha and Harish-Chandra[5,6] to overcome the problem of infinite mass renormalization. They proposed modifications of the energy tensor to remove its strongest singularities without changing its divergence off the world line. It was suggested that in place of $\theta^{\mu\nu}$ one should use

$$\theta^{\mu\nu}_{\text{mod}} = \theta^{\mu\nu} - \partial_\alpha K^{\alpha\mu\nu}.$$

It was found that K could be chosen to be antisymmetric in its first two superscripts

$$K^{\alpha\mu\nu} = -K^{\mu\alpha\nu} \quad \text{(off the world line)}$$

and such that its divergence was symmetric in the last two superscripts (off the world line), and precisely canceled the nonintegrable singularities of θ. The antisymmetry that K possesses ensures that $\partial_\mu\partial_\alpha K^{\alpha\mu\nu} = 0$ (off the world line). Thus experimental effects off the world line are not changed. But the physical picture that the theory provides us with, the picture of a momentum flow described by the tensor θ, is changed radically. The Larmor radiation may not be given by θ_{mod} (this depends on K). A very small spherical charge e would have a completely different energy tensor from a point charge e. Consequently the theory did not gain universal acceptance. In order to keep a continuity of physical description, we must not change θ off the world line.

Recently van Weert[7] and Villarroel[8] have revived interest in the third-rank tensor K, and used it to derive Teitelboim's bound four-momentum. For van Weert the divergence of K is a convenient form in which to express the strongest singularities in θ, which tensor itself is not changed. Consequently, in his calculation of the integral of θ over the spacelike plane with normal v (as for Teitelboim), the divergent self-energy term still appears. In the present paper a development of van Weert's use of K will be made which eliminates self-energy problems.

The tensor K is not as singular as θ. Part of it (K_2) is integrable, and the remainder (K_1) can be written as the derivative of an integrable function. This circumstance invites a distribution theory interpretation of the derivatives. If this is done the distributions $\partial \cdot K_1$ and $\partial \cdot K_2$ produce a surprise: They are not symmetric on the world line (although they are symmetric off it). In order to regain the symmetry of θ, simple symmetrizing distributions, concentrated on the world line, must be added to $\partial \cdot K_1$ and $\partial \cdot K_2$.

This procedure leads to a decomposition of the part of the energy tensor determined solely by a particle's retarded field,

$$\theta_{\text{ret}} = \theta_1 + \theta_2 + \theta_3 .$$

The decomposition has remarkable properties.

(1) Off the world line θ_{ret} reduces to the usual expression.

(2) Each term in the decomposition has, separately, a vanishing divergence off the world line, so that the momentum flow splits into three completely independent components. Only on the world line is there the possibility of momentum exchange.

(3) θ_3 describes the flow of radiation. It has a finite flux in asymptotic regions and accounts for the Larmor radiation. Its integral over an arbitrary spacelike plane equals the total radiated momentum up to the point where the world line cuts the plane.

(4) θ_2 is the Schott tensor or acceleration-energy tensor. Its divergence is nonzero on the world line. Its integral over an *arbitrary* spacelike plane equals the *finite* part of Teitelboim's bound momentum. This is one of the crucial results of the present paper.

(5) θ_1 is strictly conserved, everywhere, even on the world line. The flow of "momentum" described by θ_1 is totally independent; it never takes part in exchanges with the rest of the system. θ_1 contains the strongest singularities of θ_{ret} and for the case of a freely moving particle with a straight world line the two tensors are identical. Therefore θ_1 may be called the self-energy tensor. The integral of θ_1 over a spacelike plane is zero.

The fact that $\partial \cdot \theta_1 = 0$ means that the "self-momentum" is decoupled from the rest of the system. We might picture this momentum as flowing along with the particle, under, so to speak, its own steam, never being added to, never being subtracted from, never requiring a force to carry it through whatever accelerations the particle suffers. It behaves like nothing physical, but rather like a purely mathematical structure having a free ride. This suggests that θ_1 should be dropped altogether from the energy tensor θ_{ret} and

$$\theta'_{\text{ret}} = \theta_{\text{ret}} - \theta_1 = \theta_2 + \theta_3$$

should be used instead. Because the divergence of θ'_{ret} is the same as that of θ_{ret}, and their integrals over spacelike planes are the same, there would be no change in particle electrodynamics. The only objection is that the values $\theta_{\text{ret}}(x)$ and $\theta'_{\text{ret}}(x)$ differ off the world line. These values are part of most interpretations of classical continuous electromagnetism, and also of gravitational theory, but it is unclear to what extent they are actually fundamental.

Whether θ_1 is dropped or not, the total energy of a charged particle at rest is simply its rest mass. There is never any question of distinguishing between an infinite "bare" mass and a finite experimental mass. This is another consequence of the vanishing of the integral of θ_1, the definition of which requires distribution theory.

(6) The divergence of θ_{ret} (and also of θ'_{ret}),

$$\partial \cdot \theta_{\text{ret}} = \partial \cdot \theta'_{\text{ret}} = \int d\tau\, \tfrac{2}{3}(a^2 v - \dot{a})\delta(x-z) ,$$

equals the negative of the reactive force density on the particle, and the Lorentz-Dirac equation is a consequence. No infinities appear, and since momentum conservation is invoked in differential form, no restrictions on the world line in the distant past are needed.

(7) The definition of θ_{ret} as the distribution theory extension of the usual form of θ_{ret} off the world line is essentially unique. Any change in the definition, of the form

$$e^2 \int d\tau \;\; (\text{symmetric tensor}) \;\; \delta(x-z) ,$$

would lead to normal dependent momentum integrals for the new θ_{ret}. This would be physically unacceptable since it would entail abandoning the usual concept of total momentum and abandoning differential momentum conservation in the form $\partial \cdot T = 0$.

(8) The decomposition of θ_{ret} into its three terms *requires* distribution theory. Although θ_1 and θ_2 are ordinary functions off the world line, they

cannot be defined solely in terms of their values there (as θ_3 can be). Distribution theory is needed to achieve the *physical* decomposition. Distribution theory appears to play an even more essential and important role in displaying the structure of the energy tensor θ than it does in providing the solution of Maxwell's equations with a δ-function source.

The present distribution definition of θ_{ret} is an alternative to one given earlier.[9] The previous definition was awkward to work with, but the present one is extraordinarily simple. When θ_2 is expressed in terms of K_2, not only is $\partial_\mu\partial_\alpha K_2^{\alpha\mu\nu}=0$ true as a distribution equation, but K_2 makes no contribution to integrals over spacelike planes. So the use of the third-rank tensor leads to technical efficiency. One is effectively working with simple distributions concentrated on the world line.

The plan of this paper is as follows. After notation is established in Sec. II, electrodynamic relations valid off the world line are dealt with in Sec. III. The distribution extension of θ_{ret} is made in Sec. IV, and the divergences calculated there are used to derive the Lorentz-Dirac equation in Sec. V. Momentum integrals are discussed in Sec. VI. The theory is reduced to electrostatics in Sec. VII. Conclusions are drawn in Sec. VIII where also the uniqueness of the definition of θ_{ret} is considered. A reformulation of the theory is discussed in Sec. IX.

II. NOTATION

We take $c=1$ and use a metric tensor g with diagonal components $(-1,+1,+1,+1)$. A general point in space-time is denoted by x with $x^0=t$, the particle's world line by $z(\tau)$, a function of the proper time τ, $v\equiv\dot{z}$, $a\equiv\dot{v}$, $v^2=-1, v\cdot a=0$. Tensors of the form $[a,v]\equiv av-va$ and $(a,v)\equiv av+va$ have contravariant components $a^\mu v^\nu \mp v^\mu a^\nu$. The electromagnetic conventions are that $F^{0k}=E_k$, $F^{ij}=\epsilon^{ijk}B_k$, and that the energy tensor has components

$$\theta^{\mu\nu}=\frac{1}{4\pi}(-F^\mu{}_\alpha F^{\alpha\nu}+\tfrac{1}{4}g^{\mu\nu}F^{\beta\alpha}F_{\alpha\beta}) \tag{2}$$

so $+\theta^{00}$ is the energy density. Consequently the energy-momentum content in a spacelike plane σ with future-pointing unit normal vector n ($n^2=-1$, $n^0>0$) is written (if it exists)

$$P=-\int_\sigma n\cdot\theta dV\,, \tag{3}$$

where dV is the invariant differential volume element in the plane.

The backward (opening into the past) light cone, whose vertex is a general point x, is intersected by the particle's world line at a unique point $z(\tau)$. The vector $R\equiv x-z(\tau)$ is lightlike and future pointing: $R^2=0$, $R^0\geq 0$. The distance between x and $z(\tau)$ in the retarded rest frame has the invariant definition $\rho\equiv -v(\tau)\cdot R$, and the unit spacelike vector pointing from $\vec{z}$ to $\vec{x}$ in the rest frame has also a coordinate-independent definition

$$u\equiv -v+R/\rho \quad (u^2=1, u\cdot v=0, u\cdot R=\rho)\,.$$

In terms of a definite world line, the one the particle actually follows, the constructions given above define R and τ throughout space-time. They have become functions of x and their derivatives may be calculated by differentiating the defining relations

$$\partial(R^2)=2(\partial R)\cdot R=0\,,$$

$$\partial R=\partial x-(\partial\tau)\dot{z}=g-(\partial\tau)v\,.$$

These give, using $\rho=-v\cdot R$ and $g\cdot R=R$,

$$\partial\tau=-R/\rho,\quad \partial R=g+Rv/\rho\,. \tag{4}$$

It is convenient to use the dyadic notation but care must be taken to maintain the proper order of factors. The covariant components of ∂ are $\partial_\mu=\partial/\partial x^\mu$ as has been used tacitly in $\partial x=g$, whose component forms are variously $\partial_\mu x^\nu=g_\mu^\nu$, $\partial^\mu x^\nu=g^{\mu\nu}$, etc. It turns out that all derivatives in electrodynamics can be calculated in terms of the two contained in (4). One immediate consequence is particularly useful:

$$\begin{aligned}\partial\rho&=u+a\cdot uR\\&=R/\rho-v+(a\cdot R)R/\rho\,.\end{aligned} \tag{5}$$

It follows that $R\cdot\partial\rho=\rho$.

The notation outlined here [except for the signs in (2) and (3)], together with the basic formulas of Sec. III, is described in Rohrlich's book.[10]

III. ELECTROMAGNETIC FIELD TENSOR AND THE ENERGY TENSOR

Maxwell's equations with a point particle source of charge e following the world line $z(\tau)$ are

$$\partial\cdot F=-4\pi j,\quad \partial\cdot F^*=0\,, \tag{6}$$

where

$$j(x)=e\int d\tau\, v(\tau)\delta(x-z(\tau))\,. \tag{7}$$

The sources of external fields are supposed to be beyond the region of immediate interest and are not included in (6). But their effects are included in the decomposition of the field tensor

$$F=F_{\text{ext}}+F_{\text{ret}}\,, \tag{8}$$

where F_{ret} is the retarded solution of (6) and F_{ext} satisfies the homogeneous equations. Corresponding to (8), the electromagnetic energy tensor θ decomposes into three parts:

$$\theta = \theta_{ext} + \theta_{mix} + \theta_{ret} \,. \tag{9}$$

Outside the external sources, we get from (2) and (6),

$$\partial \cdot \theta_{ext} = 0, \quad \partial \cdot \theta_{mix} = -F_{ext} \cdot j \,. \tag{10}$$

The retarded solution of (6) can be expressed in terms of the Liênard-Wiechert potential $A(x) = ev(\tau)/\rho$ $(\partial \cdot A = 0)$:

$$F_{ret}(x) = [\partial, A] = \frac{e}{\rho^2}\left[v, \frac{R}{\rho}\right] + \frac{e}{\rho}\left[a + \frac{a\cdot R}{\rho} v, \frac{R}{\rho}\right]. \tag{11}$$

Because $R = \rho(u+v)$ and both u and v are unit vectors, the relative size of the two terms in (11) is largely determined by the powers of ρ outside the brackets. We call F_{vel} the part that dominates for small ρ, and F_{rad} (which vanishes for $a = 0$) the part that dominates for large ρ:

$$F_{ret} = F_{vel} + F_{rad} \,. \tag{12}$$

As one can easily check using (4),

$$F_{vel} = \left[\partial, \frac{eR}{\rho^2}\right], \quad F_{rad} = \left[\partial, \frac{-eu}{\rho}\right]. \tag{13}$$

The potentials $A_{vel} = eR/\rho^2$, $A_{rad} = -eu/\rho$ are not divergence-free, though their sum A is.

The part of the energy tensor (9) due to F_{ret} alone may be decomposed according to the degree of singularity of its terms at $\rho = 0$:

$$\theta_{ret} = \theta_I + \theta_{II} + \theta_{III} \quad (\rho \neq 0)\,, \tag{14}$$

where

$$\theta_I = \frac{e^2}{4\pi}\left[\tfrac{1}{2}g + \frac{(v,R)}{\rho} - \frac{RR}{\rho^2}\right]\frac{1}{\rho^4} \quad (\rho \neq 0)\,, \tag{15}$$

$$\theta_{II} = \frac{e^2}{4\pi}\left[\frac{(a,R)}{\rho} - \frac{2a\cdot RRR}{\rho^3} + \frac{a\cdot R(v,R)}{\rho^2}\right]\frac{1}{\rho^3} \quad (\rho \neq 0)\,, \tag{16}$$

$$\theta_{III} = \frac{e^2}{4\pi}\left[a^2 - \frac{(a\cdot R)^2}{\rho^2}\right]\frac{RR}{\rho^4}\,. \tag{17}$$

Notice that each term is symmetric. A distinction has been made between θ_{III} on the one hand, and θ_I, θ_{II} on the other. θ_{III}, like F_{ret}, although singular, is not too singular to be integrated over. It can be used to form a regular functional by integrating it with a smooth test function ϕ with compact support

$$(\theta_{III}, \phi) \equiv \int d^4x\, \theta_{III}\phi \,. \tag{18}$$

The same is not true of θ_I and θ_{II}; they must be considered to be incompletely defined by (15) and (16), hence the insistence on $\rho \neq 0$.

The term θ_{III} is distinguished in a second respect: It has a clear physical significance. It comes entirely from F_{rad} and it accounts for the whole of the particle's asymptotic radiation. However, neither θ_I nor θ_{II} appears to have separate physical significance. Teitelboim's concept[2] of bound momentum comes from the sum θ_{I+II}. The expression that Harish-Chandra[6] and van Weert[7] propose provides a different decomposition (the formula can be checked by evaluating the derivatives):

$$\theta^{\mu\nu}_{I+II} = \partial_\alpha K^{\alpha\mu\nu} = \partial_\alpha K_1^{\alpha\mu\nu} + \partial_\alpha K_2^{\alpha\mu\nu} \quad (\rho \neq 0)\,, \tag{19}$$

where

$$K_1^{\alpha\mu\nu} = \frac{e^2}{4\pi}\left\{\frac{3}{4}\frac{[v,R]^{\alpha\mu}R^\nu}{\rho^5} + \frac{R^\mu g^{\alpha\nu} - R^\alpha g^{\mu\nu}}{4\rho^4}\right\} \quad (\rho \neq 0) \tag{20}$$

and

$$K_2^{\alpha\mu\nu} = \frac{e^2}{4\pi}\left\{\frac{[a,R]^{\alpha\mu}R^\nu}{\rho^4} - \frac{(a\cdot R)[R,v]^{\alpha\mu}R^\nu}{\rho^5}\right\}. \tag{21}$$

Notice that each term is antisymmetric in α and μ. The tensor K has been separated into two pieces K_1 and K_2 with singularities ρ^{-3} and ρ^{-2}, respectively. We find that

$$\theta_1 \equiv \partial \cdot K_1 = \theta_I + \frac{e^2}{4\pi}\frac{2a\cdot RRR}{\rho^6} \quad (\rho \neq 0) \tag{22}$$

and

$$\theta_2 \equiv \partial \cdot K_2 = \theta_{II} - \frac{e^2}{4\pi}\frac{2a\cdot RRR}{\rho^6} \quad (\rho \neq 0)\,. \tag{23}$$

Notice that both θ_1 and θ_2 are symmetric for $\rho \neq 0$.

We now have a decomposition of θ_{ret}, valid off the world line (we write $\theta_3 \equiv \theta_{III}$ henceforth)

$$\theta_{ret} = \theta_1 + \theta_2 + \theta_3 \quad (\rho \neq 0)\,, \tag{24}$$

in which each term is divergence-free: $\partial \cdot \theta_1 = \partial \cdot \theta_2 = 0$ for $\rho \neq 0$ because of the antisymmetry of K_1 and K_2; $\partial \cdot \theta_3 = 0$ for $\rho \neq 0$ by a short calculation using (4) and (5). That is, off the world line θ_{ret} is separated into three independent components, three independent flows of electromagnetic energy-momentum which have no exchanges with each other. The tensor θ_3 describes asymptotic radiation; θ_1 is a generalization of the energy tensor for the case of zero acceleration (when $a = 0$, $\theta_{ret} = \theta_I = \theta_1$) which might be called the self-energy part; we will find that the Schott term, describing "acceleration energy," comes from θ_2.

The singularity in K_2 is ρ^{-2}, which is integrable. The singularity in K_1 is not integrable but one can differentiate $(\rho \neq 0)$ to show that

$$K_1^{\alpha\mu\nu}=\frac{e^2}{16\pi}\left[\partial^\alpha\left(\frac{R^\mu R^\nu}{\rho^4}\right)-\partial^\mu\left(\frac{R^\alpha R^\nu}{\rho^4}\right)\right], \tag{25}$$

in which form K_1 is represented as the derivative of an integrable function.

IV. DISTRIBUTION DEFINITION OF THE ENERGY TENSOR

In this section the definitions of θ_1 and θ_2, partially given by (22) and (23), will be completed, and their divergences calculated. All that will be required from the theory of distributions is contained in the first 40 pages of Ref. 11. To illustrate the method the divergence of θ_3, although the result is known, will be treated first.

Equation (18) is the definition of $\theta_3=\theta_{\rm III}$ regarded as a distribution. It is a regular linear functional on the space of infinitely differentiable test functions ϕ which vanish outside a bounded region. By definition the derivative is given by

$$(\partial_\mu\theta_3^{\mu\nu},\phi)\equiv-(\theta_3^{\mu\nu},\partial_\mu\phi)=-\int d^4x\,\theta_3^{\mu\nu}\partial_\mu\phi(x). \tag{26}$$

The integral appearing here is convergent and so it may be written as the limit of the integral with a small region around the world line excluded $[\vartheta(\xi)=1,\ \xi>0;\ \vartheta(\xi)=0,\ \xi<0]$:

$$(\partial_\mu\theta_3^{\mu\nu},\phi)=-\lim_{\epsilon\to0}\int d^4x\vartheta(\rho-\epsilon)\theta_3^{\mu\nu}\partial_\mu\phi. \tag{27}$$

Integrating by parts and using $\partial\cdot\theta_3=0$ $(\rho\neq0)$ gives

$$(\partial_\mu\theta_3^{\mu\nu},\phi)=+\lim_{\epsilon\to0}\int d^4x\,\delta(\rho-\epsilon)(\partial_\mu\rho)\theta_3^{\mu\nu}\phi. \tag{28}$$

The integrated terms do not appear because ϕ has compact support.

To go further we must change variables in the integral (28). From Refs. 9 and 12 we have

$$\int d^4x f(x)=\int d\tau\int\rho^2d\rho\,d\Omega f(z+R), \tag{29}$$

where the integral on the right is ordered and the integration over angles Ω must be done in the rest system at the retarded time τ. Using (5) and the explicit expression (17) for θ_3 we now find

$$(\partial_\mu\theta_3^{\mu\nu},\phi)=+\lim_{\epsilon\to0}\int d\tau\,\rho^2d\rho\,d\Omega\,\delta(\rho-\epsilon)\frac{e^2}{4\pi}\times[a^2-(a\cdot u)^2]\frac{(u+v)^\nu}{\rho^2}\phi.$$

With the help of the integrals

$$\int\frac{d\Omega}{4\pi}u=\int\frac{d\Omega}{4\pi}uuu=0,\qquad \int\frac{d\Omega}{4\pi}uu=\tfrac{1}{3}(g+vv), \tag{30}$$

which may be evaluated in the rest frame and then expressed in a general frame, we get

$$(\partial\cdot\theta_3,\phi)=\int d\tau\tfrac{2}{3}e^2a^2v(\tau)\phi(z(\tau)). \tag{31}$$

We may express (31) in a slightly more formal form corresponding to the expression (7) for the particle current

$$\partial\cdot\theta_3(x)=\int d\tau\tfrac{2}{3}e^2a^2v\delta(x-z(\tau)). \tag{32}$$

In order to get precise distribution theory definitions of θ_1 and θ_2 we try to implement the idea that they can be written as distribution derivatives of integrable functions. That is, we test whether the formula (23), $\theta_2=\partial\cdot K_2$ $(\rho\neq0)$ with integrable K_2, can be extended to a distribution theory formula. Similarly we test $\theta_1=\partial\cdot K_1$ $(\rho\neq0)$ with K_1 given as the derivative of an integrable function by (25), to see whether it can be extended. We find in both cases that these definitions are unsuitable because they are not symmetric on the world line. But they may be simply symmetrized without changing their values off the world line.

Consider first $\partial\cdot K_2$ in which the derivative is interpreted in a distribution theory sense. We have, because K_2 is integrable,

$$\begin{aligned}(\partial_\alpha K_2^{\alpha\mu\nu},\phi)&\equiv-(K_2^{\alpha\mu\nu},\partial_\alpha\phi)\\&=-\int d^4x K_2^{\alpha\mu\nu}\partial_\alpha\phi\\&=-\lim_{\epsilon\to0}\int d^4x\,\vartheta(\rho-\epsilon)K_2^{\alpha\mu\nu}\partial_\alpha\phi.\end{aligned}$$

From now on the limits $\epsilon\to0$ will not be written explicitly but will simply be understood. Integrating by parts,

$$(\partial_\alpha K_2^{\alpha\mu\nu},\phi)=\int d^4x\,\vartheta(\rho-\epsilon)(\partial_\alpha K_2^{\alpha\mu\nu})\phi+\int d^4x\,\delta(\rho-\epsilon)(\partial_\alpha\rho)K_2^{\alpha\mu\nu}\phi. \tag{33}$$

The first term is symmetric in μ and ν in view of Eq. (23) since only the region $\rho>0$ is involved, but the second term is not:

$$\int d^4x\,\delta(\rho-\epsilon)\frac{e^2}{4\pi}(-a+a\cdot uu)^\mu(u+v)^\nu\phi(x)=-\int d\tau\tfrac{2}{3}e^2a^\mu v^\nu\phi(z). \tag{34}$$

The lack of symmetry occurs only on the world line—the distribution (34) depends only on the values of the test function ϕ at points $x=z$. For a test function ϕ which vanished in a neighborhood of the world line, only the first term in (33) would be present.

To form an everywhere symmetrical θ_2 we may add to $\partial\cdot K_2$ a symmetrizing term corresponding to (34). The final definition of θ_2 is then

$$(\theta_2^{\mu\nu}, \phi) \equiv (\partial_\alpha K_2^{\alpha\mu\nu}, \phi) - \int d\tau \tfrac{2}{3} e^2 a^\nu v^\mu \phi(z) . \tag{35}$$

This agrees with (23) for $\rho \neq 0$, that is, for test functions such that $\phi(z) = 0$. Although (35) is symmetric in μ and ν, its form makes it much simpler to calculate $\partial_\mu \theta_2^{\mu\nu}$ than $\partial_\nu \theta_2^{\mu\nu}$ because in the former case we can use the antisymmetry of $K_2^{\alpha\mu\nu}$ in α and μ:

$$\begin{aligned}(\partial_\mu \theta_2^{\mu\nu}, \phi) &= (\partial_\mu \partial_\alpha K_2^{\alpha\mu\nu}, \phi) + \int d\tau \tfrac{2}{3} e^2 a^\nu v_\mu \partial^\mu \phi(z) \\ &= (K_2^{\alpha\mu\nu}, \partial_\mu \partial_\alpha \phi) + \int d\tau \tfrac{2}{3} e^2 a^\nu \frac{d\phi}{d\tau} \\ &= -\int d\tau \tfrac{2}{3} e^2 \dot{a}^\nu \phi(z) . \end{aligned} \tag{36}$$

Therefore

$$\partial \circ \theta_2(x) = -\int d\tau \tfrac{2}{3} e^2 \dot{a}\, \delta(x - z) . \tag{37}$$

The calculation with K_1 is similar. Using (25) we interpret the right-hand side of

$$\partial_\alpha K_1^{\alpha\mu\nu} = \frac{e^2}{16\pi}\left(\partial^2 \frac{R^\mu R^\nu}{\rho^4} - \partial_\alpha \partial^\mu \frac{R^\alpha R^\nu}{\rho^4}\right) \tag{38}$$

as distribution theory derivatives of integrable functions. That is,

$$(\partial_\alpha K_1^{\alpha\mu\nu}, \phi) \equiv \frac{e^2}{16\pi} \int d^4x \frac{R^\mu R^\nu}{\rho^4} \partial^2 \phi - \frac{e^2}{16\pi} \int d^4x \frac{R^\alpha R^\nu}{\rho^4} \partial_\alpha \partial^\mu \phi . \tag{39}$$

The first term in (39) is symmetric but the second is not. Introducing $\vartheta(\rho - \epsilon)$ into the second integral, integrating by parts, and using $\partial_\alpha (R^\alpha R^\nu / \rho^4) = 0$ ($\rho \neq 0$) we find

$$\int d^4x \frac{e^2}{16\pi} \delta(\rho - \epsilon) \frac{(u+v)^\nu}{\rho^2} \partial^\mu \phi(x) = \frac{e^2}{4} \int d\tau\, v^\nu \partial^\mu \phi(z) . \tag{40}$$

The final form of θ_1 is obtained by adding to $\partial \circ K_1$ the symmetrizing term corresponding to (40):

$$(\theta_1^{\mu\nu}, \phi) \equiv (\partial_\alpha K_1^{\alpha\mu\nu}, \phi) + \frac{e^2}{4} \int d\tau\, v^\mu \partial^\nu \phi(z) . \tag{41}$$

The divergence of θ_1 is given by

$$(\partial_\mu \theta_1^{\mu\nu}, \phi) = -(\partial_\alpha K_1^{\alpha\mu\nu}, \partial_\mu \phi) - \frac{e^2}{4} \int d\tau\, v^\mu \partial_\mu \partial^\nu \phi .$$

The first term vanishes by using (39) with the substitution $\phi \to \partial_\mu \phi$. The second term vanishes because ϕ has compact support

$$(\partial_\mu \theta_1^{\mu\nu}, \phi) = -\frac{e^2}{4} \int d\tau \frac{d}{d\tau} \partial^\nu \phi(z) = 0 . \tag{42}$$

Therefore,

$$\partial \cdot \theta_1 = 0 . \tag{43}$$

To sum up, we now have precise distribution definitions for the decomposition

$$\theta_{\text{ret}} = \theta_1 + \theta_2 + \theta_3 , \tag{44}$$

in which θ_1 is defined by (41) and its divergence is zero, θ_2 is defined by (35) and its divergence is $-\int d\tau \frac{2}{3} e^2 \dot{a} \delta(x - z)$, θ_3 is defined by (18) and its divergence is $+\int d\tau \frac{2}{3} e^2 a^2 v \delta(x - z)$. Furthermore, the definitions of θ_1, θ_2, and θ_3 are essentially unique, as will be discussed in Sec. VIII.

One can show, although it requires some calculation, that according to the present definitions $\theta_1 + \theta_2$ equals $\theta_{\text{I+II}}$ of Ref. 9 [Eq. (65)].

V. LORENTZ-DIRAC EQUATION

The energy tensor for a particle of experimental mass m and world line $z(\tau)$ is

$$K_{\text{part}} \equiv \int d\tau\, mvv\, \delta(x - z(\tau)) , \tag{45}$$

and its divergence is

$$\begin{aligned}\partial \cdot K_{\text{part}} &= \int d\tau\, mvv \cdot \partial \delta(x - z) \\ &= -\int d\tau\, mv \frac{d}{d\tau} \delta(x - z) \\ &= \int d\tau\, ma \delta(x - z) . \end{aligned} \tag{46}$$

We adopt as fundamental the conservation of energy-momentum expressed in differential form by

$$\partial \cdot (K_{\text{part}} + \theta) = \partial \circ (K_{\text{part}} + \theta_{\text{ext}} + \theta_{\text{mix}} + \theta_{\text{ret}}) = 0 . \tag{47}$$

Using the previously calculated divergences we get

$$\int d\tau [ma - eF_{\text{ext}} \circ v - \tfrac{2}{3} e^2 (\dot{a} - a^2 v)] \delta(x - z) = 0 . \tag{48}$$

Therefore the Lorentz-Dirac equation follows:

$$ma = eF_{\text{ext}} \circ v + \tfrac{2}{3} e^2 (\dot{a} - a^2 v) . \tag{1}$$

In this formulation Eq. (1) is a local relation between v, a, $\dot{a}$ and the external field which does not depend on asymptotic properties of the system (such as the restriction to the case $a = 0$ for τ sufficiently negative).

The explicit deduction of (1) from (48) will illustrate a form of calculation to be used in Sec. VI. Equation (48) is a formal expression of

$$\int d\tau [ma - eF_{\text{ext}} \cdot v - \tfrac{2}{3} e^2 (\dot{a} - a^2 v)] \phi(z) = 0 . \tag{49}$$

Consider a spacelike plane with future-pointing unit normal n ($n^0>0$, $n^2=-1$) which intersects the world line at $z(\tau')$ for some particular τ'. For $\phi(x)$ choose

$$\phi(x)=\delta(-n\cdot[x-z(\tau')]) \tag{50}$$

(this is equivalent to taking a limit of differentiable ϕ's with compact support) so that ϕ is a δ function in the time variable of the coordinate system with origin $z(\tau')$ and time axis n. Changing the integration variable in (49) to

$$t\equiv -n\circ(z(\tau)-z(\tau')),\ \ \frac{dt}{d\tau}=-n\circ v(\tau)>0\,, \tag{51}$$

we get (1) (at proper time τ') since the n dependence $1/n\circ v(\tau')$ can be canceled out.

VI. MOMENTUM INTEGRALS

In this section momentum integrals, of the form

$$P=-\int_\sigma n\cdot\theta\, dV\,, \tag{3}$$

over a spacelike plane σ with future-pointing unit normal vector n, will be defined and calculated for the separate pieces θ_1, θ_2, and θ_3 of $\theta_{\rm ret}$. The results may then be applied to a system of charged particles interacting electromagnetically.

The theory developed in Sec. III was not taken far enough to include momentum integrals of the form (3). In fact the only integral over one of the θ's that has appeared so far was used in the definition (18) of the functional θ_3 in terms of the integrable function $\theta_3(x)$:

$$(\theta_3^{\mu\nu},\phi)\equiv\int d^4x\,\theta_3^{\mu\nu}(x)\phi(x)\,. \tag{18}$$

Since we now have definitions of (θ,ϕ) we may invert this procedure to provide definitions of the integrals over each of the θ's weighted with the test function ϕ:

$$\int d^4x\,\theta^{\mu\nu}\phi\equiv(\theta^{\mu\nu},\phi)\,. \tag{52}$$

For θ_1 or θ_2 this is a new definition of the left-hand side in terms of the right-hand side given by (41) or (35).

In order to extract the momentum integral (3) from (52) we must take the limit (if it exists) of (52) with a sequence of infinitely differentiable test functions ϕ, of compact support, which approach

$$\phi(x)\to -n_\mu\delta(-n\circ[x-z(\tau')])\,, \tag{53}$$

and then sum over μ. When θ is integrable in the classical sense, as θ_3 is, this process reduces (52) to the classical integral (3) over the plane σ, with normal n, which cuts the world line at $z(\tau')$. When θ is not classically integrable, as θ_1 and θ_2 are not, the process gives a new definition for (3).

The new definition of the integral (3) agrees with the old definition when the latter works. The new definition is also consistent with the natural generalization of Gauss's theorem to the present situation. If χ is the characteristic function for a four-dimensional region in space-time V_4 (one inside V_4, zero outside), then, provided we can take the limits of functionals as the test functions approach discontinuous forms on the boundary of V_4,

$$\begin{aligned}\int_{V_4} d^4x\,\partial_\mu\theta^{\mu\nu} &= \int d^4x\,(\partial_\mu\theta^{\mu\nu})\chi\\ &=(\partial_\mu\theta^{\mu\nu},\chi)\\ &=-(\theta^{\mu\nu},\partial_\mu\chi)\\ &=-\int d^4x\,\theta^{\mu\nu}\partial_\mu\chi\,.\end{aligned}$$

The surface integral which appears here on the right is the general form of which our construction of the momentum integral (3) is a special case.

We now consider the existence and calculation of the momentum integral P_3 over the piece θ_3. Because the radiative effects of the particle's acceleration in the past propagate along light cones, the form of θ_3,

$$\theta_3=\frac{e^2}{4\pi}\left[a^2-\frac{(a\cdot R)^2}{\rho^2}\right]\frac{RR}{\rho^4}\,, \tag{17}$$

shows that it will be nonzero on the plane σ and of order a^2r^{-2} [r is the spatial distance in the plane from the origin $z(\tau')$] unless there is a proper time in the past previous to which the acceleration is zero. To keep the calculation simple we suppose that $a=0$ previous to some instant in the past [this condition could be weakened, and appropriate changes in the argument made to arrive at the result (55) for world lines whose acceleration vanished sufficiently fast in the asymptotic past]. Equation (17) then shows that θ_3 will vanish outside a corresponding bounded region in the plane σ and the integral

$$P_3=-\int_\sigma n\cdot\theta_3 dV$$

will exist.

In order to calculate P_3 the simplest procedure is to use

$$(\partial_\mu\theta_3^{\mu\nu},\phi)=-(\theta_3^{\mu\nu},\partial_\mu\phi)$$

with a sequence of ϕ's which in finite regions approach

$$\phi(x) \to \vartheta(+n\cdot[x-z(\tau')]), \tag{54}$$

and which vanish in the distant past and in distant spatial directions where θ_3 itself is zero. This sequence of ϕ's approaches the characteristic function for a region of space-time bounded by the plane σ where θ_3 is not zero, and by a three-dimensional surface which encloses the region in the past of σ in which $\theta_3 \neq 0$. The limit function (54) is zero after the plane σ and one earlier than σ. Its gradient equals minus the limit (53)

$$-\partial_\mu \vartheta(+n\cdot[x-z(\tau')]) = -n_\mu \delta(-n\cdot[x-z(\tau')]).$$

Using the divergence of θ_3 given by (31), and (54),

$$(\partial\cdot\theta_3, \phi) \to \int_{-\infty}^{\infty} d\tau \tfrac{2}{3}e^2a^2 v\vartheta(+n\cdot[z(\tau)-z(\tau')])$$
$$= \int_{-\infty}^{\tau'} d\tau \tfrac{2}{3}e^2a^2 v.$$

On the other hand,

$$-(\theta_3, \cdot\partial\phi) \to -\int d^4x n\cdot\theta_3 \delta(-n\cdot[x-z(\tau')])$$
$$= -\int_\sigma dV n\cdot\theta_3 \equiv P_3.$$

Therefore

$$P_3(\tau') = \int_{-\infty}^{\tau'} d\tau \tfrac{2}{3}e^2a^2 v. \tag{55}$$

This integral, which was calculated by Teitelboim,[2] represents the sum of four-momentum radiated by the particle up to the proper time τ'. It is independent of the normal n to the plane σ. It depends on the whole history of the particle up to the point at which the particle cuts the plane. It satisfies

$$\frac{dP_3}{d\tau} = \tfrac{2}{3}e^2a^2 v. \tag{56}$$

We turn to the calculation of P_2. Provided that $a=0$ previous to some instant in the distant past, K_2, like θ_3, will vanish in distant spatial regions on the plane σ. We may therefore take a limit of (52) with a δ-convergent sequence of smooth functions δ_ϵ of the variable $t_n \equiv -n\circ[x-z(\tau')]$:

$$-n_\mu\delta_\epsilon(-n\circ[x-z(\tau')]) \to -n_\mu\delta(-n\circ[x-z(\tau')]). \tag{53$'$}$$

Functions of t_n alone cannot be of compact support since they depend only on the orthogonal distance from the plane σ, but we may take it that outside the region where $\theta_2 \neq 0$ the functions δ_ϵ depend on r as well and are smoothly extinguished. This will not affect (52) which is calculated from (35):

$$P_2^\nu(\tau') = (\theta_2^{\mu\nu}, -n_\mu\delta(-n\cdot[x-z(\tau')]))$$
$$= \int d^4x K_2^{\alpha\mu\nu} n_\mu \partial_\alpha \delta(-n\cdot[x-z(\tau')])$$
$$+ \int d\tau \tfrac{2}{3}e^2 a^\nu v^\mu n_\mu \delta(-n\circ[z(\tau)-z(\tau')]). \tag{57}$$

The first integral in (57) vanishes because $K_2^{\alpha\mu\nu} n_\mu n_\alpha = 0$, and the second, after the change of variable $t \equiv -n\cdot[z(\tau)-z(\tau')]$, gives $-\tfrac{2}{3}e^2a^\nu(\tau')$. We therefore have

$$P_2(\tau') = -\int_\sigma n\circ\theta_2 dV = -\tfrac{2}{3}e^2 a(\tau') \tag{58}$$

and

$$\frac{dP_2}{d\tau} = -\tfrac{2}{3}e^2\mathring{a}\ . \tag{59}$$

Like P_3, the momentum P_2 is independent of the normal to the plane. It is determined by the point of intersection with the world line. Unlike P_3, P_2 is a state function: It is a multiple of the local value of the acceleration at the point $z(\tau')$.

The result (58) is partly a refinement, partly a generalization of Teitelboim's bound momentum integral.[2] Teitelboim considered the specific plane with normal $n = v(\tau')$ passing through $z(\tau')$, not a general one; and he considered the combined effect of $\theta_1+\theta_2$ which produced an infinite self-mass because his integral was the "limit" of an ordinary integral with a hole cut out of the plane around the particle. It is the new definition of the integral, the limit of (52) with (53), that accounts for the improvement.

In a formal way we might calculate P_1 in the same manner that P_2 was calculated in (57). Using the definition (41) with (39) we would get

$$P_1^\nu(\tau') = (\theta_1^{\mu\nu}, -n_\mu\delta(-n\cdot[x-z(\tau')]))$$
$$= -\frac{e^2}{16\pi}\int d^4x \frac{R^\nu}{\rho^4}(n\cdot R\partial^2\delta - R\cdot\partial n\cdot\partial\delta)$$
$$-\frac{e^2}{4}\int d\tau\, v\circ n \partial^\nu \delta(-n\cdot[z(\tau)-z(\tau')]).$$

Because $\partial^2\delta = n^2\delta'' = -\delta''$, and $R\cdot\partial n\cdot\partial\delta = -R\cdot n\delta''$, the first integral vanishes. The proper-time integral is zero as is evident after the substitution $t = -n\cdot[z(\tau)-z(\tau')]$. The formal result is therefore

$$P_1 = 0. \tag{60}$$

The calculation leading to $P_1 = 0$ is only formal because the factor RR/ρ^4 in the integrand does not vanish in distant regions of the plane σ even when

$a=0$ in the distant past. The strongest singularities of $\theta_{\rm ret}$ persist for a free particle. Therefore the argument allowing the use of (53′) in the calculation of P_2 cannot be repeated for P_1. The result may be justified, however, if we use, in place of (53′), a sequence of functions cut off at large spatial distances:

$$-n_\mu \vartheta(E-r)\delta_\epsilon(-n\cdot[x-z(\tau')]) \to -n_\mu \vartheta(E-r)\,\delta(-n\cdot[x-z(\tau')]). \quad (53'')$$

As before r is the spatial polar distance from the origin of the coordinates with time axis n, and E is a large constant which will eventually tend to infinity.

The sequence (53″) can be used with (52) to give a definition of an integral over the bounded region $r \leq E$ in the plane σ. Letting $E\to\infty$ then serves as a definition of an "improper integral" if we copy the language of Riemann integral theory. Schwartz[13] gave a definition of an integral of a distribution with compact support. In (53′) a generalization of his definition which reduces the dimension of the domain of integration was implied. The sequence (53″) represents a further generalization which is necessary for distributions without compact support. The functions $\vartheta(E-r)$ are a shorthand for a limiting process using infinitely differentiable test functions which equal unity for $r\leq E$ and vanish just outside this region (see Ref. 11, p. 142).

If we use (53″) instead of (53′) to calculate P_1, we get

$$P_1(\tau') = -\frac{e^2}{16\pi}\int d^4x\,\frac{R}{\rho^4}\,[n\cdot R\partial^2(\vartheta\delta) - R\cdot\partial n\cdot\partial(\vartheta\delta)]$$

$$-\frac{e^2}{4}\int d\tau\, v\circ n\partial\delta(-n\circ[z(\tau)-z(\tau')]).$$

The proper-time integral vanishes again since (53′) and (53″) are the same near the world line. To analyze the remaining integral we calculate the derivatives in terms of r and $t_n = -n\cdot[x-z(\tau')]$. We have

$$\partial^2(\vartheta\delta) = \left(\nabla^2 - \frac{\partial^2}{\partial t_n^2}\right)\vartheta(E-r)\delta(t_n)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}r^2\delta(r-E)\delta(t_n) - \vartheta(E-r)\delta''(t_n)$$

and

$$R\cdot\partial n\cdot\partial(\vartheta\delta) = R\cdot\partial[\vartheta(E-r)\delta'(t_n)]$$

$$= -R\cdot e_r\delta(r-E)\delta'(t_n)$$

$$-R\cdot n\vartheta(E-r)\delta''(t_n),$$

in which e_r is the radial unit four-vector perpendicular to n and satisfying

$$x - z(\tau') = nt_n + e_r r.$$

The expression for P_1 reduces to

$$P_1(\tau') = -\frac{e^2}{16\pi}\int dV\,\delta(r-E)\left[\frac{\partial}{\partial r}\left(\frac{RR\cdot n}{\rho^4}\right) - \frac{\partial}{\partial t_n}\left(\frac{RR\cdot e_r}{\rho^4}\right)\right].$$

The derivatives here may be calculated using

$$e_r\cdot\partial = \frac{\partial}{\partial r}, \quad n\cdot\partial = \frac{\partial}{\partial t_n}$$

together with (4) and (5)

$$\partial R = g + \frac{Rv}{\rho}, \quad \partial\rho = u + a\cdot uR.$$

But because we are only interested in the limit $E\to\infty$ we can ignore terms of the form

$$\int dV\frac{1}{\rho^3}\delta(r-E) \sim \frac{1}{E}\to 0.$$

The remaining contributions, which do not tend to zero as E^{-1}, cancel:

$$-\frac{e^2}{16\pi}\int dV\,\delta(r-E)\left(-4\frac{RR\cdot n}{\rho^5}\,a\cdot u\,e_r\cdot R + 4\frac{RR\cdot e_r}{\rho^5}\,a\cdot u n\cdot R\right) = 0.$$

If we had supposed that $a=0$ in the distant past these terms would have vanished separately for sufficiently large E, but as we have just seen this assumption is not necessary and

$$P_1 = -\int_\sigma n\cdot\theta_1 dV = 0 \quad (60')$$

whether a vanishes in the distant past or not.

The fact that the integrals for P_1, P_2, and P_3 are defined and finite for any spacelike plane σ allows us to construct the total momentum integral for a closed system of several charged particles interacting electromagnetically. The total energy tensor for such a system is

$$T = \sum_A \int d\tau_A m_A v_A v_A \delta(x - z_A) + \theta, \quad (61)$$

where, schematically,

$$\theta = \left(\sum_A F_{\text{ret}(A)}\right)\left(\sum_B F_{\text{ret}(B)}\right)$$

$$= \sum_A \theta_{\text{ret}(A)} + \sum_{A<B}\theta_{\text{int}(AB)}. \quad (62)$$

The tensor T, which may now be written

$$T = \sum_A (K_{\text{part}(A)} + \theta_{\text{ret}(A)}) + \sum_{A<B}\theta_{\text{int}(AB)}, \quad (63)$$

is conserved differentially if each particle obeys a Lorentz-Dirac equation in the retarded fields of the others.

Provided that all the particles have straight world lines in the sufficiently distant past, we can calculate the total momentum associated with T:

$$P=-\int_\sigma n\cdot T\,dV$$
$$=\sum_A[m_A v_A(\tau'_A)-\tfrac{2}{3}e_A^2 a_A(\tau'_A)]$$
$$+\sum_A\int_{-\infty}^{\tau'_A}d\tau_A\tfrac{2}{3}e_A^2 a_A^2 v_A-\sum_{A<B}\int_\sigma dV\,n\cdot\theta_{\text{int}(AB)}\,. \tag{64}$$

In the expression (64), τ'_A is A's proper time at the point of intersection $z_A(\tau'_A)$ of A's world line and the plane σ. The total momentum P is finite and conserved, that is, it is unaltered if the plane is tilted or translated.

Since the divergence of θ_1 is zero and the momentum integral P_1 associated with it also vanishes, we get the same physical results by working with the truncated form of θ_{ret}

$$\theta'_{\text{ret}}=\theta_2+\theta_3\,, \tag{65}$$

and the analogous expression for the total energy tensor without self-energy terms

$$T'=\sum_A(K_{\text{part}(A)}+\theta'_{\text{ret}(A)})+\sum_{A<B}\theta_{\text{int}(AB)}\,. \tag{66}$$

The total momentum is the same whether one uses T or T':

$$P=-\int_\sigma n\cdot T\,dV=-\int_\sigma n\cdot T'\,dV;$$

and the condition $\partial\cdot T'=0$ requires that each particle satisfy a Lorentz-Dirac equation.

We return to the consideration of the energy tensor created by one particle. We can relate the momentum integrals calculated in (60′) and (58) to the integrals of θ_1 and θ_2 over the plane with a hole cut out around the particle.

Let us use the notation

$$\vartheta(\rho-\epsilon)\theta_2^{\mu\nu}=\vartheta(\rho-\epsilon)\partial_\alpha K_2^{\alpha\mu\nu}$$
$$=\vartheta(\rho-\epsilon)\left(\theta_{\text{II}}-\frac{e^2}{4\pi}\frac{2a\cdot RRR}{\rho^6}\right)^{\mu\nu} \tag{67}$$

to refer to the distribution defined by the first integral in Eq. (33). It is a generalization of the Cauchy principal value. It is noteworthy that the limit $\epsilon\to0$ (which is understood) exists. This may be checked explicitly, but it is already clear since the left-hand side of (33) is well defined, and the second integral on the right-hand side of (33) is shown to exist in (34). Using (33), (34), and (35),

$$\theta_2=\vartheta(\rho-\epsilon)\theta_2-\int d\tau\tfrac{2}{3}e^2(a,v)\delta(x-z)\,. \tag{68}$$

Equation (68) makes the symmetry of θ_2 manifest.

By evaluating (68) on the "test function" (53) we get the following relation:

$$P_2=-\int_\sigma dV\,n\cdot\theta_2$$
$$=-\int_\sigma dV\,\vartheta(\rho-\epsilon)n\cdot\theta_2$$
$$+\int d\tau\tfrac{2}{3}e^2 n\cdot(a,v)\delta(-n\cdot[z(\tau)-z(\tau')])$$
$$=-\int_\sigma dV\,\vartheta(\rho-\epsilon)n\cdot\theta_2-\tfrac{2}{3}e^2\frac{n\cdot av+n\cdot va}{n\cdot v}$$
$$=-\tfrac{2}{3}e^2 a\,. \tag{69}$$

Therefore

$$-\int_\sigma dV\,\vartheta(\rho-\epsilon)n\cdot\theta_2=\tfrac{2}{3}e^2\frac{n\cdot a}{n\cdot v}v\,. \tag{70}$$

This is the integral of θ_2 over the plane σ with the region $\rho<\epsilon$ omitted, in the limit $\epsilon\to0$. The integral exists for all normal directions n; it vanishes when the plane is orthogonal to the world line ($n\cdot a=0$ if $n=v$).

From Eqs. (41) and (39) one finds, after some calculation, a form for θ_1 which is similar to (68):

$$\theta_1=\vartheta(\rho-\epsilon)\theta_1-\frac{e^2}{2\epsilon}\int d\tau(\tfrac{1}{3}g+\tfrac{4}{3}vv)\delta(x-z)$$
$$+\int d\tau\tfrac{2}{3}e^2(a,v)\delta(x-z)\,. \tag{71}$$

In (71) the first two terms must be taken together before $\epsilon\to0$. Evaluating (71) on the "test function" (53)—the formal calculation gives the correct result if a vanishes in the distant past—one gets

$$P_1=0=-\int_\sigma dV\,n\cdot\theta_1$$
$$=-\int_\sigma dV\,\vartheta(\rho-\epsilon)n\cdot\theta_1+\frac{e^2}{2\epsilon}\int d\tau\left(\frac{n}{3}+\tfrac{4}{3}n\cdot vv\right)\delta(-n\cdot[z(\tau)-z(\tau')])-\int d\tau'\tfrac{2}{3}e^2 n\cdot(a,v)\delta(-n\cdot[z(\tau)-z(\tau')])$$
$$=-\int_\sigma dV\,\vartheta(\rho-\epsilon)n\cdot\theta_1-\frac{e^2}{2\epsilon}\frac{n+4n\cdot vv}{3n\cdot v}+\tfrac{2}{3}e^2\frac{n\cdot av+n\cdot va}{n\cdot v}\,.$$

If one suspends the understanding that the limit $\epsilon \to 0$ is to be taken, so that the first two terms on the right can be considered separately,

$$-\int_\sigma dV\,\vartheta(\rho-\epsilon)n\cdot\theta_1=\frac{e^2}{2\epsilon}\frac{n+4n\cdot vv}{3n\cdot v}-\tfrac{2}{3}e^2\left(a+\frac{n\cdot a}{n\cdot v}v\right), \qquad (72)$$

for finite ϵ but with the neglect of terms which vanish in the $\epsilon \to 0$ limit.

Adding (72) and (70),

$$-\int_\sigma dV\,\vartheta(\rho-\epsilon)n\cdot(\theta_1+\theta_2)=-\tfrac{2}{3}e^2 a+\frac{e^2}{2\epsilon}\frac{n+4n\cdot vv}{3n\cdot v}. \qquad (73)$$

If $n=v$ one gets Teitelboim's momentum integral

$$-\int_\sigma dV\,\vartheta(\rho-\epsilon)v\cdot(\theta_1+\theta_2)=-\tfrac{2}{3}e^2 a+\frac{e^2}{2\epsilon}v. \qquad (74)$$

VII. ELECTROSTATICS OF POINT PARTICLES

Before simplifying to the case of electrostatics, notice that the distribution structure of θ_1 may be written, using (38) and (41), in the form

$$\theta_1^{\mu\nu}=\frac{e^2}{16\pi}\left[\partial^2\left(\frac{R^\mu R^\nu}{\rho^4}\right)-\partial^\mu\partial_\alpha\left(\frac{R^\alpha R^\nu}{\rho^4}\right)\right]-\frac{e^2}{4}\int d\tau\, v^\mu\partial^\nu\delta(x-z). \qquad (75)$$

This form, in which θ_1 is written using distribution theory derivatives of integrable functions, makes manifest $\partial_\mu\theta_1^{\mu\nu}=0$. But θ_1 is symmetric, although this is not obvious in (75), so of course $\partial_\nu\theta_1^{\mu\nu}=0$ as well.

If the particle is at rest, we use the rest frame coordinates in which it is at the origin, and revert to three-vector notation: $z^0=\tau$, $\vec{z}=0$, $v^0=1$, $\vec{v}=0$, $\rho=r=R^0$, $\vec{R}=\vec{x}$. The components (75) of θ_1 are independent of time $x^0=t$. The component θ_1^{00} is

$$\theta_1^{00}=\frac{e^2}{16\pi}\nabla^2\frac{1}{r^2}. \qquad (76)$$

In a dyadic notation, the space-space components are

$$\theta_1^{ss}=\frac{e^2}{16\pi}\left[\nabla^2\left(\frac{\vec{x}\vec{x}}{r^4}\right)-\vec{\nabla}\vec{\nabla}\cdot\left(\frac{\vec{x}\vec{x}}{r^4}\right)\right]. \qquad (77)$$

Equation (77) is not obviously symmetric, but because $\vec{\nabla}\cdot(\vec{x}\vec{x}/r^4)=0$ $(r\neq 0)$, we have

$$\int dV\,\phi(\vec{x})\vec{\nabla}\cdot\left(\frac{\vec{x}\vec{x}}{r^4}\right)\equiv-\int dV\,\vec{\nabla}\phi\cdot\frac{\vec{x}\vec{x}}{r^4}$$
$$=-\int dV\,\vartheta(r-\epsilon)\vec{\nabla}\phi\cdot\frac{\vec{x}\vec{x}}{r^4}$$
$$=+\int dV\,\phi\,\delta(r-\epsilon)\frac{\vec{x}}{r^3}=0.$$

The second term in (77) is zero. At the same time we have shown that the (three) divergence of θ_1^{ss} vanishes:

$$\theta_1^{ss}=\frac{e^2}{16\pi}\nabla^2\frac{\vec{x}\vec{x}}{r^4}, \qquad (78)$$

$$\vec{\nabla}\cdot\theta_1^{ss}=0. \qquad (79)$$

The time-space components of θ_1 form a three-vector,

$$\vec{\theta}_1=\frac{e^2}{16\pi}\nabla^2\frac{\vec{x}}{r^3}-\frac{e^2}{4}\vec{\nabla}\delta(\vec{x}). \qquad (80)$$

The two possible expressions for $\vec{\theta}_1$ which arise from θ_1^{0k} and θ_1^{k0} are the same because $\vec{\nabla}\cdot(\vec{x}/r^3)=4\pi\delta(\vec{x})$. This relation also shows that

$$\vec{\nabla}\cdot\vec{\theta}_1=0. \qquad (81)$$

By performing the derivatives indicated in (76), (78), and (80) for $r\neq 0$ we see that ($\vec{1}$ = unit dyadic)

$$\theta_1^{00}=\frac{e^2}{8\pi}\frac{1}{r^4}=\frac{1}{8\pi}\vec{E}^2 \quad (r\neq 0), \qquad (82)$$

$$\theta_1^{ss}=\frac{e^2}{8\pi}\frac{\vec{1}}{r^4}-\frac{e^2}{4\pi}\frac{\vec{x}\vec{x}}{r^6} \quad (r\neq 0)$$
$$=\frac{\vec{1}}{8\pi}\vec{E}^2-\frac{1}{4\pi}\vec{E}\vec{E}, \quad (r\neq 0), \qquad (83)$$

$$\vec{\theta}_1=0 \quad (r\neq 0), \qquad (84)$$

where $\vec{E}=e\vec{x}/r^3$ $(r\neq 0)$ for a point charge. Therefore the energy density, Maxwell's stress tensor, and the Poynting vector are the same, where these can be defined $(r\neq 0)$ as the components that Eq. (75) provides. But Eq. (75) goes further, because it has the distribution theory consequences $\vec{\nabla}\cdot\theta_1^{ss}=\vec{\nabla}\cdot\vec{\theta}_1=0$ which say that there is no force on the charge, and there is no change in energy. These are exactly the statements that we want a theory of a single charged particle at rest to provide.

In the present, single-particle, static case, the energy tensor $\theta_{\rm ret}$ equals θ_1 so the truncated energy tensor $\theta'_{\rm ret}$ is, from (65), zero. $\theta'_{\rm ret}$ gives the same vanishing force density as (79) and (81). The total electromagnetic energy-momentum is zero whether calculated from $\theta_{\rm ret}$ or $\theta'_{\rm ret}$. The energy integral is evaluated by integrating by parts,

$$\int dV\, \theta'^{00}_{\text{ret}} = \int dV\, \theta^{00}_{1}$$

$$= \lim_{E\to\infty} \int dV \frac{e^2}{16\pi}\left(\nabla^2 \frac{1}{r^2}\right) \vartheta(E-r) = 0\,, \qquad (85)$$

with a similar equation for the momentum.

If we have two point charges at rest, A and B, the energy tensor has the form

$$\theta = \theta_{1(A)} + \theta_{(AB)} + \theta_{1(B)}\,, \qquad (86)$$

where $\theta_{1(A)}$ and $\theta_{1(B)}$ have components of the form (76), (78), and (80), and $\theta_{(AB)}$ is the cross term. Since $\partial\cdot\theta_{1(A)} = \partial\cdot\theta_{1(B)} = 0$, the force density is

$$-\partial\cdot\theta = -\partial\cdot\theta_{(AB)} = F_{(A)}\cdot j_{(B)} + F_{(B)}\cdot j_{(A)}\,, \qquad (87)$$

whose spatial components in our static case form the vector

$$\vec{\mathrm{E}}_{(A)}(\vec{\mathrm{x}})e_B\delta(\vec{\mathrm{x}}-\vec{\mathrm{z}}_B) + \vec{\mathrm{E}}_{(B)}(\vec{\mathrm{x}})e_A\delta(\vec{\mathrm{x}}-\vec{\mathrm{z}}_A)\,. \qquad (88)$$

Each charge acts on the other but there is no self-force on either.

The same force relations hold with the modified tensor

$$\theta' = \theta_{(AB)} \qquad (89)$$

from which the total energy and momentum integrals may be calculated. The only nonzero integral in the static case is the energy

$$\int dV \frac{1}{8\pi} 2\vec{\mathrm{E}}_{(A)}(\vec{\mathrm{x}})\cdot\vec{\mathrm{E}}_{(B)}(\vec{\mathrm{x}}) = \frac{e_A e_B}{|\vec{\mathrm{z}}_A - \vec{\mathrm{z}}_B|}\,. \qquad (90)$$

This is the result of the procedure in elementary electrostatics in which the self-energy terms are "dropped."

VIII. CONCLUSIONS

The energy tensor of continuous electrodynamics can be used in the particle case to define a function off the world line

$$\theta_{\text{ret}} = \frac{1}{4\pi}\left(-F_{\text{ret}}\cdot F_{\text{ret}} + \tfrac{1}{4}gF_{\text{ret}}\cdot\cdot F_{\text{ret}}\right) \quad (\rho\neq 0)\,. \qquad (14)$$

In Sec. IV a distribution

$$\theta_{\text{ret}} = \theta_1 + \theta_2 + \theta_3 \qquad (44)$$

was defined with the following properties:

(a) θ_{ret} (distribution) $= \theta_{\text{ret}}$ (function)$(\rho\neq 0)$. The local values off the world line are the same. Therefore calculations of Larmor radiation, for example, are the same. The energy tensor for a small but continuous charge distribution with the same total charge would nearly equal θ_{ret} off the world line.

(b) The self-energy tensor θ_1 is rigorously conserved: $\partial\cdot\theta_1 = 0$, whatever the acceleration of the charge. Applied to the static case of a single charge this solves the self-stress problem. Maxwell's stress tensor (83) is well defined except at the position of the particle, so although the associated force density $\vec{\mathrm{f}} = -\vec{\nabla}\cdot\theta_1^{ss} = 0$ for $r\neq 0$, it is undefined precisely where it is needed. However, with the distribution definition (78) for the stress tensor, $\vec{\nabla}\cdot\theta_1^{ss}$ is rigorously zero; the force is therefore zero and the theory is satisfactory.

According to the new definition of momentum integrals provided by (52) and (53″),

$$P_1 = -\int_\sigma n\cdot\theta_1 dV = 0\,. \qquad (60')$$

This provides a solution to the classical self-energy problem.

These results give a strong invitation to drop θ_1 from θ_{ret}. This, however, would involve abandoning the form (2) for θ in terms of the field tensor F, and property (a) would no longer be preserved.

(c) The acceleration energy tensor θ_2 satisfies

$$\partial\cdot\theta_2 = -\int d\tau \tfrac{2}{3}e^2\dot{a}\delta(x-z) \qquad (37)$$

and

$$P_2 = -\int_\sigma n\cdot\theta_2 dV = -\tfrac{2}{3}e^2 a\,. \qquad (58)$$

This is a refinement of Teitelboim's bound momentum. Teitelboim expressed both the self-energy momentum and the acceleration-energy momentum in a single formula. He was able to do this only for the plane with $n=v$. Equation (58) generalizes the acceleration part to arbitrary planes.

(d) The radiation tensor θ_3 satisfies

$$\partial\cdot\theta_3 = \int d\tau \tfrac{2}{3}e^2a^2v\delta(x-z) \qquad (32)$$

and

$$P_3 = \int_{-\infty}^{\tau'} d\tau \tfrac{2}{3}e^2a^2v\,. \qquad (55)$$

The tensor θ_3 describes the asymptotic radiation. The properties of the tensor have been described fully by Teitelboim.[2]

(e) The Lorentz-Dirac equation follows from differential energy-momentum conservation

$$\partial\cdot(K_{\text{part}} + \theta_{\text{ext}} + \theta_{\text{mix}} + \theta_{\text{ret}}) = 0\,. \qquad (47)$$

This is the solution to the self-stress problem for a moving particle.

(f) The integral for the total momentum of a system of charged particles interacting electromagnetically exists and is finite for any spacelike plane σ. It is given by (64) in terms of T, but can equally well be computed with T' of Eq. (66) in which the θ_1 contributions are omitted.

The question of the uniqueness of θ_{ret} must be considered. Is it possible to find a different definition of θ_{ret}, an alternative to (44), with acceptable properties? It will be recalled that when the definitions of θ_1 and θ_2 were given in Eqs. (41) and (35) by the addition of symmetrizing terms, concentrated on the world line, to $\partial \cdot K_1$ and $\partial \cdot K_2$, only the simplest possibilities were set down. Would it be possible therefore to find a different but still physically acceptable θ_{ret} by adding to (44) a new term, concentrated on the world line, of the form

$$e^2 \int d\tau(\text{symmetric tensor})\ \delta(x-z)?$$

We are not at the moment considering changes in θ_{ret} which would modify its functional form off the world line and which would therefore violate property (a).

The tensor θ_{ret} is a physical quantity with dimensions, and the dimension of (symmetric tensor) in the integrand of the proposed addition would have to be L^{-1}. We suppose that (symmetric tensor) is built up from the metric tensor $g(L^0)$ and the various geometrical vectors in the theory, which, except for the three simplest have $L^{<-1}$: $v(L^0)$, $a(L^{-1})$, $\partial(L^{-1})$, $\dot{a}(L^{-2})$, etc. If the point particle had spin or a multipole structure there would be other possibilities. If we remember that for a free particle, $a=\dot{a}=\ddot{a}=0$, etc. it is easy to see that the available scalars, $v^2=-1$, $v\cdot a=0$, $v\cdot\partial(L^{-1})$, $a^2(L^{-2})$, etc., cannot be used to increase the dimension of a term. There are no physical constants, available within classical electromagnetism, of dimension L. Since θ is purely electromagnetic, it cannot depend on the mass of the particle, so that in classical theory $e^2/m(L)$ is excluded (the argument would not hold in quantum electrodynamics). Neither τ nor z can be used since they depend on arbitrary origins. The remaining independent nonzero possibilities for (symmetric tensor) are numerical combinations of

$$(v,a) \text{ and } (v,\partial).$$

The two possible symmetric tensors that survive the dimensional analysis may be excluded if we insist on the existence of a normal-independent momentum integral over a plane, that is, if we insist on the existence of a momentum independent of n over the set of planes σ that pass through a fixed point $z(\tau')$ on the world line. Since $\partial\cdot\theta_{\text{ret}}=0$ off the world line, two integrals of the form

$$P=-\int_\sigma n\cdot\theta_{\text{ret}}dV$$

would be the same by Gauss's theorem if we have a suitable definition of θ_{ret}.

If we consider the case of an additional term with (v,a), the new integral over a plane would differ from the old by

$$e^2\int d^4x\int d\tau\ \delta(x-z)(n\cdot av+n\cdot va)\delta(-n\cdot[x-z(\tau')])$$
$$=e^2\int dt(n\cdot av+n\cdot va)\frac{\delta(t)}{-n\cdot v}$$
$$=-e^2\left(a+\frac{n\cdot a}{n\cdot v}v\right)_{\tau=\tau'}.$$

With the additional term the momentum would depend on the tilt of the plane. The same happens with (v,∂). Another way of putting the objection: If we had the new terms in θ_{ret}, Eq. (47) would not produce an equation of motion, not simply not the Lorentz-Dirac equation, but no equation at all. The analysis similar to that in (49), (50), and (51) would produce a plane-dependent relation inconsistent with any world line [case of (v,∂)], or any world line except that of a free particle [case of (v,a)]. Otherwise $\partial\cdot T=0$ could not be satisfied.

Therefore if we retain the form of θ_{ret} off the world line, stay within the classical electrodynamics of a spinless point particle, and insist on the existence of momentum, the distribution form of θ_{ret} is uniquely determined to be (44).

If we drop the requirement (a) there are other possibilities. Pryce's tensor[4]

$$\theta_{\text{Pryce}}=\frac{e^2}{4\pi}\left[a^2-9(a\cdot u)^2-\frac{2(\dot{a}\cdot R)}{\rho}\right]\frac{RR}{\rho^4}$$

satisfies

$$\partial\cdot\theta_{\text{Pryce}}=\int d\tau\tfrac{2}{3}e^2(a^2v-\dot{a})\delta(x-z)=\partial\cdot\theta_{\text{ret}}, \tag{91}$$

so that it yields the Lorentz-Dirac equation from (47). But as Villarroel[8] has noted, it does not produce the Larmor formula. This is not an insuperable objection. Equation (91) is sufficient to effect the same response in particles as the previous theory. The distinction is this: Suppose we had a black electromagnetic absorber in the asymptotic field of a radiating particle. In both theories the absorber would suffer the same momentum change. Do we want a theory in which the same momentum flow is present when the absorber is not there? Equation (44) provides such a theory and Pryce's does not. Pure, mechanistic electrodynamics cannot decide, but the physical picture in the former case is helpful and intuitive. The sucess of the idea of photons gives further support to the picture of an asymptotic electromagnetic field with "real" momentum.

IX. WIDER CONTEXT

In point particle electrodynamics, the closed theory of the electromagnetic interaction of charged particles, Maxwell's equations together with the causal boundary condition, determine the electromagnetic field tensor F in terms of the electric current j. The current is a function of the particle world lines, which in classical physics are supposed to be observable. The field tensor F is observable to the extent that it appears in the Lorentz-Dirac equation which restricts the particles' world lines. In this theory, the energy tensor T, given by (61) or (63), appears at an unobservable level. The condition

$$\partial \cdot T = \partial \cdot (K_{\text{parts}} + \theta) = 0$$

entails that the particle world lines obey Lorentz-Dirac equations. But T itself is not observable.

Despite some paradoxes, we are accustomed, from experience in continuous electromagnetism, to thinking of the components of $\theta(x)$ as "real" densities, $\theta^{0\mu}$ as a four-momentum density, θ^{ij} as a momentum flux. The notion certainly seems to be consistent and usable for plane waves and for the radiation tensor θ_3 of a single particle. But the idea cannot be confirmed within electrodynamics. The motion of a particle in a detection apparatus may be found to satisfy a Lorentz-Dirac equation, but it will provide no confirmation that, say, θ^{00} is an energy density.

Although the question is "metaphysical" from the point of view of pure electrodynamics, it is nonetheless of interest to consider to what extent the $\theta^{\mu\nu}(x)$ can be taken to be "real" densities. One of the senses in which "real" can be understood would require that the total momentum could be added up, bit by bit, as an experimental approximation to a Riemann integral

$$-\sum \theta(x) \cdot n\delta V \to -\int_\sigma \vartheta(\rho-\epsilon) n \cdot \theta dV .$$

The distribution theory definition of the integrals of θ_1 and θ_2 is not of this form. We have seen in Sec. VI. that

$$P_2 = -\int_\sigma n \cdot \theta_2 dV = -\tfrac{2}{3}e^2 a \neq -\int_\sigma n \cdot \theta_2 \vartheta(\rho-\epsilon) dV ,$$

$$P_1 = -\int_\sigma n \cdot \theta_1 dV = 0 \neq -\int_\sigma n \cdot \theta_1 \vartheta(\rho-\epsilon) dV .$$

With the theory in the form given in the previous sections we cannot therefore interpret $\theta^{\mu\nu}(x)$ as a measurable density of a physical quantity localized at the point x. To highlight this point consider the integral (85) for finite E. As a function, for $r \neq 0$, $e^2/8\pi r^4$ is positive-definite, but

$$\int dV \frac{e^2}{16\pi}\left(\nabla^2 \frac{1}{r^2}\right)\vartheta(E-r) = -\frac{e^2}{2E} \tag{92}$$

with the definition (52) and (53″).

We will consider how the theory might be changed so it includes genuine densities but leaves the verifiable results of electrodynamics unaltered.

Let us consider one situation where a change seems called for: the case of a single point charge at rest. Off the world line $\theta^{00} = \vec{E}^2/8\pi = e^2/8\pi r^4$. The fact that the integral

$$\int dV\, \vartheta(r-\epsilon)\theta^{00} = e^2/2\epsilon \tag{93}$$

diverges as $\epsilon \to 0$ seems consistent with the infinite amount of work that has to be done to compress an initially widely distributed charge to a point. However, one does not in practice make point charges this way, and infinite energy for a point charge is inconsistent with the fact that charged particles are not infinitely massive in gravitational fields. It is also inconsistent with the result (60′), $P_1 = 0$, or, on a more intuitive level, the fact that $\partial \cdot \theta_1 = 0$ allows the self-field to follow the particle without the application of any force.

The last two inconsistencies would be resolved for charged point particles if we dropped the term θ_1 from θ_{ret}. Such a change would not alter any force relationships in particle electrodynamics because

$$\partial \cdot \theta_{\text{ret}} = \partial \cdot (\theta_{\text{ret}} - \theta_1) = \partial \cdot \theta'_{\text{ret}} ,$$

and it would not alter total momentum integrals because $P_1 = 0$. We were always free to drop θ_1 once it was seen to be decoupled from the rest of the system. The point now is that it appears necessary to drop it in order that we should have any hope of interpreting the residual θ' as a genuine physical density.

If we do drop θ_1 for a point particle we lose the positive definiteness of the energy density [look at the integrand in Eq. (90)]. We also lose the connection (2) between the field tensor F and the energy tensor θ. One would expect corresponding changes to be needed in the energy tensor due to continuous charges, if only because the fields outside a spherically symmetric charge distribution are the same as those of a point charge. These changes would presumably involve relocating the work of compression (93) of a finite charge.

The total momentum integral (64) for a system of point charges in electromagnetic interaction must be taken into account if we want to try to understand localization. For the case of two charges, A and B, (64) has the form

$$P=-\int_\sigma n\cdot T'dV$$
$$=\sum_{A,B}\left[mv(\tau')-\tfrac{2}{3}e^2a(\tau')\right]$$
$$+\sum_{A,B}\int_{-\infty}^{\tau'}d\tau\,\tfrac{2}{3}e^2a^2v-\int_\sigma dVn\cdot\theta_{\text{int}(AB)}\,. \quad (64)$$

If we integrate the equation

$$\partial\cdot\theta_{\text{int}(AB)}=-F_{(A)}\cdot j_{(B)}-F_{(B)}\cdot j_{(A)}\,,$$

which is of the same form as (10) and (87), between a spacelike plane σ_0 with normal n_0 and the plane σ with normal n, we get

$$\int_{\sigma_0}^{\sigma}d^4x\,\partial\cdot\theta_{\text{int}(AB)}=-\int_\sigma dVn\cdot\theta_{\text{int}(AB)}$$
$$+\int_{\sigma_0}dV_0n_0\cdot\theta_{\text{int}(AB)}$$
$$=-\int_{\tau'_{B0}}^{\tau'_B}d\tau_Be_BF_{(A)}(z_B)\cdot v_B$$
$$-\int_{\tau'_{A0}}^{\tau'_A}d\tau_Ae_AF_{(B)}(z_A)\cdot v_A\,,$$

provided that the world lines are straight in the remote past so that $\theta_{\text{int}}\sim r^{-4}$ at sufficiently large spatial distances. Then from (64),

$$\Delta P=P(\sigma)-P(\sigma_0)=0$$

gives

$$\Delta\sum_{A,B}\left(mv-\tfrac{2}{3}e^2a\right)=-\sum_{A,B}\int_{\tau'_0}^{\tau'}d\tau\,\tfrac{2}{3}e^2a^2v$$
$$+\int_{\tau'_{B0}}^{\tau'_B}d\tau_Be_BF_{(A)}(z_B)\cdot v_B$$
$$+\int_{\tau'_{A0}}^{\tau'_A}d\tau_Ae_AF_{(B)}(z_A)\cdot v_A\,. \quad (94)$$

The change in the sum of the particle "momenta" $mv-\frac{2}{3}e^2a$ equals the negative of the radiated momentum between the planes plus the four-vector work done by the Lorentz force of each particle on the other (between the planes).

Equation (64) is *suggestive*, but *only* suggestive, of a form of the theory which would contain "real" densities. Equation (64) arises from

$$T'=\sum\int d\tau\,mvv\,\delta(x-z)+\sum\theta_2$$
$$+\sum\theta_3+\theta_{\text{int}}\,. \quad (66)$$

According to Eq. (35),

$$\theta_2^{\mu\nu}=\partial_\alpha K_2^{\alpha\mu\nu}-\int d\tau\,\tfrac{2}{3}e^2v^\mu a^\nu\delta(x-z)\,. \quad (35)$$

If one takes the divergence with respect to the left index μ, the first term makes no contribution since $\partial_\mu\partial_\alpha K_2^{\alpha\mu\nu}=0$; and if one calculates a surface integral taking the scalar product with the normal on the left, the first term makes no contribution:

$$-\int_\sigma dVn_\mu\partial_\alpha K_2^{\alpha\mu\nu}=0$$

[see Eq. (57)]. This suggests that we drop the first term in (35) and use the unsymmetrical energy tensor

$$T''=\sum\int d\tau\,v(mv-\tfrac{2}{3}e^2a)\,\delta(x-z)$$
$$+\sum\theta_3+\theta_{\text{int}}\,. \quad (95)$$

The condition

$$\partial\cdot T''=0\quad\text{(divergence on the left)} \quad (96)$$

entails the Lorentz-Dirac equations, and the total momentum P satisfies

$$P=-\int_\sigma dVn\cdot T''\quad\text{(scalar product on the left)}\,. \quad (97)$$

In this formulation the momentum

$$p=mv-\tfrac{2}{3}e^2a \quad (98)$$

is located at the particle, and the remaining terms in T'' could be interpreted as physical densities since their integrals are Riemann integrals.

We would speak of the new particle energy tensor

$$K''_{\text{parts}}=\sum\int d\tau\,vp\,\delta(x-z) \quad (99)$$

as describing flows of momentum p along the world lines z. Since

$$\partial\cdot K''_{\text{parts}}=\sum\int d\tau\,\dot p\,\delta(x-z)$$

the Lorentz-Dirac equations, the consequences of (96), would take the form

$$\dot p+\tfrac{2}{3}e^2a^2v-eF_{\text{ext}}\cdot v=0\,.$$

This is, effectively, the form of the theory used by Teitelboim[2] in his derivation of the Lorentz-Dirac equation. Various forms of the theory with particle momentum different from mv were noticed by Pryce.[4]

Note added in proof. Since this paper was written, two relevant references have come to the author's attention. In Phys. Rev. Lett. 12, 375 (1964), Rohrlich gave a solution to the self-energy problem. The essence of his solution is that the Lorentz-Dirac equation follows from an expression of the form $\partial \cdot (K_{\mathrm{PART}} + \theta_D) = 0$. The tensor θ_D is integrable; it has the form of the cross term in an electromagnetic energy tensor with the field tensor decomposed as $F = \overline{F} + F_+$, $F = F_{\mathrm{EXT}} + F_-$, $F_\pm = \frac{1}{2}(F_{\mathrm{RET}} \pm F_{\mathrm{ADV}})$. He does not discuss the uniqueness question. From the point of view of the present paper it is difficult to justify the unphysical division of the field tensor, or the use of the cross term.

In a set of lecture notes on the geometry of basic physics in flat space-time, Princeton, 1975 (unpublished), J. Milnor quotes a distribution theory expression for θ (identical with that used in the present paper), and shows that the Lorentz-Dirac equation implies $\partial \circ (K_{\mathrm{PART}} + \theta) = 0$. He also uses the integral notation [as in (52)] for the value of distributions on test functions—he describes them as "fake" integrals—but does not discuss their limits.

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