Gauge-invariant subtraction scheme for massive quantum electrodynamics

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A momentum-space subtraction scheme for massive quantum electrodynamics is proposed which respects gauge invariance, in contrast to ordinary normal-product techniques. As a consequence the dependence of Green's functions on the ghost mass becomes very simple and formally gauge-invariant normal products of degree up to four, when subtracted according to the proposed scheme, are automatically gauge invariant. As an application we discuss the proof of the Adler-Bardeen theorem. Zero-mass limits can be taken for Green's functions after the integration over intermediate states has been carried out.

I. INTRODUCTION

The gauge invariance of massive quantum electrodynamics has been analyzed by Lowenstein and Schroer[1] (LS) using normal-product methods. The effective Lagrangian in this model is

$$\begin{aligned}\mathcal{L}_{\rm eff}&=\tfrac{1}{2}i\bar\psi\gamma^\mu\overleftrightarrow{\partial}_\mu\psi-(M-c)\bar\psi\psi-\tfrac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu\\&\quad+\tfrac{1}{2}(m^2+a)A_\mu A^\mu+e\bar\psi\gamma^\mu\psi A_\mu\\&\quad+\tfrac{1}{2}[1-(m^2+a)/m_0^{\,2}](\partial_\mu A^\mu)^2\\&=\mathcal{L}_0+\mathcal{L}_{\rm int}, \qquad (1)\\ \mathcal{L}_0&=\tfrac{1}{2}i\bar\psi\gamma^\mu\overleftrightarrow{\partial}_\mu\psi-M\bar\psi\psi-\tfrac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu\\&\quad+\tfrac{1}{2}m^2A_\mu A^\mu+\tfrac{1}{2}(1-m^2/m_0^{\,2})(\partial_\mu A^\mu)^2 .\end{aligned}$$

In order to obtain a renormalizable theory, a non-negative parameter $m_0^{\,2}$ has been introduced so that the free vector-meson propagator becomes

$$-i\left[\frac{g_{\mu\nu}}{k^2-m^2+i\epsilon}-\frac{k_\mu k_\nu}{m^2}\left(\frac{1}{k^2-m^2+i\epsilon}-\frac{1}{k^2-m_0^{\,2}+i\epsilon}\right)\right]. \qquad (2)$$

Green's functions now depend on $m_0^{\,2}$ and describe an indefinite-metric Hilbert space with ghost particles of mass $m_0^{\,2}$:

$$(\Box+m_0^{\,2})\partial_\mu A^\mu=0\,. \qquad (3)$$

Observables of the theory are those quantities which commute with $\partial_\mu A^\mu$ and are independent of the ghost mass m_0. Lowenstein and Schroer have given two criteria for checking gauge invariance, which will also be adopted here. As shown by LS the first criterion is satisfied by all formally gauge-invariant normal products, e.g., normal products made up of equal numbers of ψ and $\bar\psi$ fields, plus combinations of $F_{\mu\nu}$, $(\partial_\mu-ieA_\mu)\psi$, $(\partial_\mu+ieA_\mu)\bar\psi$. These normal products commute with $\partial_\mu A^\mu$.

The second criterion states the $m_0^{\,2}$ independence of observables, and the whole Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ) machinery[2,3] is necessary to establish it.

It turns out[1] that Zimmermann's normal products do not automatically satisfy the second criterion, which means that the usual BPHZ subtraction prescription destroys gauge invariance. $m_0^{\,2}$-independent normal products are then constructed by taking linear combinations of normal products with the same quantum numbers and dimensions.

This process becomes rather cumbersome for Green's functions with many normal products; besides this, one would like to maintain gauge invariance at every step as e.g. in the case of Pauli-Villars regularization. It is thus comforting that by modifying the BPHZ subtraction procedure it is possible to define normal products,[4] automatically satisfying the second gauge criterion.

The crucial point to be observed is the following: in order to show $m_0^{\,2}$ independence one has to study the normal products $N_4[(\partial_\mu A^\mu)^2]$. Since A_μ is coupled to $\bar\psi\gamma_\mu\psi$, one is led to apply the equation of motion of the fermion fields in the normal product $N_5[A\partial^\mu(\bar\psi\gamma_\mu\psi)]$, where A is a scalar field with free propagator $i/(k^2-m_0^{\,2}+i\epsilon)$. As is well known, anisotropies appear in the equation of motion of the above trilinear normal product, which in turn introduce the unwanted $m_0^{\,2}$ dependence.

This paper is divided as follows: in Sec. II we define the subtraction operators; in Sec. III the infrared convergence proof is developed (the ultraviolet case is entirely analogous to Ref. 12). Finally, Sec. IV is a discussion of the application of the scheme in the proof of the Adler-Bardeen theorem, and the statement of the existence of the zero-mass limit.

II. SUBTRACTION OPERATOR

Going through the derivation of the second gauge criterion of LS, one observes that anisotropies arise because the BPHZ scheme[3] uses Taylor operators in the external momenta p_μ^γ of the graph γ, around $p_\mu^\gamma=0$, such that one has

$$t_{p^\gamma}^{\delta(\gamma)}[p^\gamma f(p^\gamma, k^\gamma, M)] = p^\gamma t_{p^\gamma}^{\delta(\gamma)-1}[f(p^\gamma, k^\gamma, M)]. \tag{4}$$

In contradistinction the mass M is left untouched. In order to eliminate anisotropies and avoid infrared divergences, one has to show that it is possible to devise a subtraction scheme in which the subtraction operators $\tau_{p\gamma_M\gamma}$ act also on masses[5] such that the following conditions hold:

(a) For any divergent one-particle-irreducible (1PI) graph γ, one must have besides (4) also

$$\tau_{p\gamma_M\gamma}^{\delta(\gamma)}[M^\gamma f(p^\gamma, k^\gamma, M^\gamma)] = M^\gamma \tau_{p\gamma_M\gamma}^{\delta(\gamma)-1}[f(p^\gamma, k^\gamma, M^\gamma)], \tag{5}$$

where M^γ is the mass associated with an internal fermion, which does not belong to a closed fermion loop.

(b) If an internal fermion line f_i belongs to a set $\mathfrak{A}$ of internal fermion lines forming a closed fermion loop, then all masses M_i associated with lines belonging to $\mathfrak{A}$ have to be subtracted with the same operator at the same point.

Besides this we will use only Taylor operators in order to preserve certain product structures, when a fermion line is reduced to a point. The subtraction scheme of Ref. 6 does not have this property and would again lead to anisotropies.

With this in mind we propose the following subtraction scheme. The subtracted integrand for 1PI graph Γ is defined by a modified version of Zimmermann's forest formula:

$$R_{\Gamma_\epsilon}(p, k, M) = S_\Gamma \sum_{U\in\mathfrak{F}_\Gamma} \prod_{\gamma\in U} (-\tau_\gamma S_\gamma) I_{\Gamma_\epsilon}(U) \tag{6}$$

where $\mathfrak{F}_\Gamma$ is the set of Γ– forests (families of 1PI nontrivial nonoverlapping subgraphs of Γ), and
$p = p_1, \ldots, p_n$ = basis for external momenta;
$k = k_1, \ldots, k_l$ = basis for internal momenta;
$M = M_1, \ldots, M_v$ = set of fermion masses associated with internal fermion lines (they have been labelled for subtraction purposes).

S_γ is a substitution operator shifting from the variables of $\lambda\in U$ to those of $\gamma\in U$ if $\gamma\supset\lambda$. S_Γ in addition sets all masses M_i equal to M, and makes the replacement $w_1 \to m-\mu$, $w_2 \to M$, $w_3 \to M-\mu$, where the w_i, $i=1,2,3$ are defined after Eq. (7).

Up to here everything is standard. See e.g. Ref. 6. The Taylor operators τ_γ are defined as follows:

$$\tau_\gamma^{\delta(\gamma)} I(p, k, w) = \sum_{n=0}^{\delta(\gamma)} \frac{1}{n!} \left(p_1^{\mu_1} \frac{\partial}{\partial \bar{p}_1^{\mu_1}} + \cdots + p_j^{\mu_j} \frac{\partial}{\partial \bar{p}_j^{\mu_j}} + w_1 \frac{\partial}{\partial \bar{w}_1} + w_2 \frac{\partial}{\partial \bar{w}_2} + w_3 \frac{\partial}{\partial \bar{w}_3} \right)^n I(\bar{p}, \bar{k}, \bar{w}) \Bigg|_{\substack{\bar{p}=0 \\ \bar{w}_i=0}},$$
$$\delta(\gamma) = 4 - \tfrac{3}{2} F_\gamma - B_\gamma - N_\gamma^c \tag{7}$$

where N_γ^c is the number of c-type (fermion mass) counterterms of γ, and w_i, $i=1,2,3$ are mass parameters defined as follows:

(1) The vector-meson mass m is put equal to $m = w_1 + \mu$ when it appears in the transverse part of the propagator (2); the masses m and m_0 in the longitudinal part do not participate in the subtraction scheme.

(2) The fermion masses M_i are treated as follows:

(a) $M = w_2$, if either M_i does not belong to a closed divergent fermion loop, or M_i belongs to a closed fermion loop which has at least one internal vector-meson line (Fig. 1).

(b) For the two undecorated divergent closed fermion loop graphs γ_1 and γ_2 of Fig. 2, we set

if $M_i\in\gamma_1$, $M_i = w_3^{\gamma_1} + \mu$

if $M_i\in\gamma_2$, in $\tau_{\gamma_2}^{(0)}$ we set $M_i = w_3^{\gamma_2} + \mu$
in τ_G, $G\supset\gamma_2$ we set $M_i = w_2$.

Thus the masses of γ_2 are treated differently

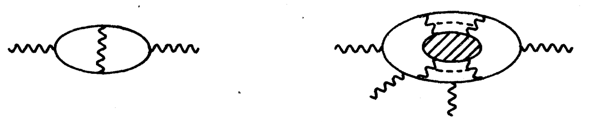

FIG. 1. Graphs with closed fermion loops L_i, where the masses M_i associated with L_i are set at $M_i=0$ in the subtraction procedure.

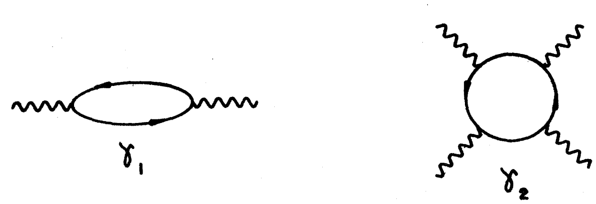

FIG. 2. Graphs γ_1 and γ_2 satisfy normalization conditions at $p_i=0$ and $M_i=\mu$.

according to the forest to which γ_2 belongs. We indicate this by a shift operator $\tilde{S}$ defined as follows:

If $M_i \in \gamma_2$ has been identified with $w_3 + \Gamma$ and if $G \supset \gamma_2$, then before applying $\tau_G^{\delta(G)}$ we act with $\tilde{S}$:

$$\tilde{S}(w_3^{\gamma_2} + \mu) = w_2^G . \quad (8)$$

In (6) we incorporate $\tilde{S}$ into S_γ, whenever $\gamma \supset \gamma_2$.

(3) The counterterm c multiplies $N_3[\bar{\psi}\psi]$. To allow smooth zero-mass limits the counterterms a and c enforce the normalization conditions

$$\Pi(m^2) = 0 , \quad (9a)$$

$$\Sigma(p)|_{\not p = M} = 0 , \quad (9b)$$

where $\Pi(p^2)$ and $\Sigma(p)$ are the nontrivial parts of the two-vector-meson and two-fermion vertex functions.

If Θ is any formal product of the basic fields and their derivatives of degree $\delta \leq 4$, where δ = operator dimension of Θ + number of mass parameters M in Θ, then normal-product Green's functions containing an arbitrary product of $N_{\delta_i}[\Theta_i]$ are defined similarly.

(1) The degree function $\delta(\gamma)$ for a subgraph γ is given by

$$\delta(\gamma) = 4 - \tfrac{3}{2}F_\gamma - B_\gamma - N_\gamma^c - \sum_{V_i \in \gamma} (4 - \delta_i) .$$

(2) The classification for subgraphs given above must be taken into account, i.e., bilinear insertions in the fermion field in γ_1 or γ_2 leave the subtraction scheme unchanged. The shift operators $\tilde{S}$ must also be used in the case of bilinear fermion insertions in γ_2, of course.

It is now straightforward to check that this subtraction scheme yields automatically gauge-invariant normal products, if they are formally gauge-invariant. This is a consequence of the fact that we have done a maximum number of subtractions at $p = 0$ and $M = 0$. The price we pay is the nontriviality in showing the infrared (and ultraviolet) absolute convergence of our renormalized Feynman integrals. We will do this by following the method and notation of Ref. 6 as closely as possible and refer the reader to that publication for details.

III. CONVERGENCE PROOF

In order to state the infrared-finiteness criterion let us repeat some definitions of Ref. 6.

Let $u_1, \ldots, u_a, v_1, \ldots, v_{n-a}$ bet a set of linearly independent elements of $\mathcal{L}(\Gamma)$, the space of linear forms in p and k with $\partial(u, v)/\partial k \neq 0$. Furthermore let C be a Γ-forest, complete with respect to S the subspace of $\mathcal{L}(\Gamma)$ spanned by $u_1, \ldots, u_a$. Then we have to show that[7]

$$\underline{\deg}_u R_{\Gamma_\epsilon}(\mathrm{C}) + 4a > 0 , \quad (10)$$

where $\underline{\deg}_u R$ denotes the lower degree in u of R.

To establish (10) we need the following power-counting lemmas:

Lemma 1.

$$\underline{\deg}_{upw} \tau_\gamma Y_\gamma \geq \underline{\deg}_{upw} Y_\gamma , \quad (11)$$

$$\underline{\deg}_u \tau_\gamma Y_\gamma \geq \underline{\deg}_{upw} Y_\gamma - \delta(\gamma), \quad \bar{\gamma} \parallel S \quad (12)$$

$$\underline{\deg}_u \tau_\gamma Y_\gamma \geq \underline{\deg}_u Y_\gamma , \quad \bar{\gamma} \not\parallel S \quad (13)$$

$$\underline{\deg}_{upw} (1 - \tau_\gamma) Y_\gamma \geq \underline{\deg}_u Y_\gamma + \delta(\gamma) + 1 , \quad \bar{\gamma} \not\parallel S . \quad (14)$$

The proof of Lemma 1 is entirely analogous to the proof of Lemma 3.2 of Ref. 6, where the definitions of $\gamma \parallel S$, $\gamma \not\parallel S$ and Y_γ can be found.

Lemma 2. Let λ be a maximal element of C properly contained in $\gamma \supset \Gamma$. Then the following inequalities hold:

$$\underline{\deg}_{upw} Y_\gamma \geq \delta(\gamma) + 1 - M(\gamma), \quad \bar{\gamma} \parallel S \quad (15)$$

$$\underline{\deg}_u Y_\gamma \geq^* - M(\gamma) + 1 , \quad (16)$$

$$\underline{\deg}_u S_\gamma \tau_\lambda Y_\lambda \geq^* - M(\lambda) + 1 , \quad (17)$$

$$\underline{\deg}_u S_\gamma (1 - \tau_\lambda) Y_\lambda \geq^* - M(\lambda) + 1 , \quad (18)$$

$$\underline{\deg}_{upw} S_\gamma \tau_\lambda Y_\lambda \geq \delta(\lambda) + 1 - M(\lambda), \quad \bar{\lambda} \parallel S \quad (19)$$

$$\underline{\deg}_{upw} S_\gamma (1 - \tau_\lambda) Y_\lambda \geq \delta(\lambda) + 1 - M(\lambda) - \delta , \quad \bar{\lambda} \not\parallel S, \ \bar{\gamma} \parallel S , \quad (20)$$

where

$$\geq^* \text{ means } \begin{cases} \geq \text{ if } M(\gamma) \neq 0 , \\ = 0 \text{ if } M(\gamma) = 0 , \end{cases} \quad (21)$$

$$M(\gamma) = 4 \times \sum_{\substack{\lambda \in \mathrm{C},\, \lambda \subseteq \gamma \\ \bar{\lambda} \parallel S}} [\text{n}^{\text{o}} \text{ of independent loops of } \bar{\lambda}(\mathrm{C})] , \quad (22)$$

and

$$\delta = \begin{cases} 1, & \text{if } \lambda = \gamma_2 \text{ of Fig. 2} \\ 0, & \text{if } \lambda \neq \gamma_2 . \end{cases} \quad (23)$$

We notice that

$$\underline{\deg}_{up\gamma_w\gamma} \tilde{S}_\gamma (1 - \tau_\lambda) \dot{I}_\lambda = \underline{\deg}_{\mu p \gamma_w \gamma} (1 - \tau_\lambda) I_\lambda - 1 , \quad \text{if } \lambda = \gamma_2 . \quad (24)$$

As in the case of Lemmas 3.3 and 3.4 of Ref. 6, the proof of Lemma 2 is by induction. (15) and (16) hold for minimal γ, since

$$\underline{\deg}_{up\gamma_w\gamma} Y_\gamma \geq \underline{\deg}_{up\gamma_w\gamma} I_\gamma \geq r(\bar{\gamma}) - M(\bar{\gamma}) \geq \delta(\gamma) + 1 - M(\gamma), \quad \bar{\gamma} \parallel S \quad (25)$$

$$\underline{\deg}_u I_\gamma \geq 0 \geq^* - M(\gamma) + 1 , \quad (26)$$

where $r(\gamma)$ is the infrared superficial divergence, generalized to include the parameters w.[6]

We now verify (17)–(20) using the inductive hypothesis[8]:

$$\underline{\deg}_u S_\gamma \tau_\gamma Y_\lambda \geqslant \underline{\deg}_u \tau_\lambda Y_\lambda \geqslant^{(12)}_{(13)} \begin{cases} \underline{\deg}_{u_p\lambda_w\lambda} Y_\lambda - \delta(\lambda) \geqslant^{(15)} -M(\lambda)+1\,, & \bar{\lambda} \parallel S \\ \underline{\deg}_u Y_\gamma \geqslant *^{(16)} -M(\lambda)+1\,, & \bar{\lambda} \not\parallel S \end{cases} \tag{27}$$

$$\begin{aligned} &\underline{\deg}_u S_\gamma (1-\tau_\lambda) Y_\lambda \geqslant *^{(12)(13)(15)(16)} -M(\lambda)+1\,, \\ &\underline{\deg}_{u_p\gamma_w\gamma} S_\gamma \tau_\lambda Y_\lambda \geqslant \underline{\deg}_{u_p\lambda_w\lambda}\, \tau_\lambda Y_\lambda \geqslant^{(11)(15)} \delta(\lambda)+1-M(\lambda)\,, \quad \bar{\lambda} \parallel S \end{aligned} \tag{28}$$

$$\underline{\deg}_{u_p\gamma_w\gamma} S_\gamma (1-\tau_\lambda) Y_\lambda \geqslant^{(24)} \underline{\deg}_{u_p\lambda_w\lambda} (1-\tau_\lambda) Y_\lambda - \delta \geqslant^{(14)} \underline{\deg}_u Y_\gamma + \delta(\gamma)+1-\delta \geqslant^{(16)} -M(\gamma)+\delta(\gamma)+1-\delta\,, \quad \bar{\lambda} \not\parallel S\,, \quad \bar{\gamma} \parallel S \tag{29}$$

where the condition $\bar{\gamma} \parallel S$ is necessary in order to remove S_γ in the first step.

To complete the inductive proof, we easily establish (15) and (16), using (19), (20), and

$$M(\gamma) = M(\bar{\gamma}) + \sum_{\substack{\text{maximal}\,\gamma_\alpha \in C \\ \gamma_\alpha \in \gamma}} M(\gamma_\alpha)\,. \tag{30}$$

It is straightforward to verify the infrared criterion (10) using Lemmas (1) and (2):

$$\underline{\deg}_u (1-\tau_\Gamma) Y_\Gamma \geqslant^{(13)}_{(12)(15)(16)} \begin{cases} \underline{\deg}_u Y_\Gamma \geqslant^{(16)} -M(\Gamma)+1 & \text{if } \bar{\Gamma} \not\parallel S \\ -M(\Gamma)+1 & \text{if } \bar{\Gamma} \parallel S\,. \end{cases} \tag{31}$$

Consequently,

$$\underline{\deg}_u R_{\Gamma_\epsilon} + 4a \geqslant^{(31)} -M(\Gamma)+1+4a > 0 \tag{32}$$

since

$$4a \geqslant M(\Gamma)\,. \tag{33}$$

The ultraviolet criterion

$$\overline{\deg}_u R_{\Gamma_\epsilon}(C) + 4a < 0 \tag{34}$$

follows immediately from the theorem of Ref. 12.

IV. DISCUSSION: THE ADLER-BARDEEN THEOREM AND THE ZERO-MASS LIMIT

As an example of the usefulness of the present scheme consider the proof of the Adler-Bardeen theorem.[9] We immediately see that no "decoration" of the fundamental fermion triangle loop will contribute to the anomaly, since there will always be internal photon lines present and consequently the fermion masses of the triangle loop will be subtracted at zero.[10]

Other possible contributions come from the graph of Fig. 3, which is proportional to

$$Y_{\mu\nu} = \int d^4k\, \Pi_{\mu\nu\alpha\beta}(p, q, k, -k-p-q)\, \epsilon^{\alpha\beta\rho\sigma} k_\rho (k+p+q)_\sigma\,, \tag{35}$$

where $\Pi_{\mu\nu\alpha\beta}(k_1, \ldots, k_4)$ is the full four-photon vertex function and $\epsilon^{\alpha\beta\rho\sigma} k_\rho (k+p+q)_\sigma$ arises from the triangle. The anomaly is calculated from

$$\left(\frac{\partial}{\partial p_\lambda} \frac{\partial}{\partial q_\eta} Y \right)_{p=q=0}.$$

To get a nonzero contribution, exactly one derivative has to act on $\Pi_{\mu\nu\alpha\beta}$; but

$$\left.\frac{\partial}{\partial p_\lambda} \Pi_{\mu\nu\alpha\beta}\right|_{p=q=0} = \left.\frac{\partial}{\partial q_\lambda} \Pi_{\mu\nu\alpha\beta}\right|_{p=q=0} = 0\,,$$

from gauge invariance.

The present scheme also has smooth $M, m \to 0$ limits [in the Landau gauge, $m_0^2 = \alpha m^2$ (see Ref. 11)] although not graph by graph. Since in this case we lose (26), the zero-mass limit can only be taken on Green's functions (after the integrations have been carried out), so that the normalization conditions for the zero-mass case,

$$\Pi(0) = \Sigma(0) = 0\,, \tag{36}$$

are satisfied.

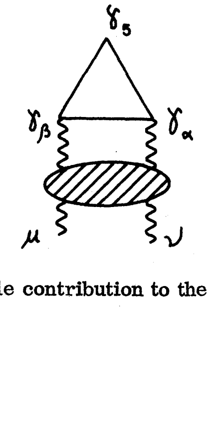

FIG. 3. Possible contribution to the Adler anomaly

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