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Massless fields with half-integral spin

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The Fierz-Pauli Lagrangian for massive particles with spin $s = n + 1/2$, n integer, is examined in the limit of vanishing mass. A considerable simplification occurs. The potential h is a Rarita-Schwinger spinor-tensor of tensorial rank n. The "spinor-trace" h', defined by $h'_{\nu\lambda\ldots} \equiv \gamma^\mu h_{\mu\nu\lambda\ldots}$ does not vanish, and neither does $h'' \equiv (h')'$; but h''' does vanish. The wave equation admits a gauge group, $h \to h + \text{sym grad}\ \xi$, with $\xi' = 0$. The most interesting feature is that the source t need not be divergence free, only the traceless part of $p^\mu t_{\mu\nu\ldots}$ must vanish. This weaker condition on t turns out to be sufficient to guarantee that only helicities $\pm s$ are transmitted between sources.

I. INTRODUCTION

This paper complements our study[1] of massless fields with higher, integer spins and demonstrates that the main results have close analogs in the case of half-integral spins. The motivation for our work will not be repeated here.

It is remarkable that the sources of massless fields with spins $\geqslant \frac{5}{2}$ need not be conserved. As in the case of integer spins we find that only the traceless part of the divergence needs to vanish. ("Traceless" here refers to the spinor trace, see below.) Again it turns out that this condition is sufficient to guarantee that real quanta have helicities $\pm s$ only.

We follow the procedure of Singh and Hagen,[2] constructing the Fierz-Pauli Lagrangian in terms of Rarita-Schwinger tensor-spinors ψ^n, ψ^{n-1} ($n = s - \frac{1}{2}$) and doublets of tensor-spinors $\psi^{n-2}, \psi^{n-3}, \ldots$; all of these are traceless in the sense that $\gamma^\mu \psi_{\mu\nu\lambda\ldots} = 0$. In this paper the term "trace" is always used in this sense; trace $\psi \equiv \psi'$, $\psi'_{\nu\lambda\ldots} \equiv \gamma^\mu \psi_{\mu\nu\lambda\ldots}$, summation on the spinor index being implied.

II. LAGRANGIAN FOR THE MASSIVE CASE

The most general Lagrangian for the free, massive field, invariant under the extended Poincaré group, is

$$\mathcal{L} = \sum_{k\leqslant n} [\bar\psi^k \alpha_k \not p \psi^k + (\bar\psi^{k-1}\beta_k p\cdot\psi^k + \text{H.c.}) + m\bar\psi^k \sigma_k \psi^k], \tag{2.1}$$

where ψ^n, ψ^{n-1} are symmetric, traceless, Dirac-Rarita-Schwinger tensor-spinors, ψ^k for $k \leqslant n-2$ is a pair of such objects; $\alpha_n, \alpha_{n-1}, \beta_n, \sigma_n, \sigma_{n-1}$ are real numbers; $\alpha_k, \beta_k, \sigma_k$ for $k \leqslant n-2$ are Hermitian 2×2 matrices and β_{n-1} is a 2×1 matrix. All spinor and tensor indices are contracted, $p_\mu \equiv -i\partial/\partial x^\mu$, $\not p \equiv \gamma^\mu p_\mu$, and $p\cdot\psi^k$ means $p^\mu \psi^k_{\mu\nu\ldots}$. The Euler-Lagrange equations are, for $k = n, n-1, \ldots,$

$$\delta\bar\psi^k \cdot (\alpha_k \not p \psi^k + \beta_{k+1} p\cdot\psi^{k+1} + \beta_k^\dagger p\psi^{k-1} + m\sigma_k\psi^k) = 0\,. \tag{2.2}$$

The variations $(\delta\bar\psi^k)^{\mu\nu\cdots}$ are symmetric and traceless and may be replaced by $\bar u y^\mu y^\nu \cdots$, where u is a Dirac spinor. The left-hand side of (2.2) becomes

$$\bar u \{\alpha_k \not p \psi^k(y) + [\beta_{k+1}/(k+1)] p\cdot\partial\psi^{k+1}(y) + \beta_k^\dagger y\cdot p\psi^{k-1}(y) + m\sigma_k\psi^k(y)\}\,, \tag{2.3}$$

where ∂_μ now stands for differentiation with respect to y^μ and $\psi^k(y)$ is the polynomial

$$\psi^k(y) \equiv y^\mu y^\nu \cdots \psi^k_{\mu\nu\cdots}\,. \tag{2.4}$$

The statement trace $\psi^k = 0$ is equivalent to $\gamma\cdot\partial\psi^k(y) = 0$. Equation (2.2) is satisfied for all traceless, symmetric $\delta\bar\psi^k$ if and only if (2.3) vanishes for all $y, \bar u$ such that $\delta^{\mu\nu} y_\mu y_\nu = 0$, $\bar u \gamma\cdot y = 0$.

The expansion of ψ^k by spin content, in the frame $\vec p = 0$, is given by

$$\psi^k(y) = \sum_{l\leqslant k} [(y^0)^{k-l} - ay\cdot\gamma\gamma_0 (y^0)^{k-l-1} - by^2 (y^0)^{k-l-2} - \cdots]\psi^{k,l}(\vec y)\,, \tag{2.5}$$

with coefficients $a = (k-l)/2(k+1)$ and $b = (k-l)(k-l-1)/4(k+1)$ determined by the constraints $\gamma\cdot\partial\psi^k(y) = 0 = \vec\gamma\cdot\vec\partial\psi^{k,l}(\vec y)$. The vanishing of (2.3) is equivalent to, for $l \leqslant k \leqslant n$,

$$\gamma_0 p_0 (\alpha_{k,l}\psi^{k,l} + \beta_{k+1,l}\psi^{k+1,l} + \beta_k^\dagger \psi^{k-1,l}) + m\sigma_k \psi^{k,l} = 0\,, \tag{2.6}$$

with

$$\alpha_{k,l} = \alpha_k (l+1)/(k+1)\,,\quad \beta_{k,l} = \beta_k (k+l+2)(k-l)/2k(k+1)\,. \tag{2.7}$$

The problem is to choose the parameters $\alpha_k, \beta_k, \sigma_k$ in such a way that (2.6) is equivalent to $(\gamma_0 p_0 - m)\psi^{n,n} = 0$ and $\psi^{k,l} = 0$ for $l \leqslant k < n$. We normalize $\sigma_n = -1$; then α_n must be unity, and the following determinants must be nonzero and independent of $z \equiv \gamma_0 p_0/m$, $l = n-1, n-2, \ldots,$:

$$\begin{vmatrix} \alpha_{n,l}z - 1 & \beta_n^\dagger z & 0 & & \\ \beta_{n,l}z & \alpha_{n-1,l}z + \sigma_{n-1} & \beta_{n-1}^\dagger z & & \\ 0 & \beta_{n-1,l}z & \alpha_{n-2,l}z + \sigma_{n-2} & & \\ & 0 & \beta_{n-2,l}z & & \\ \vdots & \vdots & 0 & \ddots & \\ & & & & \alpha_{l,l}z + \sigma_l \end{vmatrix} . \tag{2.8}$$

When $l = n-1$ we have a 2×2 determinant and find that it is necessary and sufficient that

$$\begin{aligned} &\alpha_{n-1}/\sigma_{n-1} = n/(n+1)\,, \\ &|\beta_n|^2/\sigma_{n-1} = 2n^3/(n+1)(2n+1)\,. \end{aligned} \tag{2.9}$$

When $l = n-2$ we have a 4×4 determinant,

$$\begin{vmatrix} \frac{n-1}{n+1}z - 1 & Bz & 0 \;\; 0 \\ Az & \left(\frac{n-1}{n+1}z + 1\right)\sigma_{n-1} & \beta_{n-1}^\dagger z \\ 0 & & \\ 0 & \beta_{n-1}\frac{2n-1}{2n(n-1)}z & \alpha_{n-2}z + \sigma_{n-2} \end{vmatrix}, \tag{2.10}$$

where $AB/\sigma_{n-1} = -4n^3/(n+1)(2n+1)$. In order that this be nonzero and independent of z it is necessary and sufficient that the following conditions hold:

$$\begin{aligned} &\mathrm{tr}(\alpha\epsilon\sigma\epsilon) = 0\,, \\ &(n-1)(\beta^\dagger\epsilon\sigma\epsilon\beta) = (n+1)(\beta^\dagger\epsilon\alpha\epsilon\beta) \\ &\qquad = -[2n(n+1)^2/(2n+1)]\sigma_{n-1}\,\mathrm{Det}\alpha \\ &\qquad = [(2n-1)(n^2-1)^2/2n^2(2n+1)]\sigma_{n-1}\,\mathrm{Det}\sigma\,. \end{aligned} \tag{2.11}$$

Here β is the 2×1 matrix β_{n-1}, α, σ are the 2×2 matrices α_{n-2}, σ_{n-2} and

$$\epsilon = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

These conditions do not determine the parameters α_{n-2}, β_{n-1}, σ_{n-1}, σ_{n-2} uniquely. As pointed out by Singh and Hagen,[2] we are free to choose our basis in each of the two-dimensional subspaces defined by each of the doublets ψ^{n-2}, $\psi^{n-3}, \ldots$; in other words α_{n-2}, β_{n-1}, σ_{n-2} need to be determined up to an equivalence transformation only. Nevertheless, real ambiguities seem to remain. We have not carried out a completely systematic investigation to determine all solutions to the problem. Instead we shall follow Singh and Hagen and specialize by requiring that the 2×1 matrix β_{n-1} be an eigenvector of the matrix α_{n-2}. It is easy to see that this produces a great simplification in the limit $m \to 0$; in this case only ψ^n, ψ^{n-1} and one of the two components of ψ^{n-2} remain coupled. The coupling scheme of Singh and Hagen[2] is illustrated as follows:

$$\begin{array}{ccccc} \psi^n - \psi^{n-1} - & \psi_1^{n-2} & & \psi_1^{n-3} - & \psi_1^{n-4} \\ & |\sim m & & |\sim m & |\sim m \\ & \psi_2^{n-2} - & & \psi_2^{n-3} & \psi_2^{n-4} - \cdots\,. \end{array}$$

The vertical couplings vanish in the limit $m \to 0$.

With the restriction that β_{n-1} be an eigenvector of α_{n-2} one easily completes the analysis and finds a solution that is unique up to equivalence and given by

$$\alpha_n = 1,\;\; \alpha_{n-1} = (2n+1)/2,\;\; \beta_n = n\,,$$
$$\alpha_{n-2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\;\; \beta_{n-1} = \begin{pmatrix} n+1 \\ 0 \end{pmatrix}.$$

This gives the following Lagrangian for $m = 0$:

$$\begin{aligned} \mathcal{L} = \bar\psi^n \not p \psi^n + (n+1/2)\bar\psi^{n-1}\not p\psi^{n-1} - \bar\psi^{n-2}\not p\psi^{n-2} \\ + [n\bar\psi^{n-1}p\cdot\psi^n + (n+1)\bar\psi^{n-2}p\cdot\psi^{n-1} + \mathrm{H.c.}]\,. \end{aligned} \tag{2.12}$$

Here ψ^n, ψ^{n-1}, ψ^{n-2}, are Rarita-Schwinger tensor-spinors of rank n, $n-1$, $n-2$ (we have dropped the subscript on ψ_1^{n-2} since the other component no longer appears).

III. MASSLESS FIELDS

When $m = 0$ the Fierz-Pauli program fails in the sense that the field equations no longer imply the subsidiary conditions $\psi^{k,l} = 0$ for $l \le k < n$. When the parameters are as determined above, the algebraic equations (2.6) reduce to , for $l < n$,

$$\begin{aligned} &(l+1)\psi^{n,l} + n(n+1)\psi^{n-1,l} = 0\,, \\ &(n+1)(n+l+1)(n-1-l)\psi^{n-1,l} - 2n(l+1)\psi^{n-2,l} = 0\,. \end{aligned}$$

This shows that the wave equation possesses a family of "gauge" solutions (solutions not subject to wave equations), each determined by its ψ^n component and having $\psi^{n,n} = 0$. We shall introduce a notation that allows us to write these gauge solutions in a simple and familiar form.

Let h be the symmetric Rarita-Schwinger spinor-tensor field of rank n determined by ψ^n, ψ^{n-1}, ψ^{n-2} as follows

$$\psi^n = h - a\sum_1 \gamma h' - b\sum_2 \delta h'' ,$$
$$\psi^{n-1} = d\Big(h' - c\sum_1 \gamma h''\Big) , \tag{3.1}$$
$$\psi^{n-2} = e h'' , \quad h''' = 0 .$$

Notation. The sums $\sum_1$ and $\sum_2$ include all unequal permutations of the tensor indices, thus $\sum_1$ has n terms and $\sum_2$ has $n(n-1)/2$ terms.[1] The coefficients a, b, c are found by verifying that $\mathrm{trace}\,\psi^n = \mathrm{trace}\,\psi^{n-1} = 0$; we find $a = b = 1/2(n+1)$, $c = 1/2n$. The parameters d, e will now be adjusted so that the gauge solutions discussed above take the form

$$\tilde{h} = \sum_1 p\xi , \quad \xi' = 0 . \tag{3.2}$$

We express the Lagrangian (2.12) in terms of h and require invariance under $h \to h + \sum_1 p\xi$ for $\xi' = 0$ and find that this determines $d = -n/(n+1)$, $e = -(n-1)/2$.

Having thus fixed the relationship between ψ^n, ψ^{n-1}, ψ^{n-2} and h, we write down the final form of the free Lagrangian for massless, spin-$s = n + \frac{1}{2}$ fields:

$$\mathcal{L} = \bar{h}\not{p}h + n\bar{h}'\not{p}h' - \tfrac{1}{4}n(n-1)\bar{h}''\not{p}h''$$
$$- n(\bar{h}'p \cdot h + \mathrm{H.c.})$$
$$+ \tfrac{1}{2}n(n-1)(\bar{h}''p \cdot h' + \mathrm{H.c.}) . \tag{3.3}$$

To this we next add an interaction term $\bar{h}\cdot t + \bar{t}\cdot h$, where t is an external source. Since $h''' = 0$, there is no loss of generality in requiring $t''' = 0$. The Euler-Lagrange equations are now

$$\not{p}h + \sum_1 \gamma\not{p}h' - \tfrac{1}{2}\sum_2 \delta\not{p}h'' - \sum_1 \gamma p \cdot h - \sum_1 ph'$$
$$+ \sum_2 \delta p \cdot h' + \tfrac{1}{2}\sum_2 (\gamma p + p\gamma)h'' = t . \tag{3.4}$$

One can easily check that the left side of (3.4) vanishes when h is replaced by the gauge field (3.2); hence (3.4) is gauge invariant. A concomitant of this fact is that the traceless part of the divergence of the left side of (3.4) vanishes identically. Consistency therefore requires that the traceless part of $p \cdot t$ vanish; that is, the source must satisfy the following condition:

$$p \cdot t = (1/2n)\Big(\sum_1 \gamma p \cdot t' + \sum_2 \delta p \cdot t''\Big) . \tag{3.5}$$

Thus it is seen that, as in the case of integer spins, the source need not be exactly divergence free. It is remarkable that (3.5) suffices (with $t''' = 0$) to guarantee that all transmitted quanta have helicities $\pm s = \pm(n + \frac{1}{2})$. To show this we first determine the propagator.

IV. THE PROPAGATOR

Let Eq. (3.4) be abbreviated by $Lh = t$. The solution is determined up to a gauge field $\tilde{h}$, and there exists (for each set of boundary conditions) an operator G such that the general solution takes the form $h = Gt + \tilde{h}$. The only acceptable form for G is

$$G = (1/p^2)A , \tag{4.1}$$

where A is a symmetric, first-order, differential operator. The requirement on A is that $(LA - p^2)t = 0$ for every source t that satisfies $t''' = 0$ and condition (3.5); this means that $(AL - p^2)h$ must be a gauge field for every h.

The equation $Lh = t$, Eq. (3.4), may be written as $BL_0 h = t$ with

$$L_0 h = \not{p}h - \sum_1 ph' , \tag{4.2}$$

$$Bh = h - \tfrac{1}{2}\sum_1 \gamma h' - \tfrac{1}{2}\sum_2 \delta h'' , \tag{4.3}$$

$$B^{-1}h = h - \frac{1}{2n}\sum_1 \gamma h' - \frac{1}{2n}\sum_2 \delta h'' . \tag{4.4}$$

This suggests the following ansatz for A:

$$Ah = \not{p}B^{-1}h + \alpha\sum_1 p(B^{-1}h)' .$$

Then

$$(AL - p^2)h = \sum_1 p\xi ,$$

$$\xi \equiv \not{p}h' + \alpha(2p \cdot h - 2\not{p}h' - \sum_1 ph'') .$$

The field $\sum_1 p\xi$ is a gauge field if and only if $\xi' = 0$, and this condition fixes $\alpha = -1$ and

$$Ah = \not{p}B^{-1}h + \frac{1}{n}\sum_1 ph' . \tag{4.5}$$

The operator (4.1), with A as in (4.5) will be called the propagator of the field h.

V. HELICITY THEOREM

It turns out (as in the integral spin case) that the only quanta exchanged between different parts of the source are massless quanta with helicities $\pm s$.

Theorem. If the source t satisfies $t''' = 0$ and condition (3.5), and A is given by (4.5), and $p^2 = 0$, then the only contributions to $t \cdot A \cdot t$ come from the helicity components $\pm s$ (both positive and negative energies) of t.

Proof. We first show that, when $p^2 = 0$, $t \cdot A \cdot t$ reduces to $\check{t} \cdot A \cdot \check{t}$ with $\check{t}$ effectively divergence free. This reduces the problem to one of two dimensions and then $A\cdot\check{t}$ is traceless; that is, of helicity $\pm s$. Let us put

$$\check{t}\equiv t+B\cdot\Big(\sum_1 p\xi+\not{p}\eta\Big)\,, \qquad (5.1)$$

with ξ and η to be determined. It is required that $t\cdot A\cdot(\check{t}-t)=0$; that is,

$$(t\cdot p)\cdot\Big(\sum_1 p\xi'+2p\cdot\eta-\not{p}\eta'\Big)=0\,.$$

This holds for all t satisfying the constraints if and only if the second factor is traceless, which is the case if we take

$$\xi''=0, \quad \eta''=-\xi'\,. \qquad (5.2)$$

[This also gives $(\check{t}-t)'''=n\not{p}\xi''=0$ as we should expect if $\check{t}$ is a source.] Next, we calculate

$$p\cdot\check{t}=\sum_1\gamma\,\psi+\sum_2\delta\psi'\,,$$

$$\psi\equiv\frac{1}{2n}p\cdot t'-\tfrac{1}{2}\not{p}p\cdot\xi-\tfrac{1}{2}\sum_1 p(p\cdot\xi')$$
$$-(pp\cdot\eta)+\tfrac{1}{2}\not{p}p\cdot\eta'\,.$$

It is possible to choose ξ,η so that $\psi=0$; for example by taking η to be any solution of

$$p\cdot\eta'=0, \quad pp\cdot\eta=\frac{1}{4n}\gamma\cdot(\not{p}t') \qquad (5.3)$$

and ξ to be any solution of

$$p\cdot\xi=\frac{1}{2n}t''-\tfrac{1}{2}\not{p}\eta''\,. \qquad (5.4)$$

Adopting (5.2), (5.3), (5.4) we have $p\cdot\check{t}=0$ and $t\cdot A\cdot t=t\cdot A\cdot\check{t}=\check{t}\cdot A\cdot\check{t}$.

To complete the proof, choose a coordinate system in which $p_1=p_2=0$. Then the index summations in $\check{t}\cdot A\cdot\check{t}$ run effectively over the values 1, 2 only. When the indices are interpreted this way one finds that the trace of $A\cdot\check{t}$ vanishes, therefore only the traceless part of $\check{t}$ contributes to $\check{t}\cdot A\cdot\check{t}$.

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[1]C. Fronsdal, preceding paper, Phys. Rev. D 18, 3624 (1978).

[2]L. P. S. Singh and C. R. Hagen, Phys. Rev. D 9, 910 (1974).