Massless fields with integer spin

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The Fierz-Pauli Lagrangian for massive particles with arbitrary integral spin s, first obtained by Hagen and Singh, is examined in the limit of vanishing mass. Unexpectedly, a considerable simplification occurs. The potential h is a symmetric tensor of rank s; the "trace" h', obtained by contraction of a pair of indices against the flat-space metric, does not vanish but the trace h'' of h' does. The wave equation admits a gauge group, and this implies conditions on the source. The divergence of the source need not vanish, only the traceless projection of the divergence must be zero; this is a major departure from the usual assumption and may bear on the question of the existence of a physically interesting source for fields with spin ≥ 3. This weaker condition on the source is sufficient to guarantee that only helicities $\pm s$ are transmitted between sources. A generalized Gupta program is proposed, that is, a search for a scheme for generating a theory of interacting, massless particles, consistent to all orders in the coupling constant.

I. INTRODUCTION

There are at least two reasons for a revival of interest in massless fields with higher spin. Until recently, the only known examples of coherent theories of interacting, massless fields were those of spin 0, $\frac{1}{2}$, 1, and 2. Of these, all but the first are of relevance to physics and the last two embody most of modern theoretical physics. Recently, it was discovered[1] that a consistent theory of interacting, massless fields of spin $\frac{3}{2}$ could be set up with the help of supersymmetry. Generalized supergravity[2] is currently hampered by the taboo against spins higher than 2, created by the lack of any consistent theory; this is one justification for our interest. Another is a haunting preoccupation with the idea that the massless nature of the neutrino should somehow be a strong clue to the structure of the weak interactions, just as the masslessness of the photon and the graviton are crucial to electrodynamics and to Einstein's theory of gravitation. Nobody has associated the neutrino with gauge principles,[3] and this may be due to the fact that our understanding of gauge theories is too limited. Perhaps improved understanding can come from a study of the gauge groups associated with higher spins.[4]

The theory of massive particles of higher spin was developed by Fierz and Pauli[5] in 1939. Their approach was, of course, field theoretical, and it focused on the imperative physical requirements of Lorentz invariance and positivity of the energy (after quantization). Since the paper by Wigner[6] on the unitary representations of the Poincaré group and the work of Bargmann and Wigner[7] on relativistic wave equations, it became clear that the last (positivity of energy) could be replaced by the requirement that the one-particle states carry an irreducible, unitary representation of the Poincaré group. In the case of integral spin s, the field is a symmetric tensor field ϕ^s of rank s, traceless in the sense that $\delta^{\mu\nu}\phi_{\mu\nu\cdots}=0$, divergence free,

$$\partial^\mu \phi_{\mu\cdots}=0\,, \qquad (1.1)$$

and satisfying the wave equation

$$(\partial^2+m^2)\phi^s=0\,. \qquad (1.2)$$

Here $\delta^{\mu\nu}$ are the components of the flat (Minkowski) metric tensor, $\partial_\mu \equiv \partial/\partial x^\mu$, and $\partial^2 \equiv \delta^{\mu\nu}\partial_\mu\partial_\nu \equiv \partial^\mu\partial_\mu$. So much for the free fields.

Attempts to introduce interactions by direct modification of (1.1) and/or (1.2) lead, according to Fierz and Pauli,[5] almost inevitably to an abrupt change in the number of degrees of freedom of the field, and hence to difficulties. To avoid such troubles they suggested that one begin by combining Eqs. (1.1) and (1.2) into an action principle. In order to have enough field components to vary, it is necessary to introduce certain auxiliary fields. Fierz and Pauli suggested a set of tracelsss, symmetric tensor fields of rank s-2, s-3, ..., but they did not determine how many were actually required. Later it was shown[8] that one needs a minimum of s traceless, symmetric tensor fields *in toto*. When $s=2$ it is convenient to combine ϕ^2 and ϕ^0 into a symmetric tensor field[9] $h_{\mu\nu}=\phi_{\mu\nu}+c\,\delta_{\mu\nu}\phi^0$, with c real and $\neq 0$. When $s=3$ it is not possible to combine ϕ^3, ϕ^1, and ϕ^0 into a symmetric tensor field of rank 3. It would be possible to use an auxiliary field of rank $s-1$,[10] but this is an unnecessary complication, and irrelevant for our study of the massless case—for it would become decoupled in the limit of vanishing mass. The choice of a singlet set of traceless, symmetric tensor fields of rank s, $s-2$, $s-3$, ..., 0 (one of each), proposed by Singh and Hagen,[11] is certainly the simplest viable one.

II. LAGRANGIAN FOR THE MASSIVE CASE

The total field is

$$\Phi = \{\phi^s, \phi^{s-2}, \phi^{s-3}, \ldots, \phi^0\}, \tag{2.1}$$

where ϕ^k is a traceless, symmetric tensor field of rank k. The most general Lagrangian for the free field is

$$\mathcal{L} = \sum_k [\tfrac{1}{2}\alpha_k(\partial\phi^k)\cdot(\partial\phi^k) + \tfrac{1}{2}\beta_k(\partial\cdot\phi^k)\cdot(\partial\cdot\phi^k) - \gamma_k\phi^{k-2}\cdot(\partial\partial\cdot\phi^k) + \delta_k\phi^{k-1}\cdot(\partial\cdot\phi^k) - \tfrac{1}{2}\sigma_k m^2\phi^k\cdot\phi^k]. \tag{2.2}$$

All indices will be suppressed; here they are contracted in a unique and self-evident manner.[12] The constant parameters α_k, β_k, γ_k, δ_k, σ_k must be chosen so that the Euler-Lagrange equations yield (1.1), (1.2) and $\phi^k = 0$ for $k \neq s$.

The Euler-Lagrange equations are, for $k = s$, $s-2, \ldots, 0$,[13]

$$\delta\phi^k\cdot[\alpha_k p^2\phi^k + \beta_k p(p\cdot\phi^k) + \gamma_{k+2}(pp\cdot\phi^{k+2}) + \gamma_k pp\phi^{k-2} + i\delta_{k+1}(p\cdot\phi^{k+1}) - i\delta_k p\phi^{k-1} - m^2\sigma_k\phi^k] = 0. \tag{2.3}$$

Here we have written p for $-i\partial$, that is, $p_\mu \equiv -i\partial/\partial x^\mu$. The variations $\delta\phi^k$ are symmetric and traceless, and the equations are not weakened if we specialize

$$(\delta\phi^k)^{\mu_1\cdots\mu_k} \to y^{\mu_1}\ldots y^{\mu_k},$$

where the variables y^μ satisfy $y^2 \equiv y^\mu y_\mu = 0$. Define

$$\phi^k(y) \equiv y^{\mu_1}\ldots y^{\mu_k}\phi_{\mu_1\cdots\mu_k}, \tag{2.4}$$

then Eq. (2.3) becomes

$$\alpha_k p^2\phi^k(y) + (\beta_k/k)(y\cdot p)(p\cdot\partial)\phi^k(y) + \gamma_k(p\cdot y)^2\phi^{k-2}(y) + [\gamma_{k+2}/(k+1)(k+2)](p\cdot\partial)^2\phi^{k+2}(y) - i\delta_k(p\cdot y)\phi^{k-1}(y) - i[\delta_{k+1}/(k+1)](p\cdot\partial)\phi^{k+1}(y) - m^2\sigma_k\phi^k(y) = 0. \tag{2.5}$$

Here ∂ stands for derivation with respect to y, $\partial_\mu = \partial/\partial y^\mu$. We look for plane wave solutions with $\vec{p} = 0$ and $p_0 = p > 0$. The expansion of $\phi^k(y)$ by spin content is of the form

$$\phi^k(y) = \sum_{l=0}^{k} [(y^0)^{k-l} - ay^2(y^0)^{k-l-2} - b(y^2)^2(y^0)^{k-l-4} - \ldots]\phi^{k,l}(\vec{y}) \tag{2.6}$$

with

$$a = (k-l)(k-l-1)/4k,$$
$$b = (k-l)\ldots(k-l-3)/32k(k-1).$$

The equation for $\phi^{k,l}$ is

$$p^2(\alpha_{k,l}\phi^{k,l} + \gamma_k\phi^{k-2,l} + \gamma_{k+2,l}\phi^{k+2,l}) - ip(\delta_k\phi^{k-1,l} - \delta_{k+1,l}\phi^{k+1,l}) = m^2\sigma_k\phi^{k,l} \tag{2.7}$$

with

$$\alpha_{k,l} = \alpha_k + \beta_k(k-l)(k+l+1)/2k^2,$$
$$\gamma_{k,l} = \gamma_k(k-l)(k-l-1)(k+l)(k+l+1)/4k^2(k-1)^2,$$
$$\delta_{k,l} = \delta_k(k-l)(k+l+1)/2k^2. \tag{2.8}$$

The problem is to choose the coefficients so that (2.7) is equivalent to the set $(p^2 - m^2)\phi^{s,s} = 0$, all other $\phi^{k,l} = 0$.

We start with $l = s$, the only contributing value of k is $k = s$ and (2.7) (Ref. 13) reduces to $(\alpha_s p^2 - m^2\sigma_s)\phi^{s,s} = 0$. We normalize by taking $\alpha_s = \sigma_s = 1$. When $l = s-1$ the only contributing value of k is again $k = s$ and we get $p^2(1 + \beta_s/s)\phi^{s,s-1} = m^2\phi^{s,s-1}$. This must yield $\phi^{s,s-1} = 0$ for any value of p^2, so it is necessary to put $\beta_s = -s$. When $l = s-2$, we obtain two equations by taking $k = s$, $s = 1$. Abbreviating $p^2/m^2 \equiv z$ we have

$$\begin{pmatrix} \alpha_{s,s-1}z - 1 & \gamma_s z \\ \gamma_{s,s-1}z & \alpha_{s-2}z - \sigma_{s-2} \end{pmatrix} \begin{pmatrix} \phi^{s,s-2} \\ \phi^{s-2,s-2} \end{pmatrix} = 0. \tag{2.9}$$

This must imply $\phi^{s,s-2} = \phi^{s-2,s-2} = 0$, for every value of z; therefore the determinant of the matrix must be independent of z and $\neq 0$, which gives $\sigma_{s-2} \neq 0$ and

$$\alpha_{s-2} = \sigma_{s-2}(s-1)/s,$$
$$\gamma_s^2 = -\sigma_{s-2}(s-1)^3/(2s-1). \tag{2.10}$$

When $l = s-3$ and $s-4$ one encounters similar problems with matrices of dimension 3 and 4. The result is

$$\beta_{s-2} = \sigma_{s-2}\frac{(s-1)(s-2)^2}{s(2s-1)},$$
$$\alpha_{s-3} = \sigma_{s-3}\frac{(s-2)(2s-3)}{3(s-1)^2}, \tag{2.11}$$
$$(\delta_{s-2}/m)^2 = -\sigma_{s-2}\sigma_{s-3}\frac{s(s-2)^3}{3(s-1)^2(2s-1)},$$

from $l = s-3$, and, from $l = s-4$,

$$\beta_{s-3} = \sigma_{s-3}\frac{(s-2)(s-3)^2}{3(s-1)^2},$$
$$\alpha_{s-4} = \sigma_{s-4}\frac{(s-3)(2s-5)}{2(s-2)(2s-3)},$$
$$\gamma_{s-2} = 0,$$
$$(\delta_{s-3}/m)^2 = -\sigma_{s-3}\sigma_{s-4}\frac{(s-3)^3(2s-1)}{6(s-1)(s-2)(2s-3)}.$$

When $l \leq s-1$ the dimension of the matrix is $s-l$, but matrix elements more than 2 steps away from the diagonal vanish, and it is probably not too hard to obtain general expressions for $\alpha_k, \ldots, \delta_k$. The calculation was completed by Hagen and Singh[11] by a different method, and the above formulas are in agreement with their results.[14]

The vanishing of γ_{s-2} is an unexpected boon, it produces a great simplification in the massless case and obviates the need to solve the equations for $l < s-4$. In fact, when $m=0$, we have $\delta_k = 0$ and the fields ϕ^s, ϕ^{s-2} become decoupled from all the rest. Now, for $l \leq s-2$, only two values of k contribute and (2.7) reduces to

$$p^2 \begin{bmatrix} \alpha_{s,l} & \gamma_s \\ \gamma_{s,l} & \alpha_{s-2,l} \end{bmatrix} \begin{bmatrix} \phi^{s,l} \\ \phi^{s-2,l} \end{bmatrix} = 0 \qquad (\text{for } l \leq s-2). \quad (2.12)$$

Of course, this does not imply that $\phi^{s,l}$ and $\phi^{s-2,l}$ vanish, the Fierz-Pauli program must fail when $m=0$, but it is significant that the determinant of this two-dimensional matrix vanishes for all l.[15] This guarantees the existence of a large gauge group.

III. MASSLESS FIELDS WITH INTEGER SPIN

When $m=0$ the fields ϕ^{s-3}, ϕ^{s-4}, ..., decouple and may be ignored henceforth. The equation for $\phi^{s,s}$ is $p^2\phi^{s,s}=0$, the equation for $\phi^{s,s-1}$ reduces to $0=0$, while the lower spin projections satisfy Eq. (2.12). A solution for which $\phi^{s,s}=0$ is subjected to no wave equation and will be called a gauge field. Let $\{\tilde{\phi}^s, \tilde{\phi}^{s-2}\}$ be a gauge field. The vanishing of $\tilde{\phi}^{s,s}$ is expressed by the possibility of representing $\tilde{\phi}^s$ as a gradient [compare Eq. (2.6)],

$$\tilde{\phi}^s(y) = s(p\cdot y)\xi(y) - \tfrac{1}{2}(s-1)y^2(p\cdot\xi)(y), \quad (3.1)$$

where ξ is a symmetric, traceless tensor of rank $s-1$. To find the corresponding $\tilde{\phi}^{s-2}$ we must solve (2.12), or what is the same, apply the Euler-Lagrange equation (2.5) with $k=s$. The result is

$$\tilde{\phi}^{s-2}(y) = [(s-1)/\gamma_s]p\cdot\partial\xi(y).$$

It is now convenient to combine the two traceless tensors ϕ^s, ϕ^{s-2} into a tensor h of rank s that is not traceless, $h' \equiv \text{trace}\, h \neq 0$, but whose double trace vanishes, trace $h'=0$. At this point we must abandon the notation (2.4) and reintroduce the indices. We define

$$\begin{aligned}\phi_{\mu_1\cdots\mu_s} &= h_{\mu_1\cdots\mu_s} - (1/2s)\textstyle\sum_2 \delta_{\mu_1\mu_2}h'_{\mu_3\cdots\mu_s}, \\ \phi_{\mu_1\cdots\mu_{s-2}} &= ch'_{\mu_1\cdots\mu_{s-2}},\end{aligned} \quad (3.2)$$

and choose the coefficient c to suit our convenience. Our convenience is suited by $c=(s-1)^2/2\gamma_s$, for then a gauge field is represented by the simple form

$$\tilde{h}_{\mu_1\cdots\mu_s} = \textstyle\sum_1 p_{\mu_1}\xi_{\mu_2\cdots\mu_s} \quad (\text{trace } \xi = 0). \quad (3.3)$$

The sums $\sum_2$ in (3.2) and $\sum_1$ in (3.3) symmetrize the tensors that follow by summing over all different permutations. Since g, h', and ξ are all symmetric there are s terms in $\sum_1$ and $s(s-1)/2$ terms in $\sum_2$.

Now we can write the Lagrangian (2.2) in terms of h, taking $m=0$, $\delta_k=0$, and $\alpha_k, \beta_k, \gamma_k$ from the preceding section. The result is, with $p_\mu = i\partial/\partial x^\mu$,

$$\begin{aligned}-\mathcal{L} = {}&\tfrac{1}{2}(ph)'(ph) - (s/2)(p\cdot h)\cdot(p\cdot h) \\ &- (s/2)(s-1)h'\cdot(pp\cdot h) \\ &- (s/4)(s-1)(ph')\cdot(ph') \\ &- (s/8)(s-1)(s-2)(p\cdot h')\cdot(p\cdot h'). \end{aligned}\quad (3.4)$$

We write the corresponding Euler-Lagrange equations, using the notation $\sum_1$, $\sum_2$ as defined above, and include a source term,[16]

$$\begin{aligned}&p^2h_{\mu_1\cdots} - \textstyle\sum_1 p_{\mu_1}p^\nu h_{\nu\mu_2\cdots} + \sum_2 p_{\mu_1}p_{\mu_2}h'_{\mu_3\cdots} \\ &\quad + \textstyle\sum_2 \delta_{\mu_1\mu_2}\Big(p^\nu p^\lambda h_{\nu\lambda\mu_3\cdots} - p^2h'_{\mu_3\cdots} \\ &\quad - \tfrac{1}{2}\textstyle\sum_1 p_{\mu_3}p^\nu h'_{\nu\mu_4\cdots}\Big) = t_{\mu_1\cdots}. \end{aligned}\quad (3.5)$$

The fact that the gauge field (3.3) satisfies the free field equations is expressed by the (true) statement that (3.5) is invariant under the gauge transformation

$$h_{\mu_1\cdots} \to h_{\mu_1\cdots} + \textstyle\sum_1 p_{\mu_1}\xi_{\mu_2\cdots} \quad (\text{trace } \xi = 0). \quad (3.6)$$

Because (3.5) was deduced by variation of a symmetric Lagrangian, this is equivalent to the statement that the traceless part of the divergence of the left-hand side of (3.5) vanishes; therefore, self-consistency of (3.5) requires that the traceless part of the divergence of t vanish, that is,

$$p^\nu t_{\nu\mu_2\cdots} = [\tfrac{1}{2}(s-1)]\textstyle\sum_2 \delta_{\mu_2\mu_3}p^\nu t'_{\nu\mu_4\cdots} \qquad (t' \equiv \text{trace } t). \quad (3.7)$$

In addition, trace $t'=0$.[16] Equations (3.5) and (3.7) were obtained by Schwinger[17] for the case $s=3$, but like others[18] who worked on this problem, he required that t be divergenceless. Our next task is to show that the weaker condition (3.7) is sufficient.

IV. THE PROPAGATOR

Let us abbreviate the wave equation (3.5) by

$$Lh = t \quad [\text{Eq. (3.5)}]. \quad (4.1)$$

The solution is determined up to a gauge field; thus there exists an operator G such that the general solution of (4.1) is given by[19]

$$h = Gt + \tilde{h} . \tag{4.2}$$

Of course, G is not unique. The dimension of G is the same as $1/p^2$; therefore the only acceptable form of G is

$$G = (1/p^2)A , \tag{4.3}$$

where $1/p^2$ is the usual Green's function for scalar fields and A is a constant matrix. From (4.1)–(4.3) it is seen that the operator A must satisfy $(LA - p^2)t = 0$ for every source t that satisfies the condition (3.7); this is the same as the requirement that $(AL - p^2)h$ be a gauge field for every h, and this determines A uniquely.

Equation (3.5) may be rewritten as follows:

$$BL_0 h = t ,$$

where B and L_0 are defined by

$$(L_0 h)_{\mu_1 \cdots} = p^2 h_{\mu_1 \cdots} - \sum{}_1 p_{\mu_1} p^\nu h_{\nu\mu_2 \cdots} + \sum{}_2 p_{\mu_1} p_{\mu_2} h'_{\mu_3 \cdots} , \tag{4.4}$$

$$(Bh)_{\mu_1 \cdots} = h_{\mu_1 \cdots} - \tfrac{1}{2} \sum{}_2 \delta_{\mu_1\mu_2} h'_{\mu_3 \cdots} . \tag{4.5}$$

Note that Bh is not the traceless part of h, compare Eq. (3.2), therefore B has an inverse A, namely

$$(Ah)_{\mu_1 \cdots} = h_{\mu_1 \cdots} - [1/2(s-1)] \sum{}_2 \delta_{\mu_1\mu_2} h'_{\mu_3 \cdots} . \tag{4.6}$$

Now we verify that $(AL - p^2)h$ is a gauge field,

$$[(AL - p^2)h]_{\mu_1 \cdots} = [(L_0 - p^2)h]_{\mu_1 \cdots} = \sum{}_1 p_{\mu_1} \xi_{\mu_2 \cdots} , \tag{4.7}$$

$$\xi_\mu \ldots = -(p \cdot h)_{\mu_2 \cdots} + \tfrac{1}{2} \sum{}_1 p_{\mu_2} h'_{\mu_3 \cdots} .$$

To conclude, Eq. (4.1) is solved by (4.2), with G defined by (4.3) and (4.6); this operator will be called the propagator for the field h.

We are now in a position to determine the precise dynamical content of the theory. It will be shown that the only massless quanta transmitted between sources are those of helicity $\pm s$.

V. MASSLESS PARTICLES

The elementary interaction between field and source is $h \cdot t$. If h is radiated by the source, then h is represented by (4.2) and the effective interaction between sources is $t \cdot G \cdot t$. As far as the value of this expression is concerned, the choice of G is irrelevant[19] and we may as well adopt (4.3) to obtain $(1/p^2) t \cdot A \cdot t$. The residue $t \cdot A \cdot t$ of the pole at $p^2 = 0$ gives us the amplitude for the transmission of a massless quantum between different parts of the source. By examining this residue we can determine the helicities of these quanta.

Theorem. If t satisfies Eq. (3.7), and trace $t' = 0$, and A is defined by (4.6), and $p^2 = 0$, then $t \cdot A \cdot t$ is the sum of the squares of the helicity components $\pm s$ of t.

The proof will be presented below. Of course, it it would be a trivial matter to prove this result if t were divergence free. The fact that the weaker condition (3.7) is sufficient is remarkable. It is possible to imagine that this relaxation of the condition on the divergence will facilitate the discovery of an interesting physical candidate for the source of massless fields with higher spins.

Proof of the theorem.[20] Let ξ be any traceless, symmetric tensor of rank $s-1$, then $t \cdot (\sum_1 p\xi) = 0$ (all indices contracted). Consequently,

$$t \cdot A \cdot t = t \cdot (A \cdot t + \sum{}_1 p\xi) = t \cdot A \cdot (t + B \sum{}_1 p\xi) = t \cdot A \cdot \check{t} ,$$

where $\check{t} \equiv t + B \sum_1 p\xi$ and

$$p \cdot \check{t} = [1/2(s-1)] \sum{}_2 \delta t' - \sum{}_2 \delta (pp \cdot \xi) + p^2 \xi .$$

Choose ξ to be any traceless solution of $pp \cdot \xi = t'/2(s-1)$, then $p \cdot \check{t} = p^2 \xi$, and when $p^2 = 0$:

$$t \cdot A \cdot t = t \cdot A \cdot \check{t} = \left(t \cdot A + \sum{}_1 p\xi \right) \cdot \check{t} = \check{t} \cdot A \cdot \check{t} .$$

Choose the coordinate axes so that $p_1 = p_2 = 0$, then the index summations in $\check{t} \cdot A \cdot \check{t}$ range over the index values 1,2 only. Since trace $t' = 0 = \sum \check{t}^{ijkl\cdots} \delta_{ij} \delta_{kl} (i, \ldots, l = 1, 2)$, it follows that

$$\left(\check{t} - [1/2(s-1)] \sum{}_2 \delta \check{t}' \right)^{ij\cdots} \delta_{ij} = 0 = \text{trace } A \cdot \check{t}$$

so that only the traceless part of $\check{t}$ contributes; that is, only the highest and lowest helicities contribute to $t \cdot A \cdot t$.

VI. NONLINEAR THEORY

We propose a generalized Gupta program, for arbitrary spin.[21] Let us suppose that a soure t can be found, constructed in terms of matter fields, that satisfies the condition (3.7) by virtue of the field equations for these fields, and such that trace $t' = 0$. The new interaction term $h \cdot t$ will change the field equations and (3.7) will cease to hold. To compensate, extra terms must be added to the source and a very nonlinear theory will result, just as in the case of spin 2.[22] This must lead to a deformation[23] of the gauge group defined by (3.6), and we may ask whether such deformations exist.

Recall that, when $s=2$, the deformed gauge group is the group of general coordinate transformations.[5] The infinitesimal generators form the Lie algebra of differentiable vector fields. If we write the deformed gauge transformations as

$$h^{\mu\nu}\to h^{\mu\nu}+p^{\mu}\xi^{\nu}+p^{\nu}\xi^{\mu}+\text{nonlinear terms} \tag{6.1}$$

(Ref. 24) then we can write the structure relations exactly as

$$[\xi,\eta]^{\mu}=\xi^{\nu}\partial_{\nu}\eta^{\mu}-\eta^{\nu}\partial_{\nu}\xi^{\mu}\,. \tag{6.2}$$

In the case of spin 3 it is also possible to associate the set of gauge fields with a noncommutative Lie algebra, for example,

$$[\xi,\eta]_{\mu\nu}=\omega^{\alpha\beta}(\xi_{\mu\alpha}\eta_{\beta\nu}-\eta_{\mu\alpha}\xi_{\beta\nu})\,,$$

where ω is an antisymmetric tensor field. Therefore deformations of the gauge group do exist. The next question is whether the deformed gauge group admits an affine representation of the form

$$h_{\mu\nu\lambda}\to h_{\mu\nu\lambda}+\sum{}_{1}p_{\mu}\xi_{\nu\lambda}+\text{nonlinear terms}$$

(Ref. 24). Finally, it would still remain to be seen whether a theory of interacting fields exists that is invariant under the deformed gauge group.

Some work has been done on a more conventional approach to the problem. Consider matter described by initially free, scalar fields, satisfying the Klein-Gordon equation $(\partial^2+m^2)\varphi=0$. It turns out to be possible to construct a source t that satisfies $t''=0$ and, by virtue of the Klein-Gordon equation, condition (3.7). This ensures consistency to zeroth order in the coupling. The simplest test of consistency to first order in the coupling was devised by Weinberg,[18] in S-matrix terms. Weinberg concluded, on the basis of such an analysis, that the coupling of scalar particles to massless particles of spin higher than 2 must vanish at zero energy. We have repeated this calculation and we have found that the less severe conservation law (3.7) leads to the same conclusion.

It is possible to envisage several alternative possibilities. (1) Massless particles of spin higher than 2 couple to matter, but the coupling vanishes at zero energy. (2) Couplings exist that do not vanish at zero energy, but these are of a very special kind and probably involve fields of high spin. (3) The problem of consistency is resolved cooperatively by massless fields of all spins.

The third possibility is suggested by supergravity; it seems that massless fields of spin $\frac{3}{2}$ cannot couple without the cooperation of gravity. (See Grisaru and Pendleton, Ref. 21.) Perhaps massless fields of spin 3 can couple only with the help of other massless fields of integer spin. The analogy with supergravity suggests a higher symmetry that unites massless particles of all spins. In fact, it is possible to deform the direct product of the gauge groups for spins 1, 2, ... and obtain a simple, non-Abelian gauge group that includes the gauge groups of electrodynamics and of Einstein's theory of gravitation.

APPENDIX

Here we shall show that the Fierz-Pauli program for integer spin s needs at least s traceless, symmetric tensor fields. Let the wave equation for the total field be written as

$$(pp\cdot A+p\cdot B-1)\Phi=0\,, \tag{A1}$$

where A and B are constant tensors of rank 2 and 1, respectively. Linearize by introducing $\Psi_\mu=p_\mu\Phi$,

$$\begin{pmatrix}-\delta_\nu{}^\mu & p_\nu\\ p^\lambda A_{\lambda\nu} & p\cdot B-1\end{pmatrix}\begin{pmatrix}\Psi_\mu\\ \Phi\end{pmatrix}=0$$

or

$$\begin{pmatrix}0 & p\\ p\cdot A & p\cdot B\end{pmatrix}\begin{pmatrix}\Psi\\ \Phi\end{pmatrix}=\begin{pmatrix}\Psi\\ \Phi\end{pmatrix}. \tag{A2}$$

Project this equation into the subspace with spin l, with dimension $5d_l$, then there must be no nontrivial solution for $l<s$; therefore the projected matrix M_l must be nilpotent for $l<s$. This implies that the $2d_l$ power of M_l must vanish for $l<s$, so that

$$\begin{pmatrix}0 & p\\ p\cdot A & p\cdot B\end{pmatrix}^{2d}=M_s{}^{2d}\,, \tag{A3}$$

since $d\equiv d_0$ is the largest among the d_l. Now let Θ be the projection operator defined by $\Theta\Phi=\phi^{s,s}$, the spin-s component of the tensor field ϕ^s. Then

$$M_s=\begin{pmatrix}0 & p\\ p & 0\end{pmatrix}\otimes\Theta\,,$$

$$M_s{}^{2d}=(p^2)^{d-1}\begin{pmatrix}pp & 0\\ 0 & p^2\end{pmatrix}\otimes\Theta\,.$$

Now (A2) and (A3) show that $(p^2)^d\Theta$ is a polynomial in p_μ, which implies that $d\geq s$. But d is precisely the number of traceless, symmetric tensor fields, so the stated result has been proved.[8]

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[3]See, nevertheless, the interesting attempts by M. Flato and P. Hillion, Phys. Rev. D 1, 1667 (1970); R. Penrose, Proc. R. Soc. London A284, 159 (1964); A. Salam and J. Strathdee, Phys. Lett. 49B, 465 (1974).
[4]See, e.g., J. Fang and C. Fronsdal, UCLA report, 1978 (unpublished). This is a review of the Gupta program, which aims to recover Einstein's theory of gravitation from the point of view of a theory of massless, spin-2 fields in flat space, emphasizing the role of gauge invariance.
[5]M. Fierz and W. Pauli, Proc. R. Soc. London A173, 211 (1939).
[6]E. P. Wigner, Ann. Math. 40, 149 (1939).
[7]V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. USA 34, 211 (1948).
[8]C. Fronsdal, Nuovo Cimento Suppl. 9, 416 (1958). There the discussion is limited to the case when no first-order derivatives appear in the field equations. The proof is completed in the Appendix.
[9]To simplify the printing, we leave off the superscript that indicates the rank of a tensor field whenever this can be done without creating ambiguities.
[10]M. Kawasaki and M. Kobayashi, Phys. Rev. D 17, 446 (1978).
[11]L. P. S. Singh and C. R. Hagen, Phys. Rev. D 9, 898 (1974).
[12]For example: $(\partial\phi)\cdot(\partial\phi) \equiv (\partial^\mu\phi^\nu\cdots)(\partial_\mu\phi_\nu\cdots)$.
[13]The parameters α_k, $\cdots$, σ_k are taken to vanish for $k > s$, and in addition $\gamma_{s-1} = \delta_s = \delta_{s-1} = 0$, in accordance with (2.1) and (2.2).
[14]There is a slight disagreement in the expressions for δ_{s-2} and δ_{s-3}; this is due to a misprint. L. P. S. Singh has kindly informed me that Eq. (17) of Ref. 11 should be corrected by replacing the factor $q-1$ by $q-2$.
[15]Note that the matrix elements are known for all l by (2.8) in terms of $\alpha_s = 1$, $\beta_s = -s$, and α_{s-2}, β_{s-2}, γ_s given by (2.10) and (2.11).
[16]Because we insist on a variational principle, the source term must be represented by an additional term, $h\cdot t$, in the Lagrangian density. The variation δh, like h itself, is symmetric and of vanishing double trace, trace $(\delta h)' = 0$; therefore the source term in (3.5) will automatically have these properties.
[17]J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, Mass., 1970), see p. 169.
[18]S. Weinberg, Phys. Rev. 135, B1049 (1964).
[19]We are interested in the freedom allowed by gauge invariance. Of course, the general solution can be written down as in (4.2) only in the context of specific boundary conditions; these give meaning to the symbol $1/p^2$ in (4.3).
[20]A proof consisting of direct calculation was given in a preprint of this paper.
[21]The original Gupta program, for $s = 2$, was reviewed in Ref. 4. The case $s = \frac{3}{2}$ should be of particular interest for supergravity; see M. T. Grisaru and H. N. Pendelton, Phys. Lett. 67B, 323 (1977). For the case of several fields of spin 1, see e.g., V. I. Ogievetskii and I. V. Polubarinov, Zh. Eksp. Teor. Fiz. 45, 966 (1963) [Sov. Phys.—JETP 18, 668 (1964)].
[22]S. N. Gupta, Phys. Rev. 96, 1683 (1954).
[23]A deformation in the sense of M. Gerstenhaber, Ann. Math. 79, 59 (1964).
[24]"Nonlinear" here includes terms that are bilinear in h and ξ.