Space-time orbits for interacting relativistic particles: Syntactic versus semantic observables

Arthur Komar
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033
(Received 21 November 1977)

Recent work establishing a violation of the so-called "no-interaction theorem" for interacting relativistic particles was directed primarily toward the development of the quantum theory of such systems. In that work the classical trajectories for two-particle systems span a two-dimensional region in phase space. The present paper addresses itself more directly to the intended classical content of the "no-interaction" theorem, namely, the determination of orbits in space-time for the pair of particles under consideration. The problem is nontrivial in the sense that its treatment requires the recognition of the conceptual distinction between syntactically defined observables and semantically defined observables. The latter betray Machian features in so far as they make reference to the open character of the sytem vis-a-vis the other matter of the universe. It is noteworthy and curious that such considerations were not required for the proper treatment of the quantized system. A critique of the essential features of the "no-interaction" theorem, including certain unstated (and hard to justify) assumptions, is presented.

I. INTRODUCTION

The so-called "no-interaction theorem[1]" asserts that the only relativistically covariant Hamiltonian systems such that the coordinate variables of the individual particles transform correctly under Lorentz transformations are collections of free particles. In a recent paper[2] we demonstrated that this theorem can be violated provided we are prepared to consider Hamiltonian systems of a more complex sort than those envisaged in the proof of the theorem. In particular, we showed that, for a system of two interacting relativistic particles, a manifestly covariant Hamiltonian description of the dynamics can be obtained in a 16-dimensional phase space. The new feature is that the trajectories of the system are generated by two commuting Hamiltonian constraints rather than by a single Hamiltonian. As a consequence, the points in phase space which describe trajectories span a two-dimensional region. Although this causes no conceptual difficulties when we quantize the system, the question remains as to how, should we choose to confine ourselves to the classical theory, can we recover the covariant space-time particle orbits which the "no-interaction theorem" claims cannot exist? It is the intention of this paper to demonstrate how this is accomplished. In the process we shall find that the concept of an *observable* has to be sharpened; that the usual rather loose employment of this term obscures the fact that there are two distinct conceptual usages, one syntactic and the other semantic; and most surprisingly, not only is there no unique relationship between syntactic and semantic observables, but differing identifications will in general produce physically distinct particle orbits.

Although the need for such careful considerations becomes most acute when we treat interacting systems, the problem of the identification of the two modes of observables and the effect it has upon the description of the space-time orbits occurs for free particles as well, although the linear nature of the orbits of the individual particles has tended to obscure this fact. Since, for free particles, the (two-dimensional) phase-space trajectories as well as the space-time orbits can be readily exhibited in closed form, it will be particularly clarifying if we first analyze this simple system.

II. ONE FREE PARTICLE

Let us first review how one can treat the dynamics of a single free particle in a manifestly covariant manner. As per usual, we begin at the most primitive level by the introduction of the Lorentz four-vector q^μ, by which we intend to describe the location of the particle in space and time. In order to entertain dynamical considerations, we shall next introduce the canonically conjugate four-vector p_μ. That is, we are choosing to describe the kinematics of the system in an eight-dimensional phase space with a simplectic structure given by the fundamental Poisson bracket relations

$$[q^\mu, q^\nu] = 0,\ [p_\mu, p_\nu] = 0,$$
$$[q^\mu, p_\nu] = -[p_\nu, q^\mu] = \delta^\mu_\nu . \tag{2.1}$$

The intended (i.e., "semantic" or "physical") meaning of the kinematical variables p_μ is that it will represent the energy-momentum four-vector of the particle. Poisson brackets between more general functions of the q^μ and p_μ are defined in the usual manner; namely,

$$[A(q,p), B(q,p)] \equiv \frac{\partial A}{\partial q^\mu}\frac{\partial B}{\partial p_\mu} - \frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial q^\mu}. \tag{2.2}$$

Employing this Poisson bracket algebra and introducing the Minkowski metric of signature (1, −1, −1, −1) to raise and lower indices, it easily is seen that the infinitesimal algebra of the Poincaré group is realized by the ten quantities $p^\mu, L^{\mu\nu} \equiv q^\mu p^\nu - q^\nu p^\mu$ and that q^μ and p_μ transform correctly as Lorentz vectors in this realization

Dynamics is introduced by imposing the constraint which asserts that our system is to be a free particle with (rest) mass m; namely,

$$K \equiv p^2 - m^2 = 0\,. \tag{2.3}$$

Equation (2.3) defines a seven-dimensional constraint hypersurface of two sheets in the eight-dimensional phase space. Since our considerations in this paper are entirely classical we shall confine our considerations exclusively to the sheet of positive energy and regard that sheet as the entire constraint hypersurface. Employing the Poisson bracket structure, the constraint K may be used to generate trajectories. More specifically, if we iterate the relations

$$\delta q^\mu \equiv \tau[q^\mu, K] = 2\tau p^\mu\,, \tag{2.4a}$$

$$\delta p_\mu \equiv \tau[p_\mu, K] = 0 \tag{2.4b}$$

(where τ is a constant infinitesimal parameter) we obtain a unique (one dimensional) path through each point in the phase space. If we now confine ourselves to the consideration of initial points lying on the constraint hypersurface given by Eq. (2.3), it is clear from Eq. (2.4b) that the entire path so generated lies on the constraint hypersurface. We shall call such a path a dynamical trajectory. We note that the factor space obtained by considering the points of the constraint hypersurface modulo the trajectories forms the reduced phase space of six dimensions which it is more customary to employ when treating the dynamics of a simple particle. (The induced coordinates of this reduced phase are in effect the independent Cauchy data for the trajectories as determined by Hamilton's equations of motion.)

Observables are defined purely syntactically as those functions over the phase space which commute with the constraints on the constraint hypersurface. Such *syntactic* observables are thereby constant along each trajectory and generate mappings which preserve the constraint hypersurface. If we have a sufficient number of observables we can uniquely define a trajectory by assigning values to those observables. For our example of a single free particle, in view of the fact that the reduced phase space is six dimensional, we see that in order to define a trajectory it is necessary and sufficient to specify six independent observables. One particularly simple choice is

$$p^s = a^s\,, \tag{2.5a}$$

$$L^{0s} = q^0 p^s - q^s p^0 = b^s\,. \tag{2.5b}$$

Using this choice of constants a^s and b^s, and employing the constraint equation (2.3), we may solve Eq. (2.5) for q^s; thus

$$q^s = (\vec{a}^2 + m^2)^{-1/2}(a^s q^0 - b^s)\,. \tag{2.6}$$

This equation, taken together with equation (2.5a), explicitly exhibits the one-parameter trajectory on the constraint hypersurface in phase space, where the variable q^0 is employed to label the points on the trajectory.

If we now wish to relate this trajectory with an orbit which one can observe in space-time, we must relate at least some of the phase-space variables with *semantic* observables, that is, with distances, times, and velocities as measured by rulers and clocks in the frame of reference of some observer. If we call the space-time coordinates of the observer x^μ, the natural and intended relationship implicit in the notation we have employed is

$$q^\mu = x^\mu\,, \tag{2.7}$$

This identification of the semantic observables x^μ with the phase-space variables q^μ has the virtue that it is Lorentz covariant, but it is important to note that the q^μ are *not* syntactic observables. If we now substitute Eq. (2.7) into Eq. (2.6) we obtain the usual space-time orbit for a free particle, the experimental observation of which physically identifies the system. It is evident that all the syntactic observables can be expressed as functions of semantic observables provided, in addition to Eq. (2.7), that we establish the standard intended relationship between the (syntactic observable) canonical momenta $\vec{p}$ and the semantic observable velocities $\vec{v} \equiv d\vec{x}/dx^0$; namely,

$$\vec{p} = \frac{m\vec{v}}{(1-v^2)^{1/2}}\,. \tag{2.8}$$

III. TWO FREE PARTICLES

The discussion of the previous section must at first sight seem unnecessarily involved. The need for proceeding in such detail will become clear once we consider the much more complex system of two free particles. Let us now consider a 16-dimensional phase space coordinatized by the canonical pairs of Lorentz vectors $q_i^\mu, p_{i\mu}$ $(i = 1, 2)$, which satisfy the standard Poisson bracket relations

$$[q_i^\mu, q_j^\nu] = [p_{i\mu}, p_{j\nu}] = 0\,,$$
$$[q_i^\mu, p_{j\nu}] = -[p_{j\nu}, q_i^\mu] = \delta_\nu^\mu \delta_{ij}\,. \tag{3.1}$$

Poisson brackets of more general functions are defined, as in Eq. (2.2), merely by extending the implied summation on the right-hand side over index i as well. The algebra of the Poincaré group is now realized by the Poisson bracket relations among the ten infinitesimal generators

$$P^\mu \equiv p_1^\mu + p_2^\mu , \qquad (3.2a)$$

$$L^{\mu\nu} \equiv q_1^\mu p_1^\nu - q_1^\nu p_1^\mu + q_2^\mu p_2^\nu - q_2^\nu p_2^\mu , \qquad (3.2b)$$

and it is again easily seen that q_i^μ and $p_{i\mu}$ tranform as Lorentz vectors under the associated mappings in phase space.

The dynamics of two free particles of respective rest mass m_i is most naturally introduced by imposing the two mutually commuting Lorentz-invariant constraints

$$K_i = p_i^{\,2} - m_i^{\,2} = 0 \quad (i = 1, 2) . \qquad (3.3)$$

These two constraints define a 14-dimensional hypersurface of 4 sheets in the phase space. As in the previous section, we shall only consider the sheet corresponding to the positive roots of Eq. (3.3) as the constraint hypersurface.

Dynamical trajectories are obtained by employing the constraints K_i as infinitesimal generators of canonical mappings and restricting out attention to initial points on the constraint hypersurface. More specifically, for points q_i^μ, p_i^μ of the constraint hypersurface we iterate the infinitesimal mapping

$$\begin{aligned} \delta q_i^\mu &= [q_i^\mu, \tau_1 K_1 + \tau_2 K_2] , \\ \delta p_i^\mu &= [p_i^\mu, \tau_1 K_1 + \tau_2 K_2] , \end{aligned} \qquad (3.4)$$

where τ_i are arbitrary infinitestimal constants. Since the constraints K_i commute, there is no ambiguity in exponentiating this procedure and it is evident that the resulting paths are confined in their entirety to the constraint hypersurface. Thus through each point on the constraint hypersurface a two-parametric family of paths is obtained. We now define a dynamical trajectory to be an equivalence class of points which can be connected piecewise by paths so generated. Each trajectory so defined clearly spans a two-dimensional region on the 14-dimensional constraint hypersurface, and through each point of the hypersurface there is a unique trajectory. If we now consider the equivalence class of points on the constraint hypersurface modulo trajectories we recover the 12-dimensional reduced phase space which is customarily employed in treating the two-particle system.

As in the previous section, we define *syntactic* observables as those functions of the canonical coordinates which commute with the constraints. [Clearly all the generators of the Poincare group, Eq. (3.2), are syntactic observables.] Such observables are constant throughout a trajectory and by assigning fixed values to a complete set of them (12 in the present case) a specific trajectory is uniquely determined. Analogous to Eq. (2.5) a particularly simple choice is

$$p_i^s = a_i^s , \qquad (3.5a)$$

$$q_i^0 p_i^s - q_i^s p_i^0 = b_i^s . \qquad (3.5b)$$

Similarly we employ the constraints, Eqs. (3.3), and solve Eqs. (3.5b) for q_i^s, thus

$$q_i^s = (\vec{a}_i + m_i^{\,2})^{-1/2} (a_i^s q_i^0 - b_i^s) . \qquad (3.6)$$

We see explicitly from these expressions that the trajectory in phase space is two dimensional, being parametrized by q_1^0 and q_2^0. Owing to the particularly simple nature of this system it would appear that we really have two independent linear trajectories, one for each particle, each parametrized independently by its own parameter. But this is decidedly misleading. One must remember that the path being described by Eqs. (3.5a) and (3.6) is a two-parametric sequence of points on a 14-dimensional surface in a 16-dimensional phase space. In order to relate this system to orbits in space-time we must determine the relationship between the canonical dynamical variables q_i^μ, p_i^μ and the *semantic* observables of space-time.

As in the previous section, the *semantic* observables are the quantities which can be measured in effect by clocks and rulers in some Lorentz frame of reference. That is, we now have as semantic observables the coordinates of the first particle x_1^s, the coordinates of the second particle x_2^s, the time t as measured by a standard clock, as well as derived quantities such as the velocities $v_1^s \equiv dx_1^s/dt$ and $v_2^s \equiv dx_2^s/dt$. The intended meaning of our phase-space variables is established by the relations

$$q_i^s = x_i^s , \qquad (3.7a)$$

$$q_1^0 = q_2^0 = t , \qquad (3.7b)$$

and

$$p_i^s = \frac{m_i v_i^s}{(1 - v_i^{\,2})^{1/2}} . \qquad (3.8)$$

The substitution of Eqs. (3.7) into Eqs. (3.6) will indeed give two independent linear orbits in space-time. The difficulty is that, unlike the analogous situation of Eq. (2.7), Eqs. (3.7) are not Lorentz-covariant statements. This is most easily seen by noting that the Lorentz vector $q^\mu \equiv q_1^\mu - q_2^\mu$ is required by Eq. (3.7b) to have $q^0 = 0$, a property which is not preserved under Lorentz transformations. It follows that the pair of orbits in space-time which are associated with a given phase-space trajectory depend critically on the frame of reference in which the semantic observables are related to phase-space variables, and through them to the syntactic observables. We see in fact that the phase-space trajectories form a two-dimensional domain precisely due to the extra freedom required to permit different pairings of linear space-time orbits to be associated with a given trajectory. The differing pairings result from the exploitation of the frame-dependent nature of the relationship between the semantic observables and and the syntactic observables.

IV. INTERACTING PARTICLES

When the system under consideration consists of several free particles the orbits of the individual particles are evidently linear in space and time. The multidimensional character of the trajectories appears to be an irrelevant artifice of the phase-space formalism which insists on treating the several particles as a single system. Once we introduce an interaction between the particles, it becomes clear that we are no longer dealing with an artifice, but rather that, depending upon the choice of the frame of reference in which the connection is made between the semantic and the syntactic observables, the associated pairs of space-time orbits corresponding to a given phase-space trajectory will have materially different configurations. Of course, having identified the orbit pair in some one reference frame, we are free to transform it to any other Lorentz frame and it will remain a legitimate orbit pair.

The kinematics of the system of two interacting relativistic particles is identical to that of the free particles treated in the previous section, including the realization of the Poincaré algebra given by Eqs. (3.2). The dynamics of the interaction is most conveniently expressed if we first perform a linear canonical transformation to the new set of canonical momenta

$$P^\mu = p_1^\mu + p_2^\mu , \qquad (4.1a)$$

$$p^\mu = p_1^\mu - p_2{}^\mu , \qquad (4.1b)$$

and their canonically conjugate coordinates

$$X^\mu = \tfrac{1}{2}(q_1^\mu + q_2^\mu) , \qquad (4.2a)$$

$$\chi^\mu = \tfrac{1}{2}(q_1^\mu - q_2^\mu) , \qquad (4.2b)$$

respectively. In terms of these variables the modified constraint equations are[2]

$$K_\alpha \equiv P^2 + p^2 - 2m_1{}^2 - 2m_2{}^2 - 8\mu V(r) = 0 , \qquad (4.3a)$$

$$K_\beta \equiv p \cdot P - m_1{}^2 + m_2{}^2 = 0 , \qquad (4.3b)$$

where V is an arbitrary function, and r is the Lorentz scalar given by

$$r = 2\left[\frac{(p\cdot\chi)^2}{P^2} - \chi^2\right]^{1/2} . \qquad (4.4)$$

The coefficient 8μ, where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (4.5)$$

is introduced in order that, in the nonrelativisitic limit, the function V reduces to the usual potential energy. It is easily seen that for vanishing V, the constraints of Eqs. (4.3) are completely equivalent to those of Eq. (3.3) for two free particles. Our preference for the present form of the constraints is due in part to the fact that they permit us to exhibit in closed form their canonical conjugates even in the presence of an interaction; namely, if we define the two functions

$$T_\alpha \equiv \frac{P^2 X\cdot P - p\cdot P \chi\cdot P}{P^4 - (p\cdot P)^2} , \qquad (4.6a)$$

$$T_\beta \equiv \frac{P^2 \chi\cdot P - p\cdot P X\cdot P}{P^4 - (p\cdot P)^2} , \qquad (4.6b)$$

one may check by direct computation that

$$[T_\alpha, T_\beta] = [T_\alpha, K_\beta] = [T_\beta, K_\alpha] = 0 , \qquad (4.7a)$$

$$[T_\alpha, K_\alpha] = [T_\beta, K_\beta] = 1 . \qquad (4.7b)$$

We note that whether or not we have an interaction potential the quantities T_α and T_β are linear in the configuration-space variables. It is evident from Eqs. (4.7b) that they are not observables, but rather they act as times in the sense that the numerical values which they assume serve to parametrize the points in phase space which comprise a trajectory.

For an arbitrary potential we cannot hope to exhibit a complete set of syntactic observables in closed form, for that would be equivalent to obtaining a closed-solution-form solution to the equations of motion. We can employ T_α, however, as we normally would the time in order to exhibit observables as initial data for the determination of a trajectory. As in the previous section, we require 12 functionally independent syntatic observables to determine a trajectory; then, with assigned values for T_α and T_β, and employing the two constraint relations, Eqs. (4.3), we see that we shall have exactly 16 relations to determine the coordinates of a unique point on the trajectory. In view of the fact that the generators of the Poincaré group, Eqs. (3.2), commute with the present constraint equations (4.3), we immediately have ten independent syntactic observables P^μ and $L^{\mu\nu}$. We need two more such observables and must make explicit reference to the specific form of the constraints in order to obtain them. A convenient procedure is to observe that the two Lorentz scalars, r of Eq. (4.4) and

$$s \equiv \chi^\mu \left(p_\mu - P_\mu \frac{p \cdot P}{P^2}\right) \quad (4.8)$$

both commute with the constraint K_β. It is therefore sufficient to exponeniate the action of K_α upon these scalars, employing its canonical conjugate T_α as an effective parameter. More explicitly, it is easily confirmed that the two quantities R and S, given by

$$R \equiv e^{T_\alpha [K_\alpha,]} r = r + T_\alpha [K_\alpha, r] + \frac{T_\alpha{}^2}{2!} [K_\alpha, [K_\alpha, r]] + \cdots , \quad (4.9a)$$

$$S \equiv e^{T_\alpha [K_\beta,]} s = s + T_\alpha [K_\alpha, s] + \frac{T_\alpha{}^2}{2!} [K_\alpha, [K_\alpha, s]] + \cdots , \quad (4.9b)$$

where T_α is simply the expression given by Eq. (4.6a), are syntactic observables which are functionally independent of P^μ and $L^{\mu\nu}$. We note that, if we regard the values T_α and T_β as parameters which describe the evolution of the system point along a trajectory, our complete set of observables is independent of T_β. The observables R and S are the "initial" values of r and s, respectively, in the sense of coinciding in values at points where $T_\alpha = 0$. In this intrinsic sense we have a nontrivial evolution in only one dimension despite the fact that the trajectory spans a two-dimensional region of phase space.

In order to obtain the space-time orbits associated with a given trajectory we proceed precisely as we did in the previous section, introducing the semantic observables and relating them to the phase-space coordinates via Eqs. (3.7) and (3.8). The fact that our trajectory is again a two-dimensional object is a consequence of the fact that this identification is frame dependent. There is clearly a preferred frame in which one can make the semantic identification, namely, the center-of-mass rest frame given by the condition

$$P^s = 0 . \quad (4.10)$$

The Lorentz-invariant statement of the identification of our phase-space variables q_1^0 and q_2^0 with the semantic time in the rest frame is evidently the auxiliary condition

$$p \cdot P T_\alpha + P^2 T_\beta \equiv \chi \cdot P = 0 . \quad (4.11)$$

With this additional condition we obtain a unique Lorentz-covariant pair of orbits in space-time associated with each phase-space trajectory. However, we would loose considerable insight into the nature of relativistic action at a distance if we were to discard the other pairs of orbits obtained by alternative semantic identifications which alter Eq. (4.11).

In the formal structure of the theory $\chi \cdot P$ does not commute with the constraints K_α and K_β, but only with the linear combination

$$K = K_\alpha - \frac{p \cdot P}{P^2} K_\beta . \quad (4.12)$$

Thus only the one-dimensional subset of the trajectory generated by K can be identified with the orbits of the two particles which interact instantaneously in the center-of-mass frame via the potential $V(r)$ (where r is the Euclidean distance of separation). Should we choose to identify the coordinates q_1^0 and q_2^0 with the semantic time of an observer moving relative to the center of mass of the system, it is still true that the particles interact via *the same potential* $V(r)$, *where* r *again is interpreted as a Euclidean distance of separation in the rest frame*. What has changed is that the Euclidean separation employed is no longer that of points of the two orbits *simultaneous in the center-of-mass frame*, but rather they are *simultaneous in the observer's frame*. It is therefore not surprising that with this alternative identification of semantic observables, the pair of space-time orbits associated with the given trajectory changes. It is in this fashion that action at a distance can be made consistent with relativity—namely, we have action at a distance in phase space, but the precise significance of the distance between the particles or of the times when the particles are to be regarded in interaction in space-time is ambiguous. The trajectory is the equivalence class of all such identifications. Even more surprising is that, despite the fact that we apparently have a plethora of inequivalent pairs of orbits in space-time corresponding to a given phase-space trajectory, the corresponding quantum theory is insensitive to these classical considerations, yielding a unique energy spectrum and state vector.[2] The nonobservable classical parameters are absorbed into the arbitrary phase.

V. CONCLUSION

We have seen that, contrary to the strictures of the "no-interaction" theorem,[1] not only is it possible to obtain Hamiltonian space-time orbits for interacting classical particles, we have an *embarras des richesses*. In order to reduce this wealth to familiar proportions we can choose to confine ourselves to the orbits associated with the subtrajectory generated by the Hamiltonian constraint K of Eq. (4.12). It is a particularly natural and simple choice. However, to do so would obscure the essential relativistic features of our system.

In nonrelativistic physics one could also have introduced the conceptual distinction between syntactic and semantic observables. It would, however, have been, for most purposes, forced and inconsequential. The reason for this is that there is indeed an intended meaning to the symbols q_i^μ and p_i^μ which we employ to specify our physical system, and this intended meaning is readily preserved under the Galilean group which relates the laboratory frames of equivalent classical observers. In relativistic physics we have seen that such a unique identification of our symbols with their intended standard interpretation is only possible for a single particle. Once we introduce two or more particles, we see that classical constructions, such as configuration space or phase space, become ambiguous, loosing their categoric relationship to the properties of particles in space-time. In order to give semantic content to the abstract symbols of the relativistic many-body theory we have been forced to make explicit reference to the coordinate frame of a particular observer. Stated differently, we cannot have an unambiguous interpretation of a closed relativistic classical many-particle system. In order to interpret the system physically we are compelled to open the system in a Mach-type fashion, making explicit reference to structures definable in, and to inertial properties of, the laboratory of a given observer. We emphasize that in relativistic theory this last step is no longer benign.

We should note at this point that the existence of conceptual difficulties in relating covariant quantities to laboratory (read "semantic") observables was noted in a somewhat more rudimentary form in an early exchange between Eddington[3] and Dirac, Peierls, and Pryce.[4] It was recognized in that discussion that an unambiguous Lorentz-invariant identification of four vectors with laboratory coordinates and time only becomes available for the isolated free particle. The new feature which we have observed is that for several particles, differing identifications give differing permissible space-time orbits, and that the phase-space trajectory is no longer one dimensional. Our observation that different Lorentz observers would have the particles interact at different points of their space-time orbit, was noted earlier by Thomas[5] and led him to abandon hope of a description of such systems by means of invariant space-time orbits. We have shown how such orbits may be recovered, but we are in agreement with Thomas to the extent that the orbits are not unique without the imposition of an additional condition, such as Eq. (4.11).

One last question may still puzzle the reader, namely, how we succeeded in evading the "no-interaction" theorem.[1] For, if we simply choose to confine our Hamiltonian to the linear combination given by K of Eq. (4.12), we do obtain a uniquely defined, Lorentz-invariant pair of orbits in space-time. Returning to the original derivation of the theorem we find that the critical, and suspect, condition, is the so-called world-line condition,[1] which in our present notation may be written

$$[q_i^s, L^{0t}] = q_i^t[q_i^s, P^0] - q_i^0\delta^{st} \qquad \{s, t=1,2,3, \quad i=1,2\}. \tag{5.1}$$

(We note that the last term does not occur in the literature since the authors prefer to work in a frame where $q_1^0 = q_2^0 = t = 0$.) This equation is immediately suspect since it is not Lorentz covariant. Should one Lorentz observer find it satisfied and thereby claim that his trajectory is world-line forming, a transformed observer would disagree. We must therefore carefully review the derivation of this relation.

For simplicity let us consider one particle moving in the x direction in a given Lorentz frame. The equation of its trajectory in this frame is given by some function of time $f(t)$, thus

$$x = f(t). \tag{5.2}$$

This function is obtained by integrating the syntactic differential relation for q^1 as a function of q^0:

$$\frac{dq^1}{dq^0} = [q^1, P^0] \tag{5.3}$$

(we have assumed for simplicity $[q^2, P^0] = [q^3, P^0] = 0$) and then, in this frame making the semantic identification

$$q^s = x^s, \quad q^0 = t. \tag{5.4}$$

We next consider an infinitesimal Lorentz transformation to a new frame, moving relative to the original frame with velocity v ($\ll 1$) in the x direction, and inquire how the functional form of $f(t)$ is altered by such a transformation. Thus, to order v, we have

$$x \approx \bar{x} + v\bar{t}, \qquad t \approx \bar{t} + v\bar{x}, \tag{5.5}$$

and Eq. (5.2) becomes

$$\bar{x} + v\bar{t} = f(\bar{t} + v\bar{x}) \approx f(\bar{t}) + \dot{f}(\bar{t})v\bar{x}, \tag{5.6}$$

or equivalently

$$\bar{x} \approx f(\bar{t}) + vf(\bar{t})\dot{f}(\bar{t}) - v\bar{t} \equiv \bar{f}(\bar{t}). \tag{5.7}$$

Thus the change in the functional form of $f(t)$ as a function of its argument is given by

$$\delta f \equiv \bar{f}(t) - f(t) = v(f(t)\dot{f}(t) - t). \tag{5.8}$$

Thus far the argument is unexceptionable. In order to proceed, we must at this point make a questionable assumption. It is true that the infinitesimal Lorentz transformation of Eq. (5.5) is realized on the syntactic variables via the relation

$$\delta q^1 = v[q^1, L^{01}]\,. \tag{5.9}$$

It is also true that in the original frame of reference we have identified q^1 semantically with x, and that at each instant of time t the trajectory of the particle is given by Eq. (5.2). It does *not* follow from these facts that *the functional form* of $f(t)$ as a function of its argument induces a realization of the homogeneous Lorentz transformation by means of canonical mappings in phase space, as implied by Eq. (5.9). However, if we loosely treat the phase-space variables q^1 and q^0 as synonymous with the semantic observables x and t, respectively, and even more loosely equate the observable x with the functional form $f(t)$, we obtain by substituting Eqs. (5.2), (5.4), and (5.9) into equation (5.8) the world-line condition Eq. (5.1). The lack of covariance of this condition is now seen to stem from the frame-dependent nature of the semantic identifications. The fact that in this paper we can exhibit Lorentz-invariant world lines which do not satisfy the world-line condition merely serves to show that it is not reasonable to require that the functional form of the equation for the world line in a given reference frame induce a canonical realization of the Lorentz group.

---

[1]D. G. Currie, J. Math. Phys. 4, 1470 (1963); D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 38, 350 (1963); H. Leutwyler, Nuovo Cimento 37, 556 (1965).

[2]A. Komar, Phys. Rev. D 18, 1887 (1978).

[3]A. Eddington, Proc. Comb. Phil. Soc. 35, 186 (1939).

[4]P. A. M. Dirac, R. Peierls, and M. H. L. Pryce, Proc. Camb. Phil. Soc. 38, 193 (1942).

[5]L. H. Thomas, Phys. Rev. 85, 868 (1952).