Black-hole eddy currents

Thibaut Damour
E. R. 176 du CNRS. Groupe d'Astrophysique Relativiste, Observatoire de Paris, 92190 Meudon, France
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We study dissipative test electromagnetic fields in a black-hole background. Quantities such as surface velocity, tangential electric field, normal magnetic induction, total surface current, and conduction surface current are introduced and are shown to satisfy Ohm's law with a surface resistivity of $4\pi \simeq 377$ ohms. Associated with these currents there exists a "Joule heating". These currents can exist when the black hole is inserted in an external electric circuit, but they can exist even in the absence of external currents. In particular, we study the eddy currents induced by the rotation of a black hole in an oblique uniform magnetic field, and we show how the computation of the ohmic losses allows a very simple derivation of the torque exerted on the hole.

I. INTRODUCTION

Considerable interest has recently arisen in making models using black holes as energy sources.[1-4] These models could be relevant both for galactic (γ-ray bursts, x-ray bursters) and extragalactic sources (extended radiosources, quasars). The basic motivation for believing that black holes could provide the energy supply needed in many astrophysical objects comes from the Christodoulou-Ruffini mass formula[5] which implies that up to 29% (50%) of the total energy of a bare black hole can be stored as rotational (electromagnetic), and therefore extractable, energy. The first realistic model exhibiting a process by which energy could be extracted from a rotating black hole was proposed by Ruffini and Wilson[1] and studied by Damour.[2] The ingredients needed in that model to perform the energy extraction are the rotation of the hole and a magnetized plasma. A formally related mechanism using the same ingredients though in a somewhat physically different way was proposed by Blandford and Znajek.[4] Much work is still needed in analyzing the possible magnetospheric structures around black holes.[6] It was shown by Znajek[7] that the extraction of rotational energy from the hole by means of axisymmetric magnetospheric currents gives rise to an increase of the irreducible mass. This was described as a Joule heating inside the hole as if the black hole had an effective internal resistance of order unity (i.e., 30 ohms). We shall show here more precisely how one can define, in the general nonaxisymmetric case, surface currents on the hole so that a formal vectorial Ohm's law is valid as well as the scalar Joule's law. Moreover we shall pay special attention to the case where there are no external currents and where therefore the surface currents on the hole must be considered as pure eddy currents induced by the motion of the hole in an external magnetic field. Using the solution of King and Lasota[8] describing a black hole rotating in a misaligned uniform magnetic field we shall study how the eddy currents can dissipate energy and angular momentum. This will provide a new interpretation as well as a very simple derivation of the vectorial torque exerted by the magnetic field on the hole.

This paper is organized as follows: Section II describes the geometrical and kinematical structure of a Kerr black hole and introduces the notion of surface velocity of a black hole. Section III studies the intrinsic electromagnetic structure of the horizon, consisting of a tangential electric field and a normal magnetic induction. Section IV introduces the concept of a vector surface current which is defined in order to satisfy the conservation of charge and current. The link between these quantities is given in Sec. V (vector Ohm's law) and VI (Joule's law). A simple explicit example of such externally fed currents is given in Sec. VII while Sec. VIII studies the eddy currents generated by the rotation of the hole in an external magnetic field and their dissipative effects.

II. GEOMETRY AND KINEMATICS OF THE HORIZON OF A ROTATING BLACK HOLE

We use ingoing Kerr coordinates $(v, r, \theta, \tilde{\varphi})$ to study the future horizon H of a Kerr black hole. The metric can be written as

$$ds^2 = 2\alpha^{\hat{v}}\alpha^{\hat{r}} + \alpha^{\hat{\theta}}\alpha^{\hat{\theta}} + \alpha^{\hat{\varphi}}\alpha^{\hat{\varphi}} \; , \tag{2.1}$$

where we have introduced the quasiorthonormal basis of forms:

$$\begin{aligned}
\alpha^{\hat{v}} &= (r^2 + a^2)\Sigma^{-1}(dv - a\sin^2\theta d\tilde{\varphi}) \; ,\\
\alpha^{\hat{r}} &= \Sigma(r^2 + a^2)^{-1}[dr - \tfrac{1}{2}\Delta\Sigma^{-1}(dv - a\sin^2\theta\, d\tilde{\varphi})] \; ,\\
\alpha^{\hat{\theta}} &= \Sigma^{+1/2}d\theta \, ,\\
\alpha^{\hat{\varphi}} &= \Sigma^{-1/2}\sin\theta[(r^2 + a^2)d\tilde{\varphi} - a dv] \; ,
\end{aligned} \tag{2.2}$$

with $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2Mr$

$=(r-r_+)(r-r_-)$ where $r_\pm = M\pm(M^2-a^2)^{1/2}$.

The dual basis of vectors is

$$\begin{aligned} b_{\hat v} &= \partial_v + a(r^2+a^2)^{-1}\partial_{\tilde\varphi} + \tfrac{1}{2}\Delta(r^2+a^2)^{-1}\partial_r ,\\ b_{\hat r} &= (r^2+a^2)\Sigma^{-1}\partial_r ,\\ b_{\hat\theta} &= \Sigma^{-1/2}\partial_\theta ,\\ b_{\hat\varphi} &= \Sigma^{-1/2}[(\sin\theta)^{-1}\partial_{\tilde\varphi} + a\sin\theta\,\partial_v] . \end{aligned} \tag{2.3}$$

It is such that $\alpha^{\hat a}_e b^e_{\hat c} = \delta^{\hat a}_{\hat c}$ or $b^e_{\hat c} = g^{ea} g_{\hat c\hat b}\alpha^{\hat b}_a$, hence, for instance

$$b^a_{\hat v} = g^{ab}\alpha^{\hat r}_b . \tag{2.4}$$

The intrinsic geometry of a section $v=\text{const}$ of the future horizon $r=r_+$ is given by

$$ds_H{}^2 = \omega^{(\theta)}\omega^{(\theta)} + \omega^{(\varphi)}\omega^{(\varphi)} , \tag{2.5}$$

with

$$\begin{aligned} \omega^{(\theta)} &= \Sigma_+{}^{1/2} d\theta ,\\ \omega^{(\varphi)} &= \Sigma_+{}^{-1/2}(r_+{}^2+a^2)\sin\theta\, d\tilde\varphi , \end{aligned}$$

where $\Sigma_+ = r_+{}^2 + a^2\cos^2\theta$. Here and in the following the index + means replacing r by r_+.

The area element is therefore

$$dA = \omega^{(\theta)}\wedge\omega^{(\varphi)} = (r_+{}^2+a^2)\sin\theta\, d\theta\wedge d\tilde\varphi . \tag{2.6}$$

We introduce the corresponding intrinsic vectors,

$$\begin{aligned} e_{(\theta)} &= \Sigma_+{}^{-1/2}\partial_\theta ,\\ e_{(\varphi)} &= \Sigma_+{}^{1/2}(r_+{}^2+a^2)^{-1}(\sin\theta)^{-1}\partial_{\tilde\varphi} . \end{aligned} \tag{2.7}$$

We note the following: When $r\to r_+$, $b_{\hat v}$ becomes the usual null vector normal to the horizon l,

$$l^a\partial_a = b_{\hat v} = \partial_v + a(r_+{}^2+a^2)^{-1}\partial_{\tilde\varphi} ,$$

or in covariant form [see Eq. (2.4) above],

$$l_a dx^a = \alpha^{\hat r} = \Sigma_+(r_+{}^2+a^2)^{-1}dr .$$

Therefore $b_{\hat\theta}$ and $b_{\hat\varphi}$ become tangential to the horizon and we have

$$\begin{aligned} b_{\hat\theta} &= e_{(\theta)} ,\\ b_{\hat\varphi} &= e_{(\varphi)} + V_{(\varphi)} l , \end{aligned} \tag{2.8}$$

$$\begin{aligned} \alpha^{\hat\theta} &= \omega^{(\theta)} ,\\ \alpha^{\hat\varphi} &= \omega^{(\varphi)} - V^{(\varphi)}dv . \end{aligned} \tag{2.9}$$

In these formulas we have introduced the quantity

$$V_{(\varphi)} = V^{(\varphi)} = a\sin\theta\,\Sigma_+{}^{-1/2} , \tag{2.10}$$

which can be interpreted as the rotational velocity of the horizon.

Indeed the null generators of the horizon $l=dx/dv = \partial_v + \Omega\partial_{\tilde\varphi}$ [where Ω is the angular velocity of the horizon which is $a/(r_+{}^2+a^2)$ for the Kerr[9] geometry and $-g_{v\tilde\varphi}/g_{\tilde\varphi\tilde\varphi}$ in general[10]] are tilted with respect to the time-translation Killing vector ∂_v, and $V^{(\varphi)}$ is a direct measure of the tilting of l in the following sense: During the time dv (as measured at infinity) the displacement $dx = l dv$ acquires a transverse component $d\tilde\varphi = \Omega dv$ whose corresponding length (as measured locally) is $g_{\tilde\varphi\tilde\varphi}{}^{1/2}d\tilde\varphi = \Omega g_{\tilde\varphi\tilde\varphi}{}^{1/2}dv$ and the ratio (local length)/(global time) is $\Omega g_{\tilde\varphi\tilde\varphi}{}^{1/2}$ which is precisely $V^{(\varphi)}$. (We can also notice that $V^{(\varphi)} = g_{vv}{}^{1/2}$.)

Another phrasing would consist in saying that the four-vector $V = l - \partial_v$ represents the three velocity of the horizon (with respect to ∂_v), and the space-time length of $V = \Omega\partial_{\tilde\varphi}$ is precisely $V^{(\varphi)}$.

It is interesting to note that the maximum value of $V^{(\varphi)}$ is reached when $\theta = \frac{1}{2}\pi$ and $a=M$ (i.e , at the equator of a maximally rotating Kerr hole) and is equal to one. It is tempting to conjecture that this property may hold for a general black hole.

In the following we are going to make projections of tensorial quantities on the forms α and vectors b for four-dimensional entities and on the forms ω and vectors e for two-dimensional entities. (For instance $V^{(\varphi)}$ is just the geometrical component of the two-dimensional $V^{\tilde\varphi} = \Omega$.)

III. TANGENTIAL ELECTRIC FIELD AND NORMAL MAGNETIC INDUCTION

Given an electromagnetic test field F_{ab} regular on the future horizon H we define the tangential electric field and the normal magnetic induction by the restriction of the form $F = \frac{1}{2}F_{ab}dx^a\wedge dx^b$ to the horizon $r=r_+$. Namely,

$$F = (F_{\theta v}d\theta + F_{\tilde\varphi v}d\tilde\varphi)\wedge dv + F_{\theta\tilde\varphi}d\theta\wedge d\tilde\varphi ,$$

which can be written after projection on the basis ω, e

$$F = (E_{(\theta)}\omega^{(\theta)} + E_{(\varphi)}\omega^{(\varphi)})\wedge dv + B_\perp\,\omega^{(\theta)}\wedge\omega^{(\varphi)} , \tag{3.1}$$

where

$$\begin{aligned} E_{(\theta)} &= F_{(\theta)v} = \Sigma_+{}^{-1/2}F_{\theta v} ,\\ E_{(\varphi)} &= F_{(\varphi)v} = \Sigma_+{}^{1/2}(r_+{}^2+a^2)^{-1}(\sin\theta)^{-1}F_{\tilde\varphi v} ,\\ B_\perp &= F_{(\theta)(\varphi)} = (r_+{}^2+a^2)^{-1}(\sin\theta)^{-1}F_{\theta\tilde\varphi} . \end{aligned} \tag{3.2}$$

If the field is stationary we shall have the result that the tangential electric field $E_\parallel$ is the gradient of the potential A_v,

$$E_\parallel = E_{(\theta)}\omega^{(\theta)} + E_{(\varphi)}\omega^{(\varphi)} = dA_v . \tag{3.3}$$

IV. SURFACE CURRENTS

From a phenomenological point of view it is convenient to introduce a surface charge density and a surface current on the horizon. The heuristic justification for such definitions is the following: There exists a four-current

$J^a(v, r, \theta, \tilde{\varphi})$ which is defined and conserved all over space-time. However we want *not to consider* what happens inside the black hole ($r < r_+$). Yet some charge and current can go down the hole and disappear from the region $r > r_+$. Therefore if we wish to keep the charge and current conserved in the region $r \geq r_+$, we have to endow the surface $r = r_+$ with charge and current densities. Mathematically the problem is the following: Given $J^a(v, r, \theta, \tilde{\varphi})$ such that $J^a{}_{;a} = 0$ find a complementary current j^a with support on $r = r_+$ such that $J^a Y(r - r_+) + j^a$ is conserved, where Y is the Heaviside function. This problem is very easily solved by noting that the conservation of J is ensured by Maxwell equations $J^a = (4\pi)^{-1} F^{ab}{}_{;b}$. Replacing F^{ab} by $F^{ab} Y(r - r_+)$ we get the conserved current $J^a Y(r - r_+) + j^a$ where $j^a = (4\pi)^{-1} F^{ar} \delta(r - r_+)$. It is convenient to use a Dirac distribution δ_H on the horizon normalized with respect to the time at infinity v and the local proper area dA such that,

$$\int f(v, r, \theta, \tilde{\varphi}) \delta_H \, \delta(v - v_0) g^{1/2} d^4x = \int_H f(v_0, r_+, \theta, \tilde{\varphi}) dA \ .$$

One easily finds

$$\delta_H = (r^2 + a^2) \Sigma^{-1} \delta(r - r_+) \ . \tag{4.1}$$

Hence we can write the complementary current j^a, with support on the horizon, as

$$j^a = K^a \delta_H \ , \tag{4.2}$$

with

$$K^a = (4\pi)^{-1} \Sigma_+ (r_+{}^2 + a^2)^{-1} F^{ar}_+ \ . \tag{4.3}$$

We have thus defined a surface four-current density K^a which can be decomposed into a surface charge density σ (such that $\int_H \sigma dA$ yields the total charge on the hole) and the geometrical components of a surface current density $\vec{K}$,

$$\begin{aligned} &\sigma = K^v \ , \\ &K^{(\theta)} = \Sigma_+{}^{1/2} K^\theta \ , \\ &K^{(\varphi)} = \Sigma_+{}^{-1/2} (r_+{}^2 + a^2) \sin\theta \, K^{\tilde{\varphi}} \ . \end{aligned} \tag{4.4}$$

These quantities satisfy the following conservation law on the horizon:

$$\frac{\partial \sigma}{\partial v} dA + dK^* = J^{\hat{r}}_+ \, dA \ , \tag{4.5}$$

with

$$\begin{aligned} K^* &= K^{(\theta)} \omega^{(\varphi)} - K^{(\varphi)} \omega^{(\theta)} \\ &= (r_+{}^2 + a^2) \sin\theta (K^\theta d\tilde{\varphi} - K^{\tilde{\varphi}} d\theta) \ , \\ J^{\hat{r}}_+ &= (J^a \alpha_a^{\hat{r}})_+ = \Sigma_+ (r_+{}^2 + a^2)^{-1} J^r_+ \ , \end{aligned}$$

and the symbol d denotes exterior differentiation.

V. OHM'S LAW

We are now in position to exhibit a relation between the fields and the currents introduced above which can be thought of as Ohm's law for a rotating black hole. It is sufficient to consider the components $F^{\hat{\theta}\hat{r}} = F_{\hat{\theta}\hat{v}}$ and $F^{\hat{\varphi}\hat{r}} = F_{\hat{\varphi}\hat{v}}$ of the electromagnetic field in the basis (α, b). Using Eqs. (2.8) and (2.9) connecting α and b to ω, e, and $l = \partial_v + a(r_+{}^2 + a^2)^{-1} \partial_{\tilde{\varphi}}$ and taking into account the definitions of the velocity, fields, and currents on the black hole we easily get

$$\begin{aligned} &E_{(\theta)} + V_{(\varphi)} B_\perp = 4\pi K^{(\theta)} \ , \\ &E_{(\varphi)} = 4\pi [K^{(\varphi)} - \sigma V^{(\varphi)}] \ . \end{aligned} \tag{5.1}$$

This can be written in a self-explanatory two-dimensional vectorial form, which is valid intrinsically on the horizon

$$\vec{E} + \vec{V} \times \vec{B}_\perp = 4\pi(\vec{K} - \sigma \vec{V}) \ . \tag{5.2}$$

Equation (5.2) has precisely the form of the nonrelativistic Ohm's law for a moving charged conductor of surface resistivity $4\pi \simeq 377$ ohms. This result constitutes a clear confirmation of Carter's assertion[10] that a "black hole is analogous to an ordinary body (with finite viscosity and electrical conductivity)." This was conjectured starting from the equilibrium properties of black holes. The analog of the viscous dissipation was described by Hawking and Hartle[11] in terms of the increase of the area of the hole due to the surface shear of the null generators of the horizon (tidal friction). The dimensionless coefficient of viscosity was then a number of order unity.

More recently Znajek[7] has interpreted the contribution to the increase of the area of a hole arising from an external electric circuit as a Joule dissipation, the internal resistance of the hole so introduced being of order unity.

We have shown here how it is possible to define a conserved surface current on the hole so that the vectorial Ohm's law is satisfied.[12] In the following it will be found useful to introduce the notions of the surface *conduction* current $\vec{C}$ (the total current $\vec{K}$ minus the *convection* current $\sigma\vec{V}$) and of the "dragged-along" electric field $\vec{E}^*$,

$$\vec{C} = \vec{K} - \sigma \vec{V} \ , \quad \vec{E}^* = \vec{E} + \vec{V} \times \vec{B}_\perp \ . \tag{5.3}$$

We are going to show that this conduction current not only enters naturally Ohm's law, Eq. (5.2), but allows one to express very simply the analog of Joule's law.

VI. JOULE'S LAW

We can as usual[13] define the heat $đQ$ dissipated in the hole as

$$đQ=(8\pi)^{-1}\kappa dA=dM-\Omega dS_z\ ,\qquad(6.1)$$

where κ is the surface gravity,[10,14] Ω is the angular velocity, and dA, dM, and dS_z are the increases in, respectively, area, mass, and angular momentum of the hole. The total energy flux into the hole is given by an integral on the horizon[14]:

$$\dot{M}=dM/dv=+\int_H T_v{}^r g^{1/2}d\theta\, d\tilde{\varphi}=\int_H T_v{}^b l_b dA\ ,\qquad(6.2)$$

where T_{ab} is the test energy-momentum tensor at the horizon.

The angular momentum flux is

$$\dot{S}_z=dS_z/dv=-\int_H T_{\tilde{\varphi}}{}^r g^{1/2}d\theta\, d\tilde{\varphi}=-\int_H T_{\tilde{\varphi}}{}^b l_b dA\ .\qquad(6.3)$$

Hence we get the heat production as

$$\dot{Q}=\dot{M}-\Omega\dot{S}_z=\int_H (T_{vb}+\Omega T_{\tilde{\varphi}b})l^b dA=\int_H T_{ab}l^a l^b dA\ .\qquad(6.4)$$

In the case of an electromagnetic field we have on the horizon,

$$T_{ab}l^a l^b=(4\pi)^{-1}F_{ac}F_b{}^c l^a l^b\ .\qquad(6.5)$$

Projecting onto the tetrad (α, b) only the θ, φ components contribute. Hence we find easily the Joule's law,

$$\dot{Q}=\int_H 4\pi(\vec{C})^2 dA\ .\qquad(6.6)$$

The integrand in Eq. (6.6) can be written in different forms,

$$4\pi(\vec{C})^2=(4\pi)^{-1}(\vec{E}^*)^2=\vec{E}^*\cdot\vec{C}=(\vec{E}+\vec{V}\times\vec{B}_\perp)\cdot(\vec{K}-\sigma\vec{V})\ .\qquad(6.7)$$

Developing the last expression we find

$$\dot{Q}=\int_H \vec{E}\cdot\vec{K}dA-\int_H(\sigma\vec{E}+\vec{K}\times\vec{B}_\perp)\cdot\vec{V}dA\ ,\qquad(6.8)$$

which corresponds to the above splitting of $\dot{Q}$ in $\dot{M}$ and $-\Omega\dot{S}_z$. In other words this means that we can express directly the torque $\dot{S}_z$ on the black hole as due to a Laplace-Lorentz force on the surface charge and current densities:

$$\dot{S}_z=\int_H(\sigma\vec{E}+\vec{K}\times\vec{B}_\perp)\cdot(\vec{V}/\Omega)dA\ ,\qquad(6.9)$$

with a "lever arm"

$$|\vec{R}|=|\vec{V}|/\Omega=\Sigma_+^{-1/2}(r_+{}^2+a^2)\sin\theta\ .$$

VII. A SIMPLE EXAMPLE

As an illustration of the preceding concepts let us consider the solution describing the insertion of a rotating black hole in a linear current flowing along the z axis (the axis of rotation of the hole) from $+\infty$ to $-\infty$. In fact, because of the idealization of infinitely thin electrodes the hole will oppose an infinite resistance to the current, therefore we are going to consider the case of a total current I flowing from spatial infinity along the conical surface $\theta=\theta_1$ to the "northern" polar circle $\theta=\theta_1$ of the hole and then flowing out to spatial infinity along the "southern" conical surface $\theta=\theta_2$. In the limit $\theta_1\to 0$ and $\theta_2\to\pi$ we get the linear current alluded to above. Anyway the solution here presented is valid whatever the values of θ_1 and θ_2 are. Evidently we mean by positive current flowing out of the hole a stream of negative charges physically flowing into the hole. Such currents are needed to get a stationary solution without any charge accumulation onto the hole.

Mathematically the solution is a Robinson null field[15] which can be written as

$$F_{ab}+i{}^*F_{ab}=\varphi_2(\tilde{l}_a\tilde{m}_b-\tilde{l}_b\tilde{m}_a)\ ,\qquad(7.1)$$

where $\tilde{l}$ is the ingoing principal null congruence,

$$\tilde{l}^a\partial_a=\partial_r\ ,\qquad(7.2)$$

and where

$$\tilde{m}^a\partial_a=2^{-1/2}(r+ia\cos\theta)^{-1}\times[\partial_\theta+i(\sin\theta)^{-1}\partial_{\tilde{\varphi}}+ia\sin\theta\,\partial_v]\ .\qquad(7.3)$$

Following Fackerell and Ipser[16] we define

$$\Phi_2=-(r-ia\cos\theta)\sin\theta\,\varphi_2\ .\qquad(7.4)$$

An evident solution to the equations obtained in Ref. 16 is

$$\Phi_2=\text{const}=2^{3/2}I\ ,\qquad(7.5)$$

which yields our solution for a total current I flowing along the z axis. Using the covariant components of $\tilde{l}$ and $\tilde{m}$ (easily obtained from the expressions of the forms α) we get explicitly

$$F=2I(\sin\theta)^{-1}d\theta\wedge(dv-a\sin^2\theta\, d\tilde{\varphi})\ ,\qquad {}^*F=2Id\tilde{\varphi}\wedge dv\ .\qquad(7.6)$$

This solution is singular when $\sin\theta=0$ but we get a regular solution if we define the four potential,

$$A=2I[(\ln\tan\tfrac{1}{2}\theta_1)dv+a\cos\theta_1 d\tilde{\varphi}]\ ,\quad\text{if } 0\leq\theta\leq\theta_1$$
$$A=2I[(\ln\tan\tfrac{1}{2}\theta)dv+a\cos\theta d\tilde{\varphi}]\ ,\quad\text{if } \theta_1\leq\theta\leq\theta_2$$
$$A=2I[(\ln\tan\tfrac{1}{2}\theta_2)dv+a\cos\theta_2 d\tilde{\varphi}]\ ,\quad\text{if } \theta_2\leq\theta\leq\pi\ .\qquad(7.7)$$

Hence the field $F=dA$ will be given by Eq. (7.6) when $\theta_1 \leq \theta \leq \theta_2$ and will be zero otherwise. As announced this solution describes the insertion of a rotating black hole between two conical electrodes through which a total current I is flowing. It is very easy to work out in detail the tangential fields and the current flowing on the hole and we shall content ourselves by noticing that the total potential decrease between θ_1 and θ_2 is

$$A_v(\theta_2) - A_v(\theta_1) = RI \ , \tag{7.8}$$

where the total resistance R of the hole is given by

$$R = 2\ln[(\tan\tfrac{1}{2}\theta_2)/(\tan\tfrac{1}{2}\theta_1)] \tag{7.9}$$

in units of 30 ohms.

The energy delivered to the black hole is easily computed as

$$\mathring{M} = RI^2 \ . \tag{7.10}$$

This energy differs from the heat generated because of the presence of a positive torque,

$$\mathring{S}_z = +2a(\cos\theta_1 - \cos\theta_2)I^2 \ . \tag{7.11}$$

This torque is easily interpreted as coming from the impulsion which is delivered when one dissipates some energy in a moving system, and in fact although $\mathring{S}_z > 0$ one finds $\mathring{a} < 0$ and $\mathring{\Omega} < 0$.

Finally it is curious to note that the current distribution and the resistance given in Eq. (7.9) would have been the same if the black hole had been replaced by a metallic shell endowed with a surface resistivity equal to 4π. This condition on the surface resistivity is well known in engineering electromagnetism[17] as ensuring perfect absorption (because it realizes a perfect impedance matching with the vacuum).

VIII. EDDY CURRENTS

After having discussed the dissipative effects associated with the insertion of a black hole in an external electric circuit (internal resistance) we shall describe the dissipative effects that arise when a black hole is moving in an external magnetic field (eddy currents). Let us consider the surface eddy currents generated by the rotation of a black hole in an oblique uniform magnetic field. The exact test solution has been given by King and Lasota[8] using the components of the electromagnetic field on the outgoing Kinnersley tetrad which is linked to our initial tetrad by

$$\begin{aligned} l_K &= 2(r^2+a^2)\Delta^{-1} b_{\hat{v}} \ , \\ n_K &= -\tfrac{1}{2}\Delta(r^2+a^2)^{-1} b_{\hat{r}} \ , \\ m_K &= (2\Sigma)^{-1/2}(r - ia\cos\theta)(b_{\hat{\theta}} + ib_{\hat{\varphi}}) \ , \end{aligned} \tag{8.1}$$

so that we can write the φ_0^K component as

$$\begin{aligned} \varphi_0^K &= F_{ab} l_K^a m_K^b \\ &= 2^{1/2}\Sigma^{-1/2}\Delta^{-1}(r^2+a^2)(r - ia\cos\theta)(F_{\hat{v}\hat{\theta}} + iF_{\hat{v}\hat{\varphi}}) \ . \end{aligned} \tag{8.2}$$

Hence the "dragged along" tangential field $\vec{E}^* = 4\pi\vec{C}$ is computed from φ_0^K as

$$E^*_{(\theta)} + iE^*_{(\varphi)} = -2^{-1/2}(r^2+a^2)^{-1}\Sigma^{-1/2}(r + ia\cos\theta)\Delta\varphi_0^K \ , \tag{8.3}$$

in the limit $r \to r_+$, and the heat generated is given very simply by

$$\begin{aligned} \mathring{Q} &= \int_H (4\pi)^{-1}(\vec{E}^*)^2 dA \\ &= (8\pi)^{-1}\int_H |(r^2+a^2)^{-1}\Delta\varphi_0^K|^2 dA \ . \end{aligned} \tag{8.4}$$

In our problem φ_0^K is given by[8]

$$\varphi_0^K = -i(\tfrac{2}{3}\pi)B2^{1/2}\sum_{m=-1}^{+1} Y^*_{1m}(\gamma, 0)R_{1m}(r)[\partial_\theta + i(\sin\theta)^{-1}\partial_{\tilde{\varphi}}]Y_{1m}(\theta,\varphi) \ , \tag{8.5}$$

where B is the strength of the field at infinity, γ is the tilt angle of the field with the rotation axis (z axis), the Y_{1m} are the usual spherical harmonics, R_{1m} is a hypergeometric function whose behavior near $r = r_+$ is very simple, and where

$$\varphi = \tilde{\varphi} - \tfrac{1}{2}a(M^2 - a^2)^{-1/2}\ln[(r - r_+)/(r - r_-)] \ . \tag{8.6}$$

Hence, we get without computational effort,

$$E^*_{(\theta)} + iE^*_{(\varphi)} = (r_+^2 + a^2)^{-1}BaM\sin\gamma[\Sigma_+^{-1/2}(r_+ + ia\cos\theta)][\cos(\tilde{\varphi}+\alpha) - i\cos\theta\sin(\tilde{\varphi}+\alpha)] \ , \tag{8.7}$$

where α is defined by $\sin\alpha = a/M$.

Now it is trivial to compute the heat:

$$\begin{aligned} \mathring{Q} &= [(r_+^2+a^2)^{-1}BaM\sin\gamma]^2\int[\cos^2(\tilde{\varphi}+\alpha) + \cos^2\theta\sin^2(\tilde{\varphi}+\alpha)](r_+^2+a^2)\sin\theta\, d\theta\, d\tilde{\varphi}/4\pi \ , \\ \mathring{Q} &= \tfrac{2}{3}(BaM\sin\gamma)^2/(r_+^2+a^2) \ . \end{aligned} \tag{8.8}$$

Here the dissipation is entirely due to the braking torque $\overset{\circ}{S}_z$ because the energy flux in absence of external currents is given by

$$\dot{M} = \int \vec{E} \cdot \vec{K}\, dA = \int (\vec{\nabla} A_v) \cdot \vec{K}\, dA$$

$$= \int -A_v (\vec{\nabla} \cdot \vec{K}) dA = 0 \ .$$

Therefore,

$$\overset{\circ}{Q} = -\Omega \overset{\circ}{S}_z \ ,$$

which means that the z component of the torque is

$$\overset{\circ}{S}_z = -\tfrac{2}{3} M (B \sin\gamma)^2 (aM) \ . \qquad (8.9)$$

From this result we can recover the complete vectorial torque. Indeed, following Press,[18] we know that $\overset{\circ}{S}_y$ is zero by symmetry arguments (the y direction being defined as normal to the plane defined by $\vec{\Omega}$ and $\vec{B}$). Finally the last component $\overset{\circ}{S}_x$ is obtained by noting that we can create a uniform magnetic field near the hole by putting a magnetic charge $-Br^2$ at the point r, $\theta = \gamma$, $\varphi = 0$ and letting $r \to \infty$.[19] The torque on the hole will be opposite to the torque exerted on the magnetic charge in the limit $r \to \infty$.[18] But the latter torque can have no components along the direction $\theta = \gamma$, hence

$$\sin\gamma\, \overset{\circ}{S}_x + \cos\gamma\, \overset{\circ}{S}_z = 0 \ ,$$

which implies from Eq. (8.9),

$$\overset{\circ}{S}_z = +\tfrac{2}{3} M (B^2 \sin\gamma \cos\gamma)(aM) \qquad (8.10)$$

so that we recover the result,[20]

$$\overset{\circ}{\vec{S}} = \tfrac{2}{3} M (\vec{S} \times \vec{B}) \times \vec{B} \ . \qquad (8.11)$$

Moreover in the limit of small a/M we can give a very simple heuristic interpretation of the vectorial torque (8.11). Indeed the eddy currents are given by

$$4\pi (C_{(\theta)} + i C_{(\varphi)}) = r^{-2} aBM \sin\gamma\, (\cos\varphi - i \cos\theta \sin\varphi)$$

which can be written as

$$4\pi \vec{C} = \tfrac{1}{2} (\vec{\Omega} \times \vec{B}) \times \vec{r} \ , \qquad (8.12)$$

if we formally consider the horizon $r = 2M$ as a sphere embedded in a Euclidean three-space $\vec{r} = (x, y, z)$ where r, θ, φ are polar coordinates. The expression (8.12) describes precisely, as one can easily check, the current that would flow on a metallic shell (of surface resistivity 4π) slowly rotating in an oblique uniform magnetic field.

Then, not only can the vectorial torque (8.11) be interpreted as due to the Laplace force on the eddy currents $\vec{C}$, but it can be simply calculated by introducing the "magnetic moment" $\vec{D}$ owing to the currents $\vec{C}$,

$$\vec{D} = \int \tfrac{1}{2} \vec{r} \times \vec{C}\, dA \ . \qquad (8.13)$$

As the currents $\vec{C}$ are making loops around the vector $\vec{\Omega} \times \vec{B}$, we find easily

$$\vec{D} = \tfrac{1}{6} r^4 \vec{\Omega} \times \vec{B} \ . \qquad (8.14)$$

Thus we recover the well-known expression for the vectorial torque,

$$\dot{\vec{S}} = \vec{D} \times \vec{B} \ . \qquad (8.15)$$

Finally we can note that the magnetic moment (8.14) leaves an imprint at infinity. This can be explicitly seen using Pollock's solution[20]: The magnetic field at infinity contains a curl $(r^{-3} \vec{D} \times \vec{r})$ contribution which is not plagued with the same ambiguity as the dipolar fields directed along the preexistent uniform field because $\vec{D}$ is orthogonal to $\vec{B}$.

We can conclude that we have shown the heuristic value of considering the horizon of a black hole as analogous to a thin shell of a good electric conductor having a finite surface restivity equal to $4\pi \simeq 377$ ohms.

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