# Quantum field theory in anti-de Sitter space-time

S. J. Avis, C. J. Isham, and D. Storey
*Theoretical Physics, Blackett Laboratory, Imperial College of Science and Technology, London SW7, England*
(Received 16 January 1978)

We consider the problem of quantizing scalar fields propagating in anti-de Sitter space-time. This space-time is static but not globally hyperbolic and hence the usual quantization procedures are inapplicable. Nevertheless, we show that a consistent quantization scheme can be devised by carefully controlling information entering and leaving the space-time through its timelike spatial infinity.

## I. INTRODUCTION

The problem of quantizing a field propagating in a fixed, but curved, space-time has been studied at length during the last few years.[1] The subject is clearly relevant to a study of quantum gravity proper and in addition possesses substantial intrinsic interest as epitomized by Hawking's[2] famous black-hole quantum radiation. Attention has in general been focused on linear field theories but, even in these restricted cases, there is (at least for most space-times) no unique quantization scheme. Various approaches have been suggested, but in this paper we will mainly employ the "covariant quantization" method in which the Heisenberg quantum fields manifest themselves in the traditional way as operators defined on a single Hilbert space. This method has its origins in the work of Segal on quantizing arbitrary linear systems. Segal's methods relied heavily on the existence and structure of classical solutions of the field equations. In the present context this implies that the space-time manifold must be globally hyperbolic in the sense of possessing a spatial hypersurface on which Cauchy data can be freely specified. However, many space-times do not possess this property; indeed globally hyperbolic manifolds are necessarily of the form $R\times\Sigma$, where $\Sigma$ is a three-space, and are thus in many respects rather uninteresting.

It was the desire to understand quantum field theory in non-globally hyperbolic manifolds that motivated the present paper. Anti-de Sitter space-time (or "AdS" for brevity) is a famous example of such a manifold. It possesses both closed timelike curves and a timelike boundary at spatial infinity through which data can propagate. The latter property is also possessed by the universal covering space ("CAdS") and is the prime cause of the lack of hyperbolicity. Anti-de Sitter space is an especially interesting example as it has arisen in two other contexts recently, namely as the natural background in certain supergravity models[3] and as a rather unexpected solution to the $f$-$g$ theory of gravity. Indeed, it has even been suggested that solutions of wave equations in AdS may be of relevance to the problem of quark confinement.[4]

Since AdS is a homogeneous space of the group O(3, 2), it might perhaps seem natural to adopt a group-oriented approach to quantization. Such a study has in fact been made by Fronsdal *et al.* in a series of comprehensive papers.[5,6,7,8] However, from our point of view the emphasis is not ideally placed. Indeed the rôle played by the timelike infinity is not readily discussed in this approach. Note that even in the well-understood case of de Sitter space-time, the group-theoretic SO(4, 1) treatment misses thermal radiation associated with the event horizon of an inertial observer.[9] Thus we have concentrated our efforts on finding an analog of the covariant quantization scheme by coming directly to grips with the problem of controlling information entering the space-time through timelike infinity. In Secs. III and IV we do this for massless scalar fields by conformally mapping AdS into a genuine globally hyperbolic manifold (the Einstein static universe) and show that this leads to just three natural quantization schemes. Armed with this information we tackle the massive field in Sec. V and develop an essentially unique quantization. The results thus obtained may be regarded as complementary to those found using group theory.

## II. PROBLEMS ASSOCIATED WITH ANTI-DE SITTER SPACE-TIME

Although some problems of physical interpretation still remain, the formalism, at least, of linear scalar field quantization in a static, globally hyperbolic space-time[10] is now well understood.

The aim of the covariant approach is to construct a quantum field $\psi(x)$ satisfying both the classical field equation

$$[\Box + u(x)]\hat{\psi}(x) = 0 \tag{2.1}$$

and the covariant commutation relation

$$[\hat{\psi}(x), \hat{\psi}(x')] = -i\hbar\tilde{G}(x,x')\,. \tag{2.2}$$

 

In (2.1), $\Box$ is the d'Alembertian operator associated with the background space-time, and $u(x)$ is a smooth $c$-number function which, if constant, may be very loosely interpreted as the "mass squared" of the field. The unique classical commutator function $\tilde{G}(x,x')$, defined as the difference of the advanced and retarded Green's functions, evolves classical Cauchy data specified on a Cauchy hypersurface $\Sigma$ according to

$$\psi(x)=\int_\Sigma \tilde{G}(x,x')\overleftrightarrow{\partial}_\mu\psi(x')d\sigma^\mu(x'), \qquad (2.3)$$

and it is here that global hyperbolicity is seen to be an essential prerequisite.

One begins by finding a complete orthonormal (in the sense defined below) set of positive-frequency classical solutions of the field equation (2.1) of the form

$$f_j(x)=\exp(-i\omega_j t)h_j(\vec{x}), \quad \omega_j\geq 0. \qquad (2.4)$$

Here $t$ is a time coordinate such that $\partial/\partial t$ is a hypersurface orthogonal, strictly timelike Killing vector field, and $h_j(\vec{x})$ are a complete set of functions of the spatial coordinates only. The $f_j$ form an orthonormal basis of a Hilbert space $\mathcal{H}$ having the positive-definite Klein-Gordon inner product

$$\begin{aligned}B(\alpha,\beta)&\equiv i\int_\Sigma \alpha^*\overleftrightarrow{\partial}_\mu\beta d\sigma^\mu\\ &=i\int_{t=\text{const}}\alpha^*\overleftrightarrow{\partial}_0\beta g^{00}\sqrt{-g}\,d^3x, \quad \alpha,\beta\in\mathcal{H}\end{aligned} \qquad (2.5)$$

which is independent of $\Sigma$ by virtue of the field equations. For convenience $\Sigma$ is often chosen to be a surface of constant $t$.

The $f_j$ are also required to satisfy

$$\sum_j [f_j(x)f_j^*(x')-f_j^*(x)f_j(x')]=-i\tilde{G}(x,x'). \qquad (2.6)$$

If now the real classical field is expanded as

$$\psi(x)=\sum_j[a_jf_j(x)+a_j^*f_j^*(x)], \quad a_j\in\mathbf{C} \qquad (2.7)$$

and the $a_j$ are promoted to the rank of operators $\hat{a}_j$ satisfying

$$[\hat{a}_j,\hat{a}_k]=[\hat{a}_j^*,\hat{a}_k^*]=0, \quad [\hat{a}_j,\hat{a}_k^*]=\hbar\delta_{jk}, \qquad (2.8)$$

then the resulting Hermitian field operator, $\hat{\psi}(x)$, will necessarily satisfy (2.2).

The $\hat{a}_j$ and $\hat{a}_j^*$ are interpreted as annihilation and creation operators on the Fock space, constructed in the usual way as an infinite tensor product of simple-harmonic-oscillator Hilbert spaces. The Fock representation is almost inevitably used in these circumstances, since when it exists it provides the unique quantization for which the spectrum of the Hamiltonian operator (the generator of time translations) is positive definite.

The Hilbert space $\mathcal{H}$ automatically carries a unitary representation of the time-translation group. One might further require that any other isometries of the background space-time be placed on the same footing in this respect.

Anti-de Sitter space-time[10] (AdS) may be realised as the four-dimensional hyperboloid

$$(\xi^0)^2-(\xi^1)^2-(\xi^2)^2-(\xi^3)^2+(\xi^4)^2=K^{-1} \qquad (2.9)$$

in a five-dimensional space with metric

$$\begin{aligned}ds^2&=\eta^{(5)}_{\alpha\beta}d\xi^\alpha d\xi^\beta\\ &=(d\xi^0)^2-(d\xi^1)^2-(d\xi^2)^2-(d\xi^3)^2+(d\xi^4)^2.\end{aligned} \qquad (2.10)$$

AdS is a pseudo-Riemannian space of constant curvature $K$, related to the Ricci scalar curvature by

$$K=R/12 \qquad (2.11)$$

We use the conventions

$$R=g^{\mu\nu}R^\sigma{}_{\mu\sigma\nu}, \qquad (2.12)$$

$$R^\sigma{}_{\mu\tau\nu}=\Gamma^\sigma_{\mu\nu,\tau}-\cdots \qquad (2.13)$$

and signature $(+,-,-,-)$, with the result that $K$ is positive.

The isometry group of AdS is O(3,2) which is simply the "Lorentz" group of the five-dimensional embedding space. In addition, the conformal group is O(4,2), as for Minkowski space, which is of relevance when considering conformally invariant field equations.

AdS has the topology $\mathbf{S}^1$ (time)$\times\mathbf{R}^3$ (space) and hence contains closed timelike curves. "Unwrapping" the $\mathbf{S}^1$ gives the universal covering space (CAdS), which has the topology of $\mathbf{R}^4$ and contains no closed timelike curves.

For our purposes the metric of AdS, or CAdS, is most usefully written using the following parametrization:

$$\begin{aligned}&\xi^0=K^{-1/2}\cos\tau\sec\rho, \quad \xi^1=K^{-1/2}\tan\rho\cos\theta, \quad \xi^2=K^{-1/2}\tan\rho\sin\theta\cos\phi,\\ &\xi^3=K^{-1/2}\tan\rho\sin\theta\sin\phi, \quad \xi^4=K^{-1/2}\sin\tau\sec\rho,\end{aligned} \qquad (2.14)$$

$$ds^2=K^{-1}\sec^2\rho[d\tau^2-d\rho^2-\sin^2\rho(d\theta^2+\sin^2\theta d\phi^2)], \quad 0\leq\rho<\pi/2, \quad 0\leq\theta\leq\pi, \quad 0\leq\phi<2\pi. \qquad (2.15)$$

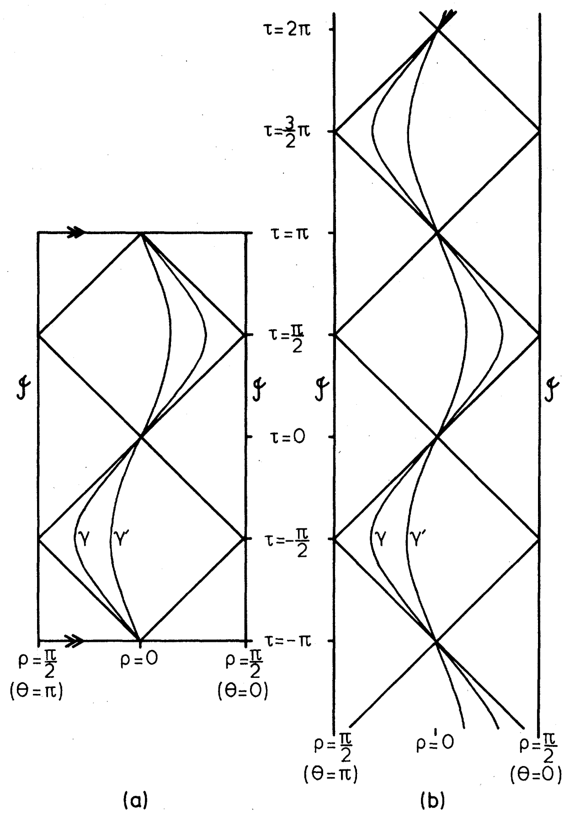


FIG. 1. Penrose diagrams for (a) anti–de Sitter space-time (top and bottom surfaces identified) and (b) its universal covering space-time. See text for discussion.

For AdS $-\pi < \tau \leq \pi$ with $\tau = -\pi$ and $\tau = \pi$ identified. For CAdS $-\infty < \tau < \infty$.

These dimensionless coordinates cover the whole of AdS and CAdS, except for the usual polar-type coordinate singularities.

In this coordinate system, spatial infinity has finite coordinate values ($\rho = \pi/2$) and AdS and CAdS are conveniently represented using Penrose diagrams,[10,11] as in Fig. 1. The coordinates $\theta$ and $\phi$ are suppressed. The null lines at $\pm 45^0$ are drawn to clarify the conformal structure; a light ray crosses AdS within half the natural period. Some timelike geodesics ($\gamma, \gamma'$) are also indicated, showing that in CAdS there is a residual effect of the time periodicity in AdS. In fact timelike geodesics emanating from any point in CAdS, which may be taken to be $\tau = \rho = 0$ since CAdS is a homogeneous space reconverge at $\rho = 0$ for $\tau = \pi, 2\pi, 3\pi$, etc.

These Penrose diagrams show clearly the two striking features of the AdS causal structure which preclude global hyperbolicity.

Firstly, AdS contains closed timelike curves, a feature lost in CAdS as already discussed.

Secondly, the surface at $\rho = \pi/2$ (i.e., at spatial infinity) is timelike, a feature shared with CAdS. The effect of this is that information may be lost to, or gained from, spatial infinity in finite coordinate time. A change of coordinates is of no avail here, since any time coordinate for which this is not so will not be globally defined (and will not give a manifestly static metric). It is this loss and gain of information which has the most disruptive effect on the Cauchy problem, and the closed timelike curves are in many ways a lesser evil.

There is another related possible source of trouble in this context, which is of a more technical nature. Rigorous quantization schemes in a globally hyperbolic space-time attach considerable importance to Cauchy data of compact support. As a consequence of global hyperbolicity, the Cauchy data on any Cauchy hypersurface will then possess this property. However, it is easily seen that in our case initial value data with compact support on one spacelike hypersurface will in general evolve in such a way that it becomes noncompact on many other spacelike hypersurfaces.

Some of the difficulties mentioned above are similar to those encountered when considering quantization in a box in Minkowski space-time. If the box is "transparent", information may escape or be thrown in from outside, and the Cauchy data within the box at a given time obviously does not uniquely determine that at other times. In fact one needs to additionally specify boundary data on the surface of the box, leading to a complicated, overdetermined system.

Of course when dealing with boxes one usually ascribes special physical properties to the walls. Typically the field, or perhaps its normal derivative, is required to vanish there, so that information is reflected and not lost. The time evolution of the Cauchy data is then unique. However, in less simple examples great care must be taken regarding the self-consistency of such mixed boundary conditions. In any case, the "walls" of AdS are at infinity and so the concept of reflecting boundary conditions is somewhat obscure. This will be clarified in Secs. IV and V.

Returning to the transparent box, one way of establishing a well-defined Cauchy problem is simply to accept that the box constitutes an incomplete manifold, and require that Cauchy data be specified on a Cauchy surface of the surrounding space-time, not just within the box. But unlike the box, AdS is complete and there is no such surrounding space-time. Nevertheless, an analog can be constructed, as explained in the next section.

## III. CONFORMALLY COUPLED MASSLESS FIELD–"TRANSPARENT" BOUNDARY CONDITIONS

To clarify the analogy between AdS and a box in Minkowski space, it is convenient to begin by considering a massless scalar field, conformally coupled to the background metric. The appropriate wave equation is

$$(\Box - \tfrac{1}{6} R)\psi = (\Box - 2K)\psi = 0 . \qquad (3.1)$$

The D'Alembertian operator, $\Box$, is given by

$$\Box\psi = g^{\mu\nu}\nabla_\mu \partial_\nu \psi = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\psi) \qquad (3.2)$$

in general, and

$$K^{-1}\Box\psi = \cos^2\rho\,\frac{\partial^2\psi}{\partial\tau^2} - \cot^2\rho\left[\cos^2\rho\frac{\partial}{\partial\rho}\left(\tan^2\rho\frac{\partial}{\partial\rho}\psi\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] \qquad (3.3)$$

in particular, for the AdS metric (2.15).

Now it so happens that CAdS may be conformally mapped into half of the Einstein static universe[10,11] (ESU), as depicted in Fig. 2. ESU may be realized as the four-dimensional cylinder

$$(\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2 + (\eta^4)^2 = K^{-1} \qquad (3.4)$$

in a five-dimensional space with metric

$$ds^2 = (d\eta^0)^2 - (d\eta^1)^2 - (d\eta^2)^2 - (d\eta^3)^2 - (d\eta^4)^2 , \qquad (3.5)$$

and hence it has the topology R (time) $\times$ $S^3$ (space). (In Fig. 2, two spatial dimensions are suppressed so that ESU appears as R $\times$ $S^1$.) The scalar curvature is

$$R^E = -6K . \qquad (3.6)$$

The ESU metric may be written in the globally defined form

$$(ds^E)^2 = K^{-1}[d\tau^2 - d\rho^2 - \sin^2\rho(d\theta^2 + \sin^2\theta\, d\phi^2)] ,$$

$$-\infty < \tau < \infty, \quad 0 \leq \rho \leq \pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi \qquad (3.7)$$

[compare (2.15)].

Our coordinate systems have been chosen to make the conformal mapping as simple as possible. In fact

$$g^E_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad (3.8)$$

where $\Omega$, the conformal factor, is given by

$$\Omega = \cos\rho . \qquad (3.9)$$

The field equation (3.1) is invariant under conformal mappings provided the field is assigned a conformal weight of $-1$, i.e.,

$$\psi^E = \Omega^{-1}\psi . \qquad (3.10)$$

So if $\psi$ is a solution of (3.1) in CAdS then $\psi^E$ is a solution of

$$(\Box^E - \tfrac{1}{6} R^E)\psi^E = (\Box^E + K)\psi^E = 0 \qquad (3.11)$$

in the appropriate half of ESU, where

$$K^{-1}\Box^E\psi^E = \frac{\partial^2\psi^E}{\partial z^2} - \frac{1}{\sin^2\rho}\left[\frac{\partial}{\partial\rho}\left(\sin^2\rho\frac{\partial\psi^E}{\partial\rho}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi^E}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi^E}{\partial\phi^2}\right] . \qquad (3.12)$$

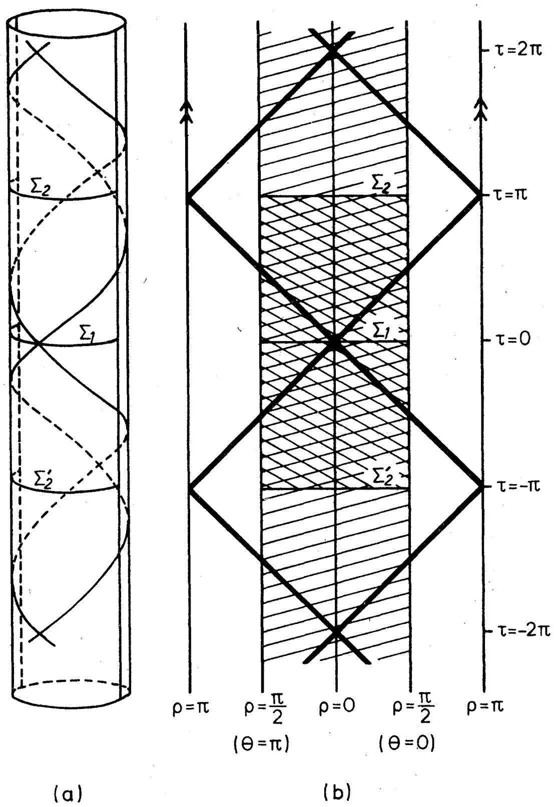


FIG. 2. (a) The Einstein static universe with two spatial dimensions suppressed is the cylinder R (time) $\times S^1$ (space). (b) As above, cut along $\rho = \pi$ and flattened out, showing the images under conformal mapping of CAdS (shaded) and AdS (double shaded, $\Sigma_2$ and $\Sigma_2'$ identified). The null lines at $\pm 45°$ are the support of $\tilde{G}^E(x, 0)$. When restricted to the image of AdS they are the image of the support of $\tilde{G}^T(x, 0)$. Note that identification of $\Sigma_2$ with $\Sigma_2'$ is commensurate with the periodicity of $\tilde{G}^E(x, 0)$ (and all other nonsingular finite-norm solutions in ESU).

Now ESU is a globally hyperbolic static space-time, and quantization therein is well known and follows the pattern of Sec. II. A summary will presently be given. We propose to use this quantization, mapped back, to give an acceptable quantum field theory in AdS (thus sidestepping the problem of how to fix the information passing into the space-time).

Separation of variables yields the following collection of positive frequency, finite $B$-norm [Eq. (2.5)] solutions of (3.11) defined on the whole of ESU:

$$\psi^E_{\omega l m}=N_{\omega l}\,e^{-i\omega\tau}(\sin\rho)^l\,C^{l+1}_{\omega-l-1}(\cos\rho)Y^m_l(\theta,\phi) \qquad (3.13)$$

where $\omega, l, m$ are integers such that $\omega-1\geq l\geq |m|$. Here $C^p_q(z)$ are Gegenbauer polynomials,[12] $Y^m_l(\theta,\phi)$ are the usual spherical harmonics and $N_{\omega l}$ are normalization constants.

The $\psi^E_{\omega l m}$ form an orthonormal basis for the Hilbert space $\mathcal{H}^E$ of all finite-norm, positive-frequency solutions of (3.11), with inner product defined by

$$B^E(\alpha,\beta)=i\int_{\tau=\text{const}}\alpha^*\overleftrightarrow{\partial}_0\beta\sqrt{-g^E}\,d^3x, \quad \alpha,\beta\in\mathcal{H}^E \qquad (3.14)$$

[cf. (2.5) and note $g^{E00}=1$]. Hence all such solutions are periodic in $\tau$ with period $2\pi$. This is related to the fact that, in the absence of interactions, a classical massless particle passing through the point $(\tau,\rho,\theta,\phi)$ will also pass through the points $(\tau+2\pi n,\rho,\theta,\phi)$ for $n=\pm1,\pm2$, etc. So the spatial "periodicity" of ESU has induced an effective temporal periodicity. Moreover, upon restricting the solutions to the image of CAdS, and mapping back using (3.10), this periodity is seen to be precisely that which allows the functions to be defined on AdS.

In addition to the periodicity discussed above, we also have

$$\psi^E(\tau,\rho,\theta,\phi)=-\psi^E(\tau+(2n+1)\pi,\pi-\rho,\pi-\theta,\phi+\pi),$$
$$n=0,\pm1,\pm2,\ldots. \qquad (3.15)$$

[A classical massless particle passing through $(\tau,\rho,\theta,\phi)$ must not only pass through $(\tau+2\pi n,\rho,\theta,\phi)$ but also through $(\tau+(2n+1)\pi,\pi-\rho,\pi-\theta,\phi+\pi)$. It is more difficult to find an intuitive classical explanation of the minus sign.] It follows that the specification of Cauchy data on the complete surface $\tau=0$ is equivalent to its specification on the pair of incomplete surfaces $\{\tau=0,\ \rho<\pi/2\}$ and $\{\tau=\pi,\ \rho<\pi/2\}$ in the following sense. If the solution is $C^\infty$ then so is the induced data on these partial surfaces. However, the converse is not strictly true since there is a consistency condition on the boundary values of the partial data to ensure that the induced solution in ESU really is $C^\infty$. On the other hand if distributional solutions are considered there is no such restriction, but it is now necessary to include the boundary at $\rho=\pi/2$ on one of the partial Cauchy surfaces in order to obtain a complete specification of the solution in terms of this partial data.

The quantization schemes in AdS that we are developing employ only those solutions in AdS whose ESU counterparts are everywhere $C^\infty$ solutions of the wave equation (3.11). In the sense defined above they are specified by their "initial value" data on the pair of surfaces $\{\tau=0,\ \rho<\pi/2\}$ and $\{\tau=\pi,\ \rho<\pi/2\}$ in AdS, denoted $\Sigma_1$ and $\Sigma_2$ respectively (see Fig. 2). (Note that with respect to the AdS metric these are *complete* surfaces.) The set of all such solutions generates a Hilbert space $\mathcal{H}^T$ with inner product

$$B^T(\alpha,\beta)=i\int_{\Sigma_1\cup\Sigma_2}\alpha^*\overleftrightarrow{\partial}_0\beta g^{00}\sqrt{-g}\,d^3x, \quad \alpha,\beta\in\mathcal{H}^T \qquad (3.16)$$

Of course by construction $\mathcal{H}^T$ is identical to $\mathcal{H}^E$, the Hilbert space of solutions in ESU equipped with the $B^E$-norm of (3.14). Indeed this norm maps conformally into (3.16) with the integration region being transferable from the single Cauchy surface in ESU to the pair of surfaces in AdS by virtue of (3.15).

To actually reconstruct the AdS solution from its "Cauchy data" we require the analog $\tilde{G}^T(x,x')$ of the classical commutator function. Just as for the basis functions this is obtained from the ESU commutator function, $\tilde{G}^E(x,x')$, by restriction and mapping back, using (3.10). Since AdS and ESU are both homogeneous spaces, $\tilde{G}^E(x,x')$ and $\tilde{G}^T(x,x')$ are characterized by their behavior as functions of a single variable $x$, with $x'$ chosen to be the coordinate origin for convenience. The commutator function $\tilde{G}^E(x,o)$ is readily constructed from the well-known Feynman function[13] (propagator) and may be written in the form

$$\tilde{G}^E(x,0)=-\frac{K}{4\pi}\delta(\cos\rho-\cos\tau)\epsilon^0(\tau)\,, \qquad (3.17)$$

where

$$\epsilon^0(\tau)\equiv\operatorname{sgn}(\sin\tau)\,. \qquad (3.18)$$

Hence (noting $\cos\rho\geq0$ in AdS)

$$\tilde{G}^T(x,0)=-\frac{K}{4\pi}\delta(1-\cos\tau\sec\rho)\epsilon^0(\tau) \qquad (3.19)$$

The supports of $\tilde{G}^E(x,0)$ and $\tilde{G}^T(x,0)$ are concentrated on the light cones through the origin in ESU and AdS respectively (see Fig. 2). This "Huygens principle" is in fact a major reason for referring to the field as "massless".[13]

The classical solution may now be constructed from the "effective Cauchy data" on $\Sigma_1$ and $\Sigma_2$

using

$$\psi(x)=\int_{\Sigma_1\cup\Sigma_2}\tilde{G}^T(x,x')\overleftrightarrow{\partial}_0\psi(x')g^{00}\sqrt{-g}\,d^3x' \tag{3.20}$$

and so $\Sigma_1\cup\Sigma_2$ will be called an "effective Cauchy surface" for AdS.

Now that the classical Cauchy problem is under control, quantization is fairly straightforward and follows the pattern outlined in Sec. II based on the field operator.

$$\hat{\psi}=\sum_{\omega l m}(\psi_{\omega l m}\hat{a}_{\omega l m}+\psi^*_{\omega l m}\hat{a}^*_{\omega l m}), \tag{3.21}$$

where $\psi_{\omega l m}$ are given by

$$\psi_{\omega l m}=\Omega\psi^E_{\omega l m}=N_{\omega l}\exp(-i\omega\tau)\cos\rho(\sin\rho)^l \times C^{l+1}_{\omega-l-1}(\cos\rho)Y^m_l(\theta,\phi) \tag{3.22}$$

and are regarded now as functions on AdS. It may be checked explicitly that the relation (2.6) survives the restriction and mapping back. This completes the quantization since we have constructed a quantum field on AdS satisfying both the field equation and our analog of the covariant commutation relation.

An alternative way of completely specifying a quantum field theory is to construct a Feynman function. Hence it is of interest to try to do so for AdS, and in particular to see if any meaning can be attached to the term "time-ordered product" in a space containing closed timelike curves.

In Minkowski space the commutator function is simply related to the real part of the Feynman function, which in turn is the boundary value of a unique analytic function of the Minkowskian invariant distance, satisfying the wave equation with a single δ-function source. To look for an analogous function in AdS it is advantageous to introduce the invariant distance $\sigma(x,x')$. This is the analog of $\frac{1}{2}[(t-t')^2-(\vec{x}-\vec{x}')^2]$ in Minkowski space and in fact is half the distance from $x$ to $x'$ in the embedding space:

$$\sigma(x,x')\equiv\tfrac{1}{2}\eta^{(5)}_{\alpha\beta}(\xi^\alpha-\xi'^\alpha)(\xi^\beta-\xi'^\beta) \tag{3.23}$$

[cf. (2.10)]. In particular

$$K\sigma(x,0)=1-\cos\tau\sec\rho\,. \tag{3.24}$$

The points $x$ satisfying $\sigma(x,x')=0$ lie on the "light cone" through $x'=(\tau',\rho',\theta',\phi')$, while those satisfying $\sigma(x,x')=2K^{-1}$ lie on the "light cone" through the antipodal point $x'_A=(\tau'+\pi,\rho',\pi-\theta',\phi'+\pi)$.

Expressed in terms of $\sigma$, (3.1) becomes

$$\left(\sigma(2-K\sigma)\frac{d^2}{d\sigma^2}+4(1-K\sigma)\frac{d}{d\sigma}-2K\right)G(\sigma)=0\,,\quad \sigma\neq 0,\frac{2}{K}\,. \tag{3.25}$$

The most general analytic solution of (3.25) is an arbitrary linear combination of $(K\sigma)^{-1}$ and $(K\sigma-2)^{-1}$. In Minkowski space the correct function is uniquely determined by demanding that the real part be causal. Although causality is an obscure notion in AdS it is nevertheless reasonable to require that the prospective Feynman function must at least look locally like the Minkowski one. With this in mind we take

$$G^T(\sigma)=\left(\frac{iK}{8\pi^2}\right)\frac{1}{K\sigma-i0} \tag{3.26}$$

as the Feynman function for "transparent" boundary conditions, which in fact solves the inhomogeneous equation ($\sigma$ real)

$$(\Box-2K)G^T(\sigma)=-\delta^4(x,x')\,. \tag{3.27}$$

With this choice the commutator function $\tilde{G}^T$ is related to $G^T$ by

$$\tilde{G}^T(x,0)=2\,\epsilon^0(\tau)\,\mathrm{Re}\,G^T(x,0) \tag{3.28}$$

[N.B. (3.18)] in close analogy with the relationship in Minkowski space.

The way in which $G^T$ can be related to a suitably defined "time-ordered product" will be explained in Sec. V, since our remarks will also apply to the Feynman functions constructed in Secs. IV and V.

Likewise, discussion of the extent to which the Hilbert space $\mathcal{H}^T$ carries a representation of the AdS isometry group will be postponed until then. However, it is convenient to discuss the related topic of conservation laws at this stage. In view of the loss of energy, angular momentum, etc. to infinity, as discussed in Sec. II, this will be of particular interest in AdS. To begin with, some remarks on the definitions of energy-momentum tensors are in order.

The Lagrangian density for a "conformally" coupled scalar field is

$$\mathcal{L}=\tfrac{1}{2}\sqrt{-g}\,[g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi-(\mu^2-\tfrac{1}{6}R)\psi^2]\,. \tag{3.29}$$

(A mass $\mu$ has been included for later use.) There are two distinct energy-momentum tensors associated with this Lagrangian density.

(1) The variational (new improved[14]) energy-momentum tensor, obtained by varying the action $S$:

$$\delta S=\delta\int\mathcal{L}\,d^4x=\int\tfrac{1}{2}T_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}\,d^4x\,. \tag{3.30}$$

From (3.29),

$$T_{\mu\nu}=\partial_\mu\psi\partial_\nu\psi-\tfrac{1}{2}g_{\mu\nu}[g^{\lambda\sigma}\partial_\lambda\psi\partial_\sigma\psi-(\mu^2-\tfrac{1}{6}R)\psi^2]+R_{\mu\nu}\frac{\psi^2}{6}+\frac{1}{6}(g_{\mu\nu}\Box-\nabla_\mu\partial_\nu)\psi^2 . \qquad (3.31)$$

For $\mu=0$, $T_{\mu\nu}$ has conformal weight $-2$ and is traceless.

(2) The canonical energy-momentum tensor

$$t_{\mu\nu}=\partial_\mu\psi\partial_\nu\psi-\tfrac{1}{2}g_{\mu\nu}[g^{\lambda\sigma}\partial_\lambda\psi\partial_\sigma\psi-(\mu^2-\tfrac{1}{6}R)\psi^2] . \qquad (3.32)$$

This may also be obtained by variation of $S$ but in this case $R$ is treated as though it were independent of $g^{\mu\nu}$. Thus $t_{\mu\nu}$ is just the same as for a minimally coupled theory with mass $(\mu^2-R/6)^{1/2}$ (which may be imaginary in our case).

Let $\xi_a^\mu$, $a=0,1,\ldots,9$ be the global Killing vector fields on AdS such that $\xi_0^\mu$ corresponds to time translation, $\xi_1^\mu$, $\xi_2^\mu$, and $\xi_3^\mu$ to spatial rotations, and the other six to "Lorentz boosts" in the five-dimensional embedding space. Define

$$Q_a(\tau)\equiv\int_{\tau=\text{const}}T_{\mu\nu}\,\xi_a^\nu\,d\sigma^\mu =\int_{\tau=\text{const}}T_{0\nu}\xi_a^\nu g^{00}\sqrt{-g}\;d\rho\,d\theta\,d\phi . \qquad (3.33)$$

The $Q_a(\tau)$ will not be independent of $\tau$ in general.

Since $T_{\mu\nu}$ has conformal weight $-2$ for a massless field the integrand of (3.33) is conformally invariant, and so (3.33) is equivalent to

$$Q_a(\tau)=\int_{\substack{\tau=\text{const.}\\ \rho<\pi/2}}T^E_{0\nu}\,\xi_a^{E\nu}\sqrt{-g^E}\;d\rho\,d\theta\,d\phi . \qquad (3.34)$$

The $\xi_a^{E\nu}$ are the vector fields induced on half of ESU by the action of the conformal mapping on the $\xi_a^\nu$. When their explicit form is computed it becomes clear that they can be extended to the whole of ESU. While $\xi_0^{E\nu},\ldots,\xi_3^{E\nu}$ still generate isometries, the other six do not, but rather correspond to proper conformal motions of ESU, i.e.,

$$\nabla^{E\mu}\xi_a^{E\nu}+\nabla^{E\nu}\xi_a^{E\mu}=\lambda_a g^{\mu\nu} , \qquad (3.35)$$

where $\lambda_a=0$ for $a=0,1,2,3$, $\lambda_a\neq 0$ for $a=4,\ldots,9$. Now

$$\nabla^{E\mu}(T^E_{\mu\nu}\,\xi_a^{E\nu})=0 \qquad (3.36)$$

by virtue of (3.35) and the fact that $T^E_{\mu\nu}$ is traceless and divergence-free. Thus, integrating (3.36) over the compact region between two constant $\tau$ hypersurfaces of ESU and applying Gauss's theorem, it follows that

$$P_a\equiv\int_{\tau=\text{const}}T^E_{0\nu}\xi_a^{E\nu}\sqrt{-g^E}\;d\rho\,d\theta\,d\phi \qquad (3.37)$$

is independent of $\tau$. Indeed these are the usual conserved quantities for a globally hyperbolic manifold. But now (3.15), along with the symmetry properties of the $\xi_a^{E\nu}$, allows $P_a$ to be decomposed as

$$P_a=Q_a(\tau)+Q_a(\tau+\pi) . \qquad (3.38)$$

In other words, although in general the one hypersurface quantities $Q_a(\tau)$ are not $\tau$ independent, the sums $Q_a(\tau)+Q_a(\tau+\pi)$ are $\tau$ independent and equal $P_a$, a conserved quantity corresponding to a global conformal motion of ESU.

Thus the effect of the "transparent" boundary conditions obtained by conformally mapping into ESU is to recirculate the energy, angular momentum, etc., lost to timelike infinity, resulting in a well-defined, if rather unusual, conservation law.

In Sec. IV we wish to consider the possibility of a "closed" quantization, analogous to a box in Minkowski space with reflecting walls. This is achieved in practice by demanding conservation of the $Q_a$, i.e., conservation of quantities integrated over a *single* hypersurface.

## IV. CONFORMALLY COUPLED MASSLESS FIELDS–"REFLECTIVE" BOUNDARY CONDITIONS

In Sec. III a quantization was discussed which involved the specification of effective Cauchy data on a suitable pair of spacelike hypersurfaces, and it was shown that most field configurations did not have conserved energy etc. as calculated by integrating the appropriate density over one surface.

In this section two alternative quantization schemes will be obtained by finding those maximal subsets of the positive-frequency solutions (3.22) which have the property that all finite linear combinations

$$\psi(x)=\sum_{\omega l m}[c_{\omega l m}\psi_{\omega l m}(x)+c^*_{\omega l m}\psi^*_{\omega l m}(x)] ,$$

$$c_{\omega l m}\in\mathbf{C} \qquad (4.1)$$

give $Q_a(\tau)$ [defined in (3.33)] independent of $\tau$, i.e., conservation laws based on a single hypersurface.

First note that from (3.36), (3.34) and Gauss's theorem

$$0=\int_{\tau_1\le\tau\le\tau_2}\nabla^{E\mu}(T^E_{\mu\nu}\,\xi_a^{E\nu})dv =Q_a(\tau_2)-Q_a(\tau_1)+\int_{\tau_1}^{\tau_2}X_a\,d\tau \qquad (4.2)$$

where

$$X_a = \int_{\rho=\pi/2} T^E_{\mu 1} g^{E11} \sqrt{-g^E}\, d\theta\, d\phi\,. \tag{4.3}$$

The requirement that $Q_a(\tau_1) = Q_a(\tau_2)$ for all $\tau_1$ and $\tau_2$ is equivalent to $X_a = 0$ (i.e., no net flux across $\rho = \pi/2$). The minimal conditions imposed on the $c_{\omega l m}$ by setting $X_0 = 0$ (energy conservation) is that for each $l$ independently either all the $c_{\omega l m}$ with $\omega$ odd must vanish or all the $c_{\omega l m}$ with $\omega$ even must vanish. No further restriction is imposed by demanding $X_1 = X_2 = X_3 = 0$ (angular momentum conservation). Finally, on requiring $X_4 = \cdots = X_9 = 0$ the complete restriction is that either all the $c_{\omega l m}$ with $\omega - l$ odd must vanish or all the $c_{\omega l m}$ with $\omega - l$ even must vanish.

Thus the requirement that all the $Q_a(\tau)$ be independent of $\tau$ decomposes the basis functions $\psi_{\omega l m}$ into two disjoint classes which are listed below together with their principal properties:

(1) $$\psi^1_{\omega l m} = \sqrt{2}\, N_{\omega l} \exp(-i\omega\tau) \cos\rho (\sin\rho)^l \times C^{l+1}_{2n}(\cos\rho) Y^m_l(\theta,\phi)\,, \tag{4.4}$$

where $\omega = l + 2n + 1$ and $n$ is a non-negative integer,

$$\psi^1_{\omega l m}(x_A) = -\psi^1_{\omega l m}(x)\,, \tag{4.5}$$

$$\frac{\partial}{\partial\rho}(\sec\rho\, \psi^1_{\omega l m}) \to 0 \text{ as } \rho \to \pi/2\,. \tag{4.6}$$

(2) $$\psi^2_{\omega l m} = \sqrt{2}\, N_{\omega l} \exp(-i\omega\tau) \cos\rho (\sin\rho)^l \times C^{l+1}_{2n+1}(\cos\rho) Y^m_l(\theta,\phi)\,, \tag{4.7}$$

where $\omega = l + 2n + 2$ and $n$ is a non-negative integer,

$$\psi^2_{\omega l m}(x_A) = \psi^2_{\omega l m}(x)\,, \tag{4.8}$$

$$\sec\rho\, \psi^2_{\omega l m} \to 0 \text{ as } \rho \to \pi/2\,. \tag{4.9}$$

Each class corresponds to a definite "parity" under the point to antipodal point transformation and a well-defined behavior at spatial infinity.

Let $\mathcal{H}^1$ and $\mathcal{H}^2$ denote the Hilbert spaces formed from the functions (1) and (2), respectively. It is clear that *all* elements of $\mathcal{H}^1$ or $\mathcal{H}^2$ have the same definite parity in the above sense, and it follows that a solution in one of these Hilbert spaces is completely determined by its initial value data on one spatial section, $\Sigma_1$ say. Indeed in view of this parity it is clear that the classical commutator functions to be used for evolving data on $\Sigma_1$ uniquely forward in time are

$$\tilde{G}^j(x,0) = \tilde{G}^T(x,0) - (-1)^j \tilde{G}^T(x_A,0)\,, \quad j = 1,2 \tag{4.10}$$

$$= -\frac{K}{4\pi}\epsilon^0(\tau)[\delta(K\sigma) - (-1)^j \delta(K\sigma - 2)]\,, \tag{4.11}$$

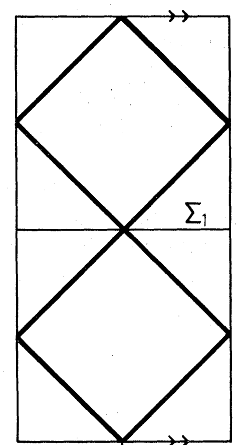


FIG. 3. The "reflective" conformal massless case. Single spacelike hypersurfaces, e.g., $\Sigma_1$, form effective Cauchy surfaces. The null lines (at ±45°) are the support of the commutation functions $\tilde{G}^1(x,0)$ and $\tilde{G}^2(x,0)$.

where $\tilde{G}^j(x,x')$ is the commutator function associated with $\mathcal{H}^j$. The support of $\tilde{G}^j(x,0)$ is indicated in Fig. 3.

The $\mathcal{H}^j$ norm may be defined in a natural way as in (2.5) but now integrated over $\Sigma_1$ only (hence the extra normalization factor $\sqrt{2}$ in the $\psi^j_{\omega l m}$).

Just as for Sec. III the quantization is implemented without difficulty now that the Cauchy problem has been taken care of. The relationship (2.6) follows easily from its "transparent" quantization counterpart, using the symmetries of the commutator functions and basis functions. The field operator

$$\hat{\psi}^j = \sum_{\omega l m} (\psi^j_{\omega l m} \hat{a}^j_{\omega l m} + \psi^{j*}_{\omega l m} \hat{a}^{j*}_{\omega l m}) \tag{4.12}$$

satisfies both the field equation and covariant commutation relation as required.

To make clear the analogy with the box in Minkowski space it is only necessary to point out that the image of CAdS (and hence AdS under identification) is effectively the interior of a box in ESU with a "wall" at $\rho = \pi/2$. For the two schemes of this section the ESU counterparts of the field satisfy

$$\frac{\partial\psi^{1E}}{\partial\rho} \to 0 \text{ as } \rho \to \pi/2 \tag{4.13}$$

in one case and

$$\psi^{2E} \to 0 \text{ as } \rho \to \pi/2 \tag{4.14}$$

in the other. These are precisely the conditions usually imposed on the boundary of a box with reflecting walls in Minkowski space, hence our description of the boundary conditions as "reflective". Of course the boundary conditions on the AdS fields themselves, (4.6) and (4.9), are more complicated, and their meaning would be much less clear in any naive approach to the problem not involving ESU.

As in the transparent case, each of the two commutator functions $\tilde{G}^j$ can be related to the real part of its corresponding "Feynman" function via

$$\tilde{G}^j(x,0)=2\epsilon^0(\tau)\,\mathrm{Re}\,G^j(x,0)\,, \tag{4.15}$$

where

$$G^j(x,0)=\frac{iK}{8\pi^2}\left[\frac{1}{K\sigma-i0}-(-1)^j\frac{1}{K\sigma-2-i0}\right] \tag{4.16}$$

with $\sigma$ as in (3.24). The $G^j$ satisfy the inhomogeneous equation

$$(\Box-2K)G^j(x,0)=-\delta^4(x)-(-1)^j\delta^4(x_A)\,. \tag{4.17}$$

The appearance of two sources in (4.17) is another manifestation of the fact that in the "reflective" schemes effective Cauchy data can only be consistently imposed on one constant $\tau$ hypersurface.

The relationship between the three quantizations for the massless scalar field in AdS is essentially summarized by the decomposition of the "transparent" one-particle Hilbert space in terms of those of the reflective cases:

$$\mathcal{H}^T=\mathcal{H}^1\oplus\mathcal{H}^2\,. \tag{4.18}$$

Consequently, the Fock spaces are related by

$$\mathcal{F}^T=\mathcal{F}^1\otimes\mathcal{F}^2\,. \tag{4.19}$$

Thus an $n$-particle "transparent" state may be written, rather symbolically, as

$$|n\rangle^T=\lambda_0|n\rangle^1\otimes|0\rangle^2\oplus\lambda_1|n-1\rangle^1\otimes|1\rangle^2\oplus\cdots\oplus\lambda_n|0\rangle^1\otimes|n\rangle^2\,, \tag{4.20}$$

where

$$\sum_0^n|\lambda_i|^2=1\,. \tag{4.21}$$

From the point of view of either "reflective" scheme this will in general appear as a mixture of $n, n-1, \ldots, 0$-particle states. In particular the "transparent" vacuum corresponds only to pure "reflective" vacuum states:

$$|0\rangle^T=|0\rangle^1\otimes|0\rangle^2 \tag{4.22}$$

while a typical one-particle "transparent" state would be interpreted in the $\mathcal{H}^1$ scheme as a mixture of one-particle and vacuum states.

## V. MASSIVE SCALAR FIELDS

The equation of motion for a "conformally" coupled massive spin-zero field in AdS is

$$[\Box+(\mu^2-2K)]\psi=0\,,\quad \mu^2>0\,. \tag{5.1}$$

Most of this section also applies to a minimally coupled field with mass $\mu'=+(\mu^2-2K)^{1/2}$ for $\mu^2\geq 2K$. The only significant difference is that for the minimal theory the canonical and variational energy-momentum tensors are identical.

It should be noted however that for $\mu\neq 0$ the field equation (5.1) is no longer conformally invariant. Thus the method of conformal mapping into ESU employed in Sec. III and IV is less appropriate here (the corresponding ESU field equation will have a position-dependent "mass"). Nevertheless it still proves useful in providing a concrete realization of spatial infinity and simplifying calculations related to conservation laws, as demonstrated for the massless case.

We will begin by considering separable positive-frequency solutions of (5.1) in AdS itself, which will be of the form $\exp(-i\omega\tau)h(\rho,\theta,\phi)$. To ensure that these are single valued in AdS, $\omega$ is required to be an integer. It is convenient to write

$$\mu^2=K(M-1)(M-2)\,,\quad M>2\,. \tag{5.2}$$

Then it is found that nonsingular, finite $B$-norm, separable solutions can only exist if $M$ satisfies either (i) $2<M<\frac{5}{2}$ or (ii) $M=3,4,5\ldots$. So we have something resembling a "mass spectrum" consisting of a small continuum and an unbounded discrete part. The corresponding solutions are

$$\text{(i)}\ \zeta^M_{\omega lm}=N'^M_{\omega l}\exp(-i\omega\tau)(\cos\rho)^M(\sin\rho)^l \times{}_2F_1\left(\tfrac{1}{2}(l+M-\omega),\tfrac{1}{2}(l+M+\omega);\tfrac{3}{2}+l;\sin^2\rho\right)Y_l^m(\theta,\phi)\,, \tag{5.3}$$

where $\omega, l, m$ are integers such that $l\geq|m|$ and ${}_2F_1(a,b;c;z)$ are hypergeometric functions,[12]

$$\text{(ii)}\ \psi^M_{\omega lm}=N^M_{\omega l}\exp(-i\omega\tau)(\cos\rho)^M(\sin\rho)^l \times P_n^{(l+1/2,\,M-3/2)}(\cos 2\rho)Y_l^m(\theta,\phi)\,, \tag{5.4}$$

where $\omega=M+l+2n$ and $l,m,n$ are integers such that $l\geq|m|$, $n\geq 0$. The normalization constants in this case are

$$N^M_{\omega l}=\frac{n!\,\Gamma(n+l+M)}{\Gamma(n+l+\frac{3}{2})\Gamma(n+M-\frac{1}{2})} \tag{5.5}$$

and the $P_n^{(\alpha,\beta)}(z)$ are Jacobi polynomials.[12]

If, as in Sec. III we were to require that the ESU counterparts of these functions be $C^\infty$ then (i) would be lost. Nor would (i) occur if only the minimally coupled case is considered. In any event, our attention will be focused mainly on the solutions (ii).

For each $M = 3, 4, 5, \ldots$ all the solutions have the same definite "parity" under the point to antipodal point transformation, and hence so do their linear combinations. In particular,

$$\psi^M(x_A) = (-1)^M \psi^M(x) . \tag{5.6}$$

When restricted to a single spacelike hypersurface the $\psi^M_{\omega l m}$ form a complete set and it is found that energy, angular momentum, etc., are conserved when integrated over such a surface.

It is clear then that for $M$ odd these cases are analogous to the massless reflective case (1) [cf. (4.5)] while for $M$ even they are analogous to the massless reflective case (2) [cf. (4.8)]. Hence the quantization of these massive fields may be modeled on the quantizations of Sec. IV. The $\psi^M_{\omega l m}$ form an orthonormal basis for the Hilbert space $\mathcal{H}^M$ with $B$-norm (2.5) (integration over $\Sigma_1$ say). They also satisfy (2.6) where the classical commutator function, which evolves "effective Cauchy data" specified on a single hypersurface, is given by

$$\tilde{G}^M(x,0) = \frac{K}{4\pi}\,\epsilon^0(\tau)\{\delta(K\sigma) - (-1)^M\delta(K\sigma - 2) + [\theta(-K\sigma) - \theta(2 - K\sigma)]P'_{M-2}(1 - K\sigma)\}, \tag{5.7}$$

where $P'_N(z)$ denotes the derivative of the Legendre polynomial of degree $N$.

The support of $\tilde{G}^M(x,0)$ is shown in Fig. 4 and reflects in a striking way the behavior of classical massive particles in AdS, all timelike geodesics through $\tau = \rho = 0$ lying entirely within the shaded regions. The fact that such geodesics reconverge and do not reach spatial infinity also offers a heuristic classical explanation for the lack of a "transparent" quantization scheme for massive fields.

In fact the massless reflective cases fit into the present scheme in a very natural way. Comparing (4.4) with (5.4) and using Eq. 10.9 (21) of Ref. 12 it is seen that the $\mathcal{H}^1$ basis functions of Sec. IV correspond to $M = 1$. Likewise, comparing (4.7) with (5.4) and using Eq. 10.9 (22) of Ref. 12, the $\mathcal{H}^2$ basis functions correspond to $M = 2$. This identification is clear cut, despite the fact that $M = 1$ and $M = 2$ are indistinguishable from the point of view of the wave equation (5.1).

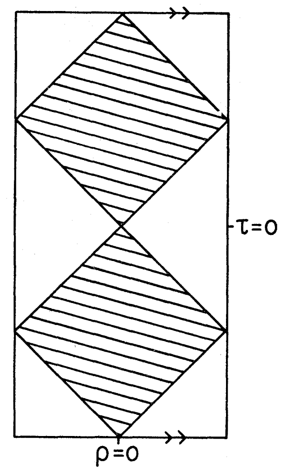


FIG. 4. The support of $\tilde{G}^M(x,0)$ for a massive field. It is regular within the shaded regions, singular on their boundary, and zero elsewhere.

Thus the quantization of these "special mass" fields is completed and it is convenient to briefly mention the relationship with the group-theory approach at this stage. Fronsdal[6] has shown, by group-theoretic arguments, that there exists a collection of irreducible representations of the universal covering group of SO(3, 2), labelled by a positive number $M$ ($E_0$ in his terminology), which correspond to solutions of the wave equation (5.1) in CAdS. Those which may be defined on AdS correspond to $M$ integral and reduce to ours, but the representation is now only faithful for SO(3, 2) itself. Thus $\mathcal{H}^M$ does carry the desired representation of the AdS isometry group.

The Feynman function generalizing those of Sec. IV is found by solving (3.25) with a suitable mass term included:

$$G^M(x,0) = \frac{iK}{4\pi^2}\,Q'_{M-2}(1 - K\sigma + i0) \tag{5.8}$$

$$= \frac{iK}{8\pi^2}\left[\left(\frac{1}{K\sigma - i0} - \frac{1}{K\sigma - 2 - i0}\right)P_{M-2}(1 - K\sigma) + [\ln(K\sigma - i0) - \ln(K\sigma - 2 - i0)]P'_{M-2}(1 - K\sigma) + 2W'_{M-3}(1 - K\sigma)\right], \tag{5.9}$$

where $Q_N(z)$ is a Legendre function of the second kind and $W_N(z)$ is a polynomial of degree $N$ given by Christoffel's formula.[12] $G^M(x,0)$ satisfies the inhomogeneous wave equation

$$[\Box + (\mu^2 - 2K)]G^M(x,0) = -\delta^4(x) - (-1)^M\delta^4(x_A) \tag{5.10}$$

and is related to the commutator function by (4.15), where $M$ and $j$ are now interchangeable.

Despite the existence of closed timelike curves, $G^M(x,x')$ can be related to the vacuum expectation value of a "time-ordered product" in the following sense:

$$-i\hbar G^M(x,x') = \langle 0|T^0\{\hat\psi^M(x)\hat\psi^M(x')\}|0\rangle \tag{5.11}$$

where

$$T^0\{\hat\psi^M(x)\hat\psi^M(x')\} \equiv \theta^0(\tau,\tau')\hat\psi^M(x)\hat\psi^M(x') + [1-\theta^0(\tau,\tau')]\hat\psi^M(x')\hat\psi^M(x) \tag{5.12}$$

and

$$\theta^0(\tau,\tau') \equiv \theta(\sin(\tau-\tau')). \tag{5.13}$$

In effect the time ordering is carried out using the smaller angle between $\tau$ and $\tau'$.

## VI. DISCUSSION

We have constructed three quantizations for a conformally coupled massless scalar field by considering CAdS as being "part of" ESU. One scheme is associated with "transparent" boundary conditions, while the other two correspond to "reflective" boundary conditions. It should be stressed that these latter conditions are those associated with a "box" only when referring to fields propagating on ESU. This remark will be important when attempting to quantize massless conformally coupled fields on other more complex, static, non-globally hyperbolic manifolds. The two reflective schemes generalize to include a sequence of massive fields for each of which there is a unique natural quantization.

In this paper the Feynman functions for these schemes were all constructed from first principles. An alternative procedure would be to start with the Feynman function for de Sitter space[15] (we have converted this to the conformally coupled field)

$$G^{dS}(x,x') = \frac{iK}{16\pi^2}\Gamma(3-M')\Gamma(M') \times {}_2F_1(3-M', M'; 2; 1-\tfrac{1}{2}K(\sigma+i0)) \tag{6.1}$$

where

$$M' = \tfrac{1}{2}[3 + (1+4\mu^2/K)^{1/2}]$$

and $\mu^2 \le 0$ for de Sitter space. [$(-\mu^2)$ is the de Sitter (mass)$^2$.] Then try to analytically continue in $\sigma$ and $\mu^2$ to their anti-de Sitter values. This is straightforward for $\mu^2 = 0$, and yields precisely the "transparent" AdS Feynman function (3.26). The "reflective" massless AdS Feynman functions are obtained as analytic continuations of a de Sitter Green function which solves the de Sitter inhomogeneous wave equation with two sources, one at $x'$ and the other at the de Sitter antipodal point to $x'$. The function (6.1) develops simple poles in $M'$ at the points $3, 4, 5, \ldots,$ and so the AdS Feynman functions, (5.8), for the allowed masses of Sec. V are not related by analytic continuation to (6.1). Indeed, these masses are precisely those for which the hypergeometric equation (3.25) (with the appropriate mass term included) has degenerate solutions.[12] Furthermore, solving (3.25) for the AdS masses corresponding to those nonintegral $M'$ greater than 2, yields Green's functions which might be interpreted as belonging to a massive field propagating in CAdS. However, demanding that such a function be a solution of the inhomogeneous CAdS wave equation having the support of its real part entirely contained within and on the light cone of $x'$ [cf. (3.28)], forces it to correspond to two sources, one at $x'$ and the other at $x'_A$. This resulting function cannot be obtained by analytic continuation from a de Sitter Green function enjoying similar properties.

A further point of interest is that both the Minkowski and de Sitter space-times may be conformally mapped into ESU in a similar manner to AdS. Moreover the four-volumes of the images of all three space-times in ESU are the same. The solutions of the conformal massless wave equation in ESU are periodic in such a way that they are uniquely determined by their behavior in any of the above images. Thus a basis for such functions in ESU may be mapped back to form a basis in anti-de Sitter, de Sitter, or Minkowski space. Indeed, mapping back the basis (3.13) to Minkowski space results in the "elementary states" of twistor theory.

## ACKNOWLEDGMENTS

One of us (S.J.A.) wishes to thank the University of London for a Mary Scharlieb Studentship. Another of us (D.S.) gratefully acknowledges receipt of a Science Research Council grant, while C. J. Isham is most grateful to Professor B. Zumino and Professor S. Deser for conversations on the role of anti-de Sitter space in supergravity which partly motivated this work.

[1]For recent reviews see B. S. DeWitt, Phys. Rep. 19C, 297 (1975); L. Parker, in *Proceedings of the Symposium on Asymptotic Properties of Space-Time*, edited by F. P. Esposito and L. Witten (Plenum, New York, 1977); C. J. Isham, in *Proceedings of the Eighth Texas Symposium on Relativistic Astrophysics*, edited by M. D. Papagiannis (New York Academy of Sciences, New York, 1977).
[2]S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).
[3]B. Zumino, Nucl. Phys. B127, 189 (1977); S. Deser and B. Zumino, Phys. Rev. Lett. 38, 1433 (1977).
[4]Abdus Salam and J. Strathdee, Phys. Lett. 67B, 429 (1977).
[5]C. Fronsdal, Rev. Mod. Phys. 37, 221 (1965).
[6]C. Fronsdal, Phys. Rev. D 10, 589 (1974).
[7]C. Fronsdal and R. B. Haugen, Phys. Rev. D 12, 3810 (1975).
[8]C. Fronsdal, Phys. Rev. D 12, 3819 (1975).
[9]G. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2738 (1977).
[10]S. W. Hawking and G. Ellis, *The Large Scale Structure of Space-time* (Cambridge Univ. Press, London, 1973).
[11]R. Penrose, in *Relativity, Groups* and *Topology*, edited by B. S. DeWitt and C. DeWitt (Gordon and Breach, London, 1964).
[12]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdély (McGraw-Hill, New York, 1953), Vols. I, II.
[13]J. S. Dowker, Ann. Phys. (N.Y.) 62, 361 (1971).
[14]C. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).
[15]P. Candelas and D. J. Raine, Phys. Rev. D 12, 965 (1975).