Effect of the self-induced torsion of the Dirac sources on gravitational singularities

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The effect of the torsion induced by the Dirac field on gravitational singularities is investigated. An exactly soluble example of the Dirac source is presented, which satisfies the energy condition for singularity theorems in the Einstein-Cartan theory. The self-induced Dirac torsion appears to enhance rather than avert singularity formation. The zero-mass limit and the neutrino limit of the energy condition for the Dirac sources are also discussed.

I. INTRODUCTION

Recently, interest in the Einstein-Cartan theory has been revived with an expectation that the introduction of intrinsic spin effects into general relativity via the torsion term may possibly avert the singularity formation in gravitational collapse and cosmology.[1,2] As is well known, the singularity theorems[3] show under very general assumptions that singularities cannot be prevented in general relativity insofar as a certain energy condition is met. In a recent paper,[4] Hehl, von der Heyde, and Kerlick obtained an energy condition for singularity theorems in the Einstein-Cartan theory

$$W=(\tilde{\sigma}_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}\tilde{\sigma}_{\lambda}{}^{\lambda})u^{\mu}u^{\nu}\geq 0\,, \tag{1}$$

where $\tilde{\sigma}_{\mu\nu}$ is the torsion-modified source tensor in the Einstein-Cartan equation,

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=(8\pi G/c^4)\tilde{\sigma}_{\mu\nu}\,, \tag{2}$$

and they have shown that all known cosmological models which are free from singularities due to the torsion effect violate the energy condition (1). In a subsequent paper,[5] Kerlick derived W for the torsion-inducing Dirac field in the form

$$W=\Sigma_{\mu\nu}u^{\mu}u^{\mu}-\tfrac{1}{2}mc^2\bar{\psi}\psi\,, \tag{3}$$

by using the modified stress-energy tensor for the Dirac Field,

$$\tilde{\sigma}_{\mu\nu}=\Sigma_{\mu\nu}+\tfrac{3}{16}\hbar c l^2 g_{\mu\nu}(\bar{\psi}\gamma_{\lambda}\gamma_5\psi)(\bar{\psi}\gamma^{\lambda}\gamma_5\psi)\,, \tag{4}$$

where

$$\Sigma_{\mu\nu}=-\tfrac{1}{2}\hbar c(\nabla_{\mu}\bar{\psi}\gamma_{\nu}\psi-\bar{\psi}\gamma_{\nu}\nabla_{\mu}\psi)\,, \tag{5}$$

∇_{μ} being the covariant differential operator with respect to the Christoffel connection. Analyzing a position-independent solution of the torsion-modified Dirac equation, he made an observation that the formation of singularities will be enhanced rather than averted when the Dirac field is taken as the source for the metric and torsion.

The purpose of this paper is to present a soluble example of the Dirac source which does support Kerlick's observation. The vanishing-mass limit and the neutrino limit of the energy condition will also be discussed. Because of the classical nature of the theory, we confine ourselves to c-number fields. As for the notations we basically follow Ref. 5. The metric tensor $g_{\mu\nu}$ when expressed on the tetrad basis takes the form $\eta_{\alpha\beta}=\mathrm{diag}(-1,1,1,1)$ and the spin matrices satisfy $\gamma_{\mu}\gamma_{\nu}+\gamma_{\nu}\gamma_{\mu}=2g_{\mu\nu}$; the Pauli adjoint of a spinor field ψ is defined by $\bar{\psi}=\psi^{\dagger}A$ with a matrix A such that $A^{\dagger}=A$, $(A\gamma_5)^{\dagger}=A\gamma_5$, $(A\gamma_{\mu})^{\dagger}=-A\gamma_{\mu}$, and $(A\gamma_{\mu}\gamma_5)^{\dagger}=A\gamma_{\mu}\gamma_5$, where $\gamma_5=(\sqrt{-g}/4!)\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}$ with $\epsilon_{0123}=1$. Note that our $\gamma^{\mu}\nabla_{\mu}\psi$ is identical to $\gamma^{\alpha}\nabla^{\{\}}_{\alpha}\psi$ of Ref. 5.

II. SPECIAL CLASS OF DIRAC SOURCES

The Dirac field ψ in the self-induced curvature and torsion is known to obey the modified Dirac equation[6]

$$\gamma^{\mu}\nabla_{\mu}\psi+\tfrac{3}{8}l^2(\bar{\psi}\gamma_{\mu}\gamma_5\psi)\gamma^{\mu}\gamma_5\psi+(mc/\hbar)\psi=0\,, \tag{6}$$

where $l^2=8\pi G\hbar/c^3$. It is not an easy matter to solve this equation exactly, but one can talk about the energy condition for a certain class of c-number solutions without knowing their explicit forms.

Consider a field ψ such that

$$\begin{aligned}\nabla_{\mu}\psi=&A(\bar{\psi}\psi)\gamma_{\mu}\psi+B(\bar{\psi}\gamma_5\psi)\gamma_{\mu}\gamma_5\psi\\&+C(\bar{\psi}\gamma_{\mu}\psi)\psi+D(\bar{\psi}\gamma_{\mu}\gamma_5\psi)\gamma_5\psi+E\gamma_{\mu}\psi\,,\end{aligned} \tag{7}$$

where A, B, C, D, and E are all real constants. This field satisfies the Dirac equation (6) if

$$4B+C+D=-3l^2/8\,, \tag{8}$$

$$(A-B)(\bar{\psi}\psi)+E=-mc/(4\hbar)\,, \tag{9}$$

as is easily checked with the aid of the identities

$$(\bar{\psi}\gamma_{\mu}\psi)\gamma^{\mu}\psi=(\bar{\psi}\psi)\psi+(\bar{\psi}\gamma_5\psi)\gamma_5\psi\,, \tag{10}$$

$$(\bar{\psi}\gamma_{\mu}\psi)\gamma^{\mu}\psi=(\bar{\psi}\gamma_{\mu}\gamma_5\psi)\gamma^{\mu}\gamma_5\psi\,, \tag{11}$$

which are proven in the Appendix. The adjoint relation of (7) is

$$\nabla_{\mu}\bar{\psi}=-A(\bar{\psi}\psi)\bar{\psi}\gamma_{\mu}+B(\bar{\psi}\gamma_5\psi)\bar{\psi}\gamma_{\mu}\gamma_5$$

$$-C(\bar{\psi}\gamma_\mu\psi)\bar{\psi}+D(\bar{\psi}\gamma_\mu\gamma_5\psi)\bar{\psi}\gamma_5-E\bar{\psi}\gamma_\mu. \tag{12}$$

The constraint (7) substantially simplifies the problem. Substituting (7) and (12) into (14) readily leads to

$$\tilde{\sigma}_{\mu\nu}=\hbar c[A(\bar{\psi}\psi)^2+B(\bar{\psi}\gamma_5\psi)^2+E(\bar{\psi}\psi) + \tfrac{3}{16}l^2(\bar{\psi}\gamma_\lambda\gamma_5\psi)(\bar{\psi}\gamma^\lambda\gamma_5\psi)]g_{\mu\nu} + \hbar c[C(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma_\nu\psi)+D(\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}\gamma_\nu\gamma_5\psi)] \tag{13}$$

and

$$W=\hbar c[A(\bar{\psi}\psi)^2+B(\bar{\psi}\gamma_5\psi)^2+E(\bar{\psi}\psi) + \tfrac{1}{2}(\tfrac{3}{8}l^2+C+D)(\bar{\psi}\gamma_\lambda\gamma_5\psi)(\bar{\psi}\gamma^\lambda\gamma_5\psi)] + \hbar c[C(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma_\nu\psi)+D(\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}\gamma_\nu\gamma_5\psi)]u^\mu u^\nu. \tag{14}$$

It is also easy to assess the integrability of (7). Since the commutator of the covariant differential operators when acting on ψ takes the form[7]

$$(\nabla_\mu\nabla_\nu-\nabla_\nu\nabla_\mu)\psi=-\tfrac{1}{4}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\psi, \tag{15}$$

with $\gamma_{\rho\sigma}=\frac{1}{2}(\gamma_\rho\gamma_\sigma-\gamma_\sigma\gamma_\rho)$, we see by computing the left-hand side of (15) for (7) that the field in question can exist in a space-time of curvature

$$R_{\mu\nu\rho\sigma}=4(AB-BD+DA)[(\bar{\psi}\gamma_\nu\gamma_5\psi)(\bar{\psi}\gamma_\rho\gamma_5\psi)g_{\mu\sigma}-(\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}\gamma_\rho\gamma_5\psi)g_{\nu\sigma}+(\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}_\sigma\gamma_5\psi)g_{\nu\rho}-(\bar{\psi}\gamma_\nu\gamma_5\psi)(\bar{\psi}\gamma_\sigma\gamma_5\psi)g_{\mu\rho}] + 4[A^2(\bar{\psi}\psi)^2+B^2(\bar{\psi}\gamma_5\psi)^2+2AE(\bar{\psi}\psi)+E^2](g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}), \tag{16}$$

provided that

$$C=0,\quad 2(A-B)D(\bar{\psi}\psi)-(B-D)E=0. \tag{17}$$

The condition (16) together with (17) in turn assures that the constraint (7) is integrable. The modified stress-energy tensor found in (13) is to serve as a source to the Einstein-Cartan equation (2), whereas the Einstein tensor resulting from the integrability condition (16) is of the form

$$G_{\mu\nu}=\{12[A^2(\bar{\psi}\psi)^2+B^2(\bar{\psi}\gamma_5\psi)^2+2AE(\bar{\psi}\psi)+E^2] - 8(AB-BD+DA)(\bar{\psi}\gamma_\lambda\gamma_5\psi)(\bar{\psi}\gamma^\lambda\gamma_5\psi)\}g_{\mu\nu} + 8(AB-BD+DA)(\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}\gamma_\nu\gamma_5\psi). \tag{18}$$

The Dirac field under the constraint (7) is therefore meaningful only when (13) and (18) are consistently linked by the Einstein-Cartan equation (2). This imposes further restrictions on the choice of the parameters.

Now we take an example belonging to this class in order to study the energy condition (1). In the case where $A=B=-D=-\frac{1}{8}l^2$ and $C=E=0$, the conditions (8) and (17) are satisfied, and (9) is met only if $m=0$. Accordingly, (7) becomes

$$\nabla_\mu\psi=-\tfrac{1}{8}l^2[(\bar{\psi}\psi)\gamma_\mu+(\bar{\psi}\gamma_5\psi)\gamma_\mu\gamma_5-(\bar{\psi}\gamma_\mu\gamma_5\psi)\gamma_5]\psi. \tag{19}$$

It is obvious that the field of this type is a solution of the Dirac equation (6) without a mass. The modified stress-energy tensor for this field, as follows from (13), is

$$\tilde{\sigma}_{\mu\nu}=\tfrac{1}{16}\hbar c l^2(\bar{\psi}\gamma_\lambda\gamma_5\psi)(\bar{\psi}\gamma^\gamma\gamma_5\psi)g_{\mu\nu} + \tfrac{1}{8}\hbar c l^2(\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}\gamma_\nu\gamma_5\psi), \tag{20}$$

and the Einstein tensor (18) resulting from the integrability condition (16) becomes

$$G_{\mu\nu}=\tfrac{1}{16}l^4(\bar{\psi}\gamma_\lambda\gamma_5\psi)(\bar{\psi}\gamma^\lambda\gamma_5\psi)g_{\mu\nu} + \tfrac{1}{8}l^4(\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}\gamma_\nu\gamma_5\psi), \tag{21}$$

which apparently equals the stress-energy tensor (20) multiplied by $l^2/\hbar c=8\pi G/c^4$. Thus the field ψ of the present choice (19), being integrable, satisfies the Einstein-Cartan equation (2). Substitution of (20) into (3) yields

$$W=\tfrac{1}{8}\hbar c l^2[(\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}\gamma_\nu\gamma_5\psi)u^\mu u^\nu + (\bar{\psi}\gamma_\lambda\gamma_5\psi)(\bar{\psi}\gamma^\lambda\gamma_5\psi)]. \tag{22}$$

In a frame with $u^\mu=(1,0,0,0)$, we obtain

$$W=\tfrac{1}{8}\hbar c l^2[(\bar{\psi}\gamma_0\gamma_5\psi)^2+(\bar{\psi}\gamma_\lambda\gamma_5\psi)(\bar{\psi}\gamma^\lambda\gamma_5\psi)] = \tfrac{1}{8}\hbar c l^2\sum_{k=1}^{3}(\bar{\psi}\gamma_k\gamma_5\psi)^2, \tag{23}$$

which is evidently positive-definite because $(\bar{\psi}\gamma_k\gamma_5\psi)$ is real for $k=1,2,3$. The result indeed supports Kerlick's expectation.

III. THE ZERO-MASS LIMIT

The example we have considered in the previous section is concerned with a torsion-including massless Dirac field whose presence could enhance singularity formation. A question may arise as to whether such a massless example can be sufficiently representative of the torsion-including Dirac sources. In the limit $m\to 0$, the Dirac equation (6) takes on the form of Heisenberg's nonlinear equation[8] defined in a Riemannian background; it becomes the neutrino equation only if the torsion term is absent. Therefore, in the Einstein-Cartan theory not all massless Dirac fields describe neutrinos. In fact, Heisenberg[8] looked for all possible states of matter in the nonlinear character of his equation without any presumed mass. The only distinct role the mass term plays in the Dirac equation (6) is to break chiral symmetry. It is conceivable that the torsion term will generate a mass as a consequence of the spontaneous chiral-symmetry breaking. Yet it seems unlikely that the constant m assigned *ab initio* to each Dirac field in the Einstein-Cartan theory plays any more significant role than an adjustable parameter. The value $m=0$ is merely an ordinary choice of the parameter. However, the neutrino limit is something which has to be considered separately.

At a glance, the mass term in the expression (3) for W appears to cause a negative effect on the energy condition. However, we must notice that, while $\psi^\dagger\psi$ is positive definite, the bilinear scalar $\bar\psi\psi$ can be negative and that the net effect of the mass term is not always negative. If in a certain case the negative effect wholly due to the mass term happens to dominate, then the limiting process $m\to 0$ will act to shift the turning points of the sign of W. In this connection, it would be interesting to take another look at Kerlick's example in a nonrotating spatially flat cosmological model, for which

$$W=\tfrac{1}{2}mc^2\bar\psi\psi+\frac{3\pi G\hbar^2}{c^2}(\bar\psi\gamma_\lambda\gamma_5\psi)(\bar\psi\gamma^\lambda\gamma_5\psi). \tag{24}$$

As readily follows from (10) and (11),

$$(\bar\psi\gamma_\lambda\gamma_5\psi)(\bar\psi\gamma^\lambda\gamma_5\psi)=(\bar\psi\psi)^2+(\bar\psi\gamma_5\psi)^2\geq 0. \tag{25}$$

Therefore it is obvious that $W\geq 0$ in the zero-mass limit. Since $\bar\psi\psi$ may be negative, the positive-definiteness of W for $m\neq 0$ is not immediately clear. The following consideration may be instructive though it provides no general assurance of the positive-definiteness of W in (24) for $m\neq 0$. Suppose $|\bar\psi\psi|=-\bar\psi\psi$ at a certain instance. Suppose the mass density ρ may be approximated by $m_N|\bar\psi\psi|$ with a nucleon mass m_N and $|\bar\psi\psi|\gg|\bar\psi\gamma_5\psi|$. Then (24) can be expressed as

$$W=-\frac{mc^2}{2m_N}\rho+\frac{3\pi G\hbar^2}{{m_N}^2c^2}\rho^2. \tag{26}$$

This is positive for $\rho>\bar\rho$ and negative for $\rho<\bar\rho$ where $\bar\rho=mm_Nc^4/(6\pi G\hbar^2)$ which is $\sim 10^{54} g/cm^3$ for $m=m_N$. In the limit $m\to 0$, the turning point $\bar\rho$ shifts to zero and the negative region disappears. Since $\rho>\bar\rho$ is the region pertinent to the gravitational collapse, (26) suggests irrespective of the limiting process that singularity formation cannot be averted. The approximation adopted above may not always be valid. If the super-relativistic approximation is employed as another extreme, then $|\bar\psi\psi|$ will assume a very small value and the positive-definite spin-density term will dominate in (27). Again we obtain $W\geq 0$.

Finally we wish to remark on the neutrino limit. The neutrino field may be characterized by the two-component condition,[9] say $\psi=i\gamma_5\psi$. In c-number theory, the two-component condition demands not only the mass term to vanish but also the torsion term to disappear.[10] As is evident from (11), $(\bar\psi\gamma_\mu\gamma_5\psi)\gamma^\mu\gamma_5\psi=0$ if and only if $\psi=i\gamma_5\psi$. This implies that the c-number two-component neutrino field will induce no torsion effect upon itself.[10,11] Furthermore the disappearance of spin-spin interactions reduces the Einstein-Cartan equation to the original Einstein equation. In the two-component limit, Kerlick's W of (24) vanishes. In contrast, the example proposed in the previous section reduces to a special class of neutrinos whose stress-energy tensor is of the form

$$\tilde\sigma_{\mu\nu}=\Sigma_{\mu\nu}=-\tfrac{1}{8}\hbar c l^2(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma_\nu\psi). \tag{27}$$

Simply taking the two-component limit of (22), we obtain for this class of neutrinos

$$W=-\tfrac{1}{8}\hbar c l^2(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma_\nu\psi)u^\mu u^\nu, \tag{28}$$

which is, unlike Kerlick's, nonzero and positive-definite. The nonvanishing mass limit of W indicates that the torsion-inducing massless Dirac field considered in our example has indeed a proper neutrino limit. The neutrinos of this type would enhance singularity formation just as the electromagnetic fields would. It may also be worth mentioning that the neutrinos with (27), which we have referred to as a restricted class of neutrinos elsewhere,[10] satisfy the Rainich-Misner-Wheeler conditions for the null electromagnetic fields,[12] and that an exact and explicit solution of the neutrino field slightly more general than that of the restricted class has been found by Griffiths and Newing.[13]

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APPENDIX: DERIVATION OF THE IDENTITIES (10) AND (11)

It is tedious but elementary to verify the identities (10) and (11) on the basis of a specific representation of γ's.[14] Here we derive them in slightly more general forms from the Pauli-Fierz identity[15],

$$\sum_{A=1}^{16} \gamma^A{}_{ab}\gamma^A{}_{cd} = 4\delta_{ad}\delta_{bc}\,, \tag{A1}$$

where $\gamma^A = \{I, \gamma_\mu, i\gamma_{\mu\nu}, \gamma_\mu\gamma_5, i\gamma_5\}$. Using two four-component spinors χ and ψ, we write (A1) as

$$\sum_A (\bar\psi P\gamma^A\psi)Q\gamma^A\psi = 4(\bar\psi P\chi)Q\psi\,, \tag{A2}$$

or more explicitly,

$$3(\bar\psi P\psi)Q\chi - (\bar\psi P\gamma_5\psi)Q\gamma_5\chi + (\bar\psi P\gamma_\mu\psi)Q\gamma^\mu\chi + (\bar\psi P\gamma_\mu\gamma_5\psi)Q\gamma^\mu\gamma_5\chi - \tfrac{1}{2}(\bar\psi P\gamma_\mu\gamma_\nu\psi)Q\gamma^\mu\gamma^\nu\chi = 4(\bar\psi P\chi)Q\psi\,, \tag{A3}$$

where P and Q are arbitrary 4×4 matrices. Insomuch as (A3) is valid, the following identity holds true:

$$3(\bar\psi P\gamma_5\psi)Q\gamma_5\chi - (\bar\psi P\psi)Q\chi + (\bar\psi P\gamma_\mu\gamma_5\psi)Q\gamma^\mu\gamma_5\chi + (\bar\psi P\gamma_\mu\psi)Q\gamma^\mu\chi - \tfrac{1}{2}(\bar\psi P\gamma_\mu\gamma_\nu\gamma_5\psi)Q\gamma^\mu\gamma^\nu\gamma_5\chi = 4(\bar\psi P\gamma_5\chi)Q\gamma_5\psi\,. \tag{A4}$$

Set $\psi=\chi$. Then (A3) less (A4) yields

$$(\bar\psi P\gamma_\mu\gamma_\nu\psi)Q\gamma^\mu\gamma^\nu\psi = (\bar\psi P\gamma_\mu\gamma_\nu\gamma_5\psi)Q\gamma^\mu\gamma^\nu\gamma_5\psi\,, \tag{A5}$$

whereas (A3) and (A4) plus (A5) results in

$$2(\bar\psi P\psi)Q\psi + 2(\bar\psi P\gamma_5\psi)Q\gamma_5\psi - 2(\bar\psi P\gamma_\mu\psi)Q\gamma^\mu\psi - 2(\bar\psi P\gamma_\mu\gamma_5\psi)Q\gamma^\mu\gamma_5\psi + (\bar\psi P\gamma_\mu\gamma_\nu\psi)Q\gamma^\mu\gamma^\nu\psi = 0\,. \tag{A6}$$

The resultant identity (A6) should hold even if P and Q are replaced respectively by $P\gamma_\lambda$ and $Q\gamma^\lambda$. Doing these replacements and summing over λ, we obtain

$$2(\bar\psi P\gamma_\lambda\psi)Q\gamma^\lambda\psi + 2(\bar\psi P\gamma_5\psi)Q\gamma^\lambda\gamma_5\psi - 2(\bar\psi P\gamma_\lambda\gamma_\mu\psi)Q\gamma^\lambda\gamma^\mu\psi - 2(\bar\psi P\gamma_\lambda\gamma_\mu\gamma_5\psi)Q\gamma^\lambda\gamma^\mu\gamma_5\psi + (\bar\psi P\gamma_\lambda\gamma_\mu\gamma_\nu\psi)Q\gamma^\lambda\gamma^\mu\gamma^\nu\psi = 0\,. \tag{A7}$$

The last term on the left of (A7) can be expressed as

$$(\bar\psi P\gamma_\lambda\gamma_\mu\gamma_\nu\psi)Q\gamma^\lambda\gamma^\mu\gamma^\nu\psi = 10(\bar\psi P\gamma_\lambda\psi)Q\gamma^\lambda\psi - 6(\bar\psi P\gamma_\lambda\gamma_5\psi)Q\gamma^\lambda\gamma_5\psi\,, \tag{A8}$$

with the help of the relation

$$\gamma_\lambda\gamma_\mu\gamma_\nu = (\sqrt{-g}/4!)\epsilon_{\lambda\mu\nu\rho}\gamma^\rho\gamma_5 + g_{\mu\nu}\gamma_\lambda - g_{\lambda\nu}\gamma_\mu + g_{\lambda\mu}\gamma_\nu\,. \tag{A9}$$

As a result, (A7) becomes

$$6(\bar\psi P\gamma_\lambda\psi)Q\gamma^\lambda\psi - 2(\bar\psi P\gamma_\mu\gamma_5\psi)Q\gamma^\lambda\gamma_5\psi - (\bar\psi P\gamma_\lambda\gamma_\mu\psi)Q\gamma^\lambda\gamma^\mu\psi - (\bar\psi P\gamma_\lambda\gamma_\mu\gamma_5\psi)Q\gamma^\lambda\gamma^\mu\gamma_5\psi = 0\,. \tag{A10}$$

Correspondingly, the following identity should also hold:

$$6(\bar\psi P\gamma_\lambda\gamma_5\psi)Q\gamma^\lambda\gamma_5\psi - 2(\bar\psi P\gamma_\lambda\psi)Q\gamma^\lambda\psi - (\bar\psi P\gamma_\lambda\gamma_\mu\gamma_5\psi)Q\gamma^\lambda\gamma^\mu\gamma_5\psi - (\bar\psi P\gamma_\lambda\gamma_\mu\psi)Q\gamma^\lambda\gamma^\mu\psi = 0\,. \tag{A11}$$

While (A10) minus (A11) equals

$$(\bar\psi P\gamma_\lambda\psi)Q\gamma^\lambda\psi = (\bar\psi P\gamma_\lambda\gamma_5\psi)Q\gamma^\lambda\gamma_5\psi\,, \tag{A12}$$

(A10) and (A11) with (A8) give us

$$(\bar\psi P\gamma_\lambda\gamma_\mu\psi)Q\gamma^\lambda\gamma^\mu\psi = (\bar\psi P\gamma_\lambda\psi)Q\gamma^\lambda\psi + (\bar\psi P\gamma_\lambda\gamma_5\psi)Q\gamma^\lambda\gamma_5\psi\,. \tag{A13}$$

Finally, making use of (A12) and (A13) in (A6), we arrive at the identity

$$(\bar\psi P\gamma_\lambda\psi)Q\gamma^\lambda\psi = (\bar\psi P\psi)Q\psi + (\bar\psi P\gamma_5\psi)Q\gamma_5\psi\,, \tag{A14}$$

which we have referred to as the Pauli-Kofink relation.[10] In particular, with $P=Q=1$, (A14) reduces to (10) and (A12) to (11).

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