General-relativistic nonlinear field: A kink solution in a generalized geometry

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A time-independent spherically symmetric solution of general-relativistic nonlinear field equations is obtained. It is shown that the nonlinear negative-energy scalar field has a localized solution with a positive mass. Wheeler's wormhole-type geometry is generated by the field. It can be regarded as a three-dimensional extension of the usual kink solution on the generalized spatial topology, connecting the two vacuum states from one asymptotically flat space to the other through the Rosen-Einstein bridge. The solution is shown to be completely singularity-free.

I. INTRODUCTION

The recent development of a nonlinear field model of elementary particles presented an interesting viewpoint on the origin of their mass spectrum and structure. A stable and static solution of the nonlinear field equation has been shown to have a particle character and is called a "kink" or "soliton".[1,2]

The simplest version of such theories is the so-called $\lambda\phi^4$ model. For the one-dimensional-space case, its kink solution has several interesting properties. One of them is that it is topologically separated from the vacuum state so that it is considered to represent a fermion in this space.[2] The idea of defining a fermion as a state topologically distinguishable from the vacuum is very interesting and useful, for example, to explain the conservation of baryon numbers.

However, unfortunately, a simple extension of this model for a three-dimensional case encounters a difficulty. The pseudovirial theorem[3,4] does not permit a static, nonsingular, spherically symmetric solution if the potential term of the scalar field is defined to be positive definite.

On the other hand, the formulation of the problem in the view of general relativity brings a new feature to the theory.[4,5] The point is that the effect of general relativity alters the curvature of the spacetime as well as its topological structure; hence, the pseudovirial theorem is also affected.

In this paper, we show that the general-relativistic treatment permits the kink-like solution of the simple $\lambda\phi^4$ source-free Lagrangian. This is possible only if we modify the topological structure of the space geometry.

In Sec. II we briefly review why the non-general-relativistic $\lambda\phi^4$ theory does not have a three-dimensional kink solution. We then show in Sec. III how the effect of general relativity alters the situation. In Sec. IV we show some numerical examples of solutions and discuss the consequences. The geometry of the spacetime is also investigated.

II. FIELD EQUATIONS

We write the Lagrangian density as

$$\kappa\mathcal{L} = (-g)^{1/2}\{\tfrac{1}{2}R + \epsilon[S_{,\alpha}S_{,\beta}g^{\alpha\beta} - V(S^2)]\}, \quad \kappa = 8\pi G/c^4, \tag{1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, R is the scalar curvature, S is a scalar field, and the notation $S_{,\alpha}$ denotes the derivative of S with respect to the coordinate x^α. V is a potential depending only on S^2, and ϵ is the signature of the field S, and takes the value $+1$ (usual field) or -1 (ghost field).

For a static and spherically symmetric case, we may choose the line element as

$$ds^2 = e^{2\eta}(dx^0)^2 - e^{2\alpha}(dr)^2 - r^2 d\Omega^2, \tag{2}$$

where η and α are functions of radial coordinate r.

Together with the definition of line element (2) Einstein's equation reduces to the following equations[4,5]:

$$\eta_1 = \epsilon r S_1^{\,2} - \alpha_1, \tag{3}$$

$$2r\alpha_1 = \epsilon r^2 S_1^{\,2} + 1 - (1 - \epsilon r^2 V)e^{2\alpha}, \tag{4}$$

$$S_{11} + (\eta_1 - \alpha_1 + 2/r)S_1 - Se^{2\alpha}dV/dS^2 = 0, \tag{5}$$

where the subscript 1 means the derivative with respect to r. In the weak gravitation limit, the equivalence of energy source and gravitation source gives the pseudovirial theorem[4] written as

$$\langle S_1^{\,2}\rangle + 3\langle V\rangle = 0, \tag{6}$$

where $\langle A\rangle$ means the total space integration of A,

$$\langle A\rangle = 4\pi\int_0^\infty r^2 dr A .$$

Thus it is clear to see that the above pseudovirial theorem does not permit a nonsingular static and spherically symmetric solution in the weak gravitation limit if V is positive definite. The situation is found to be the same even in the nonlinear limit of gravitation provided that the metrics are nonsingular everywhere.[4]

In this paper, we investigate the potential

$$V(S^2)=\frac{1}{2}\left(\frac{\mu}{f}\right)^2(1-f^2S^2)^2, \tag{7}$$

where f and μ are constants. Then the classical vacuum state of the field is given by $S_{\rm vac}=\pm f^{-1}$. Equation (7) is nothing but the usual $\lambda\phi^4$ Lagrangian term except for the additional constant $\frac{1}{2}\mu^2/f^2$, which is necessary to eliminate the gravitational source at the classical vacuum state of S.

For the sake of convenience we introduce new variables

$$x=\mu r, \tag{8}$$

$$y=fS, \tag{9}$$

$$q=f^2\mu(1-e^{-2\alpha})r, \tag{10}$$

$$v=(f/\mu)^2V. \tag{11}$$

From Eqs. (3)–(5) we get

$$y''+2y'/x=-[y(1-y^2)+y'f^{-2}(q/x^2-\epsilon xv)]e^{2\alpha}, \tag{12}$$

$$q'=\epsilon x^2(y'^2e^{-2\alpha}+v), \tag{13}$$

where a prime denotes d/dx, and

$$e^{-2\alpha}=1-f^{-2}q/x,\quad v=\tfrac{1}{2}(1-y^2)^2. \tag{14}$$

Note that in the limit of $f\to\infty$ with finite q, Eq. (12) tends to the non-general-relativistic $\lambda\phi^4$ model, and hence there is no nonsingular solution which satisfies the boundary condition. To alter the situation, the second term in the parentheses of the right-hand side of Eq. (12) should be predominant somewhere. However, it was found[4] that the smallness of f alone (strong gravity) is not sufficient to have a consistent solution as long as metrics are nonsingular. In the following section we show that the generalization of the spacetime topology permits static nonsingular solutions for y.

III. NONSINGULAR SOLUTION

As stated before, the set of Eqs. (12)–(14) does not have a nonsingular solution which satisfies the boundary condition, i.e., $|y|$ tends to unity for large x, as long as the metric potential α is finite everywhere.

Now let us drop this condition so that $e^{2\alpha}$ may have a singularity at $x=x_0$. By a simple order analysis of singularity, we find that a consistent solution is possible only if $e^{2\alpha}\propto(x-x_0)^{-1}$. This is nothing but the Schwarzschild-type singularity. However, it is well known that such a singularity in the metric does not imply any physical singularity of the spacetime structure. In Sec. IV we will discuss the geometry of the spacetime implied by our metrics. The behavior of y and q near $x=x_0$ is given as $y\propto(x-x_0)^{1/2}$ and $q=$const, respectively.

Then we write

$$y=\sqrt{\rho}\,Z(\rho), \tag{15}$$

$$e^{-2\alpha}=\rho E(\rho), \tag{16}$$

where $\rho\equiv x-x_0$, and Z and E are analytic functions of ρ near $\rho=0$. In order to maintain the order of singularity, we should have $E(0)\neq 0$.

Inserting Eqs. (15) and (16) into Eqs. (12)–(14), we get

$$\begin{aligned}Z''=&\frac{1}{\rho^2}\left[\frac{1}{4}-\frac{1}{2f^2E}\left(\frac{q}{x^2}-\epsilon xv\right)\right]Z\\&-\frac{1}{\rho}\left\{Z'+\frac{Z}{x}+\frac{1}{E}\left[Z+\frac{1}{f^2}\left(\frac{q}{x^2}-\epsilon xv\right)Z'\right]\right\}\\&+\left(\frac{Z^3}{E}-\frac{2}{x}Z'\right),\end{aligned} \tag{17}$$

$$E=\left(1-\frac{1}{f^2}\frac{q}{x}\right)\Big/\rho, \tag{18}$$

$$q'=\epsilon x^2[\tfrac{1}{4}(Z+2\rho Z')^2E+v], \tag{19}$$

with

$$v=\tfrac{1}{2}(1-\rho Z^2)^2. \tag{20}$$

The requirement of analyticity of Z and E at $\rho=0$ ($x=x_0$) gives the following boundary conditions:

$$\left[\frac{1}{4}-\frac{1}{2f^2E}\left(\frac{q}{x^2}-\epsilon xv\right)\right]_{\rho=0}=0, \tag{21}$$

$$Z(0)\left[\frac{1}{4}-\frac{1}{2f^2E}\left(\frac{q}{x^2}-\epsilon xv\right)\right]'_{\rho=0}-\left\{Z'+\frac{Z}{x}+\frac{1}{E}\left[Z+\frac{1}{f^2}\left(\frac{q}{x^2}-\epsilon xv\right)Z'\right]\right\}_{\rho=0}=0, \tag{22}$$

$$\left(1-\frac{1}{f^2}\frac{q}{x}\right)_{\rho=0}=0. \tag{23}$$

A straightforward but tedious algebra gives

$$q(0)=f^2x_0, \tag{24}$$

$$E(0)=\frac{1}{x_0}(2-\epsilon x_0^2f^{-2}), \tag{25}$$

$$x_0 f^{-2} Z^2(0) = -2\epsilon \,. \tag{26}$$

From Eq. (26) we conclude that the positive-signature case ($\epsilon = +1$) has no solution. For $\epsilon = -1$, we get

$$Z^2(0) = \frac{2}{x_0} f^2 \,, \tag{27}$$

$$E(0) = \frac{1}{x_0}(2 + x_0^2 f^{-2}) \,. \tag{28}$$

The first derivative of Z at $\rho = 0$, $Z'(0)$ is also calculated from Eq. (22) as a function of x_0 and f.

Thus for a given value of f, solutions are completely determined by specifying x_0. On the other hand, the boundary condition $y \to 1$ for $x \to \infty$ sets an eigenvalue problem for x_0. Note that if this boundary condition is satisfied, the metric $e^{2\alpha}$ automatically presents the Schwarzschild asymptotic behavior $e^{2\alpha} \to (1 - 2m/x)^{-1}$ for $x \to \infty$, where m is a constant related to the mass of the system. Thus, our potential Eq. (7) completely specifies the mass without introducing any constant of integration.

IV. SOLUTIONS AND DISCUSSION

Equations (17)–(19) together with the boundary conditions at $x = x_0$ can be solved numerically. For a given value of f, the value of x_0 for which y satisfies the boundary condition at infinity is uniquely determined. In Fig. 1, three solutions of y for different f values are shown. In Fig. 2 we show the typical dependence of q and $e^{-2\alpha}$ on x for the case of $f = 1$.

The asymptotic value of q, $q(\infty)$, is related the mass M of the system by

$$M = \frac{f^{-2}}{2\mu} \frac{c^2}{G} q(\infty) \,, \tag{29}$$

where $q(\infty)$ is a function of f. In Fig. 3, we plotted the quantity $[f^{-2} q(\infty)]^{1/2}$ versus f. We note that this quantity tends to zero linearly so that q behaves as $q \sim (f - f_0)^2$ near $f = f_0 \simeq 0.645$. For $f < f_0$, it seems that there is no solution, although we failed to confirm this because of the computational difficulty. In this figure we also plotted x_0/f^2 as a function of f. For large f, it is found that x_0 behaves as $x_0 \simeq 1.30 f^2$.

The metric potential η can be obtained from the equation

$$\eta' = \frac{1}{2} f^{-2} \left[e^{2\alpha}\left(\frac{q}{x^2} + xv\right) - xy'^2 \right] . \tag{30}$$

By virtue of Eqs. (21) and (27), we verify that η does not have a singularity at $x = x_0$. Taking the boundary condition $\eta(\infty) = 0$, we get

$$\eta = -\frac{1}{2} f^{-2} \int_x^\infty \left[e^{2\alpha}\left(\frac{q}{x^2} + xv\right) - xy'^2 \right] dx \,. \tag{31}$$

The time component of the metric $e^{2\eta}$ is also shown in Fig. 2. Because of the nonsingular behavior of the metric $e^{2\eta}$ at $x = x_0$, the structure of our spacetime is different from that of the Schwarzschild solution. The line element near

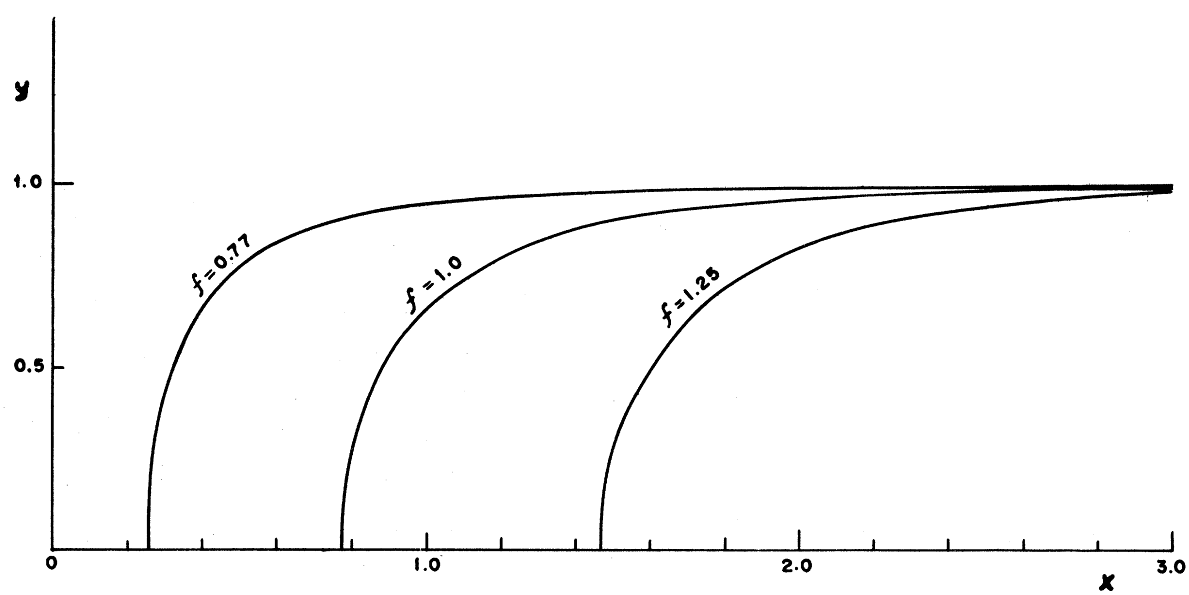

FIG. 1. Solutions of y for $f = 0.77$, 1, and 1.25 plotted versus x.

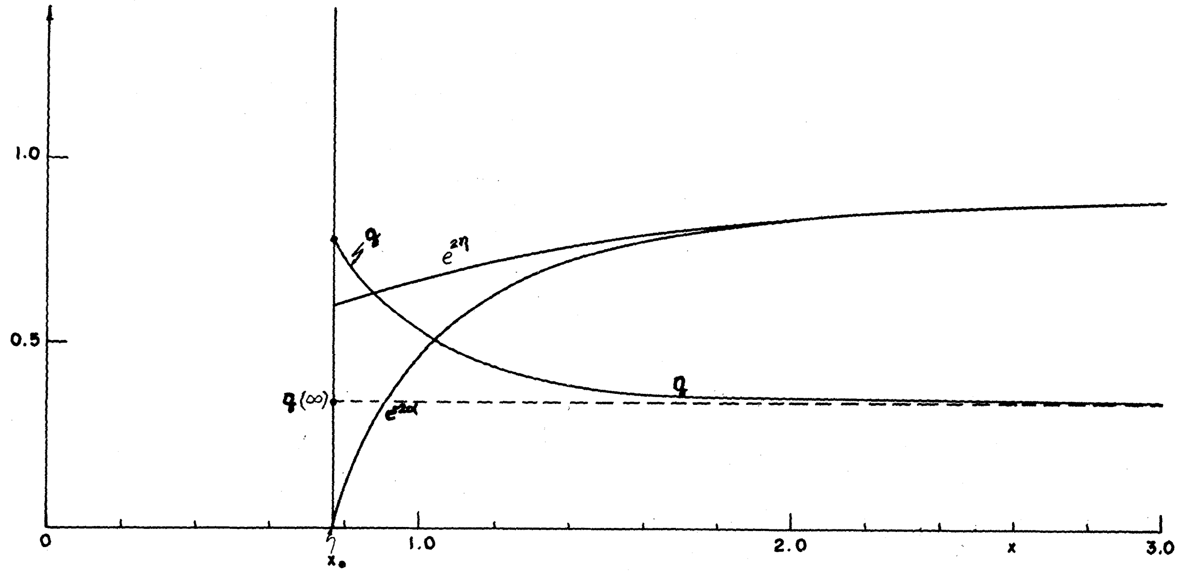

FIG. 2. The function q and metrics $e^{-2\alpha}$ and $e^{2\eta}$ plotted as functions of x for the case of $f = 1$. The asymptotic value of q, $q(\infty)$, defines the mass of the system.

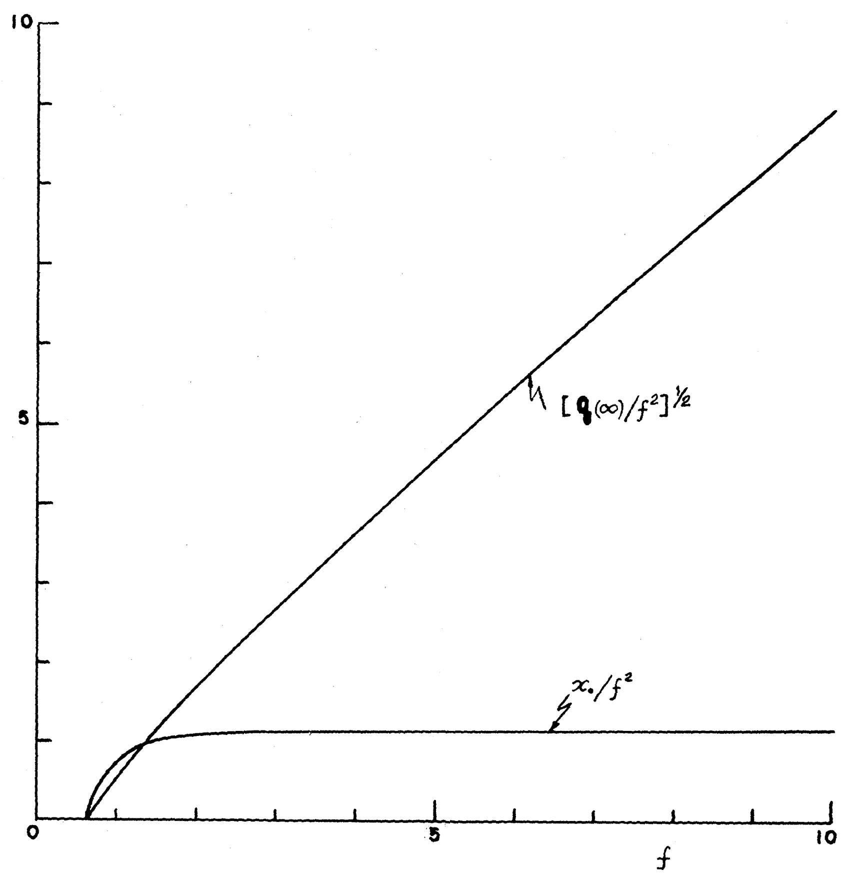

FIG. 3. Quantities $(q/f^2)^{1/2}$ and x_0/f^2 plotted versus f.

$r = r_0 \equiv x_0/\mu$ has the form

$$ds^2 \simeq A(dx^0)^2 - [(1 - r_0/r)^{-1}dr^2 + r^2 d\Omega^2] \quad (32)$$

rather than

$$ds^2 \simeq (1 - r_0/r)(dx^0)^2 - [(1 - r_0/r)^{-1}dr^2 + r^2 d\Omega^2], \quad (33)$$

where A is a constant $(0 < A < 1)$. For $r \gg r_0$, the line element has the asymptotic form

$$ds^2 \simeq (1 - 2m/r)(dx^0)^2 - [(1 - 2m/r)^{-1}dr^2 + r^2 d\Omega^2]. \quad (34)$$

In spite of the above difference, it is easy to see that our space geometry still exhibits a topological structure similar to the Rosen-Einstein bridge[6-8] on the spacelike hypersurface $x^0 = \text{const}$, i.e., two asymptotically flat spaces connected by a bridge[7,8] of radius r_0. Thus the square root of the variable ρ in Eq. (15) is equivalent to the singularity-free coordinate u of Rosen and Einstein which was used to describe the topology of the Schwarzschild geometry. More specifically, we define

$$u^2 = \rho$$

or

$$x = x_0 + u^2. \quad (35)$$

Then the line element is written as

$$ds^2 = e^{2\eta}(dx^0)^2 - \frac{4\mu^{-2}}{E(\rho)}(du)^2 - r^2 d\Omega^2. \quad (36)$$

Since $e^{2\eta}$ and $E(\rho)$ are nonsingular and have no zeros for $0 \leq \rho < \infty$, the line element is always regular.

On the other hand, the radial geodesic equations are given by

$$\frac{d}{ds}\left(e^{2\eta}\frac{dx^0}{ds}\right) = 0,$$
$$\frac{d}{ds}\left(e^{2\alpha}\frac{dr}{ds}\right) = -\eta' e^{2\eta}\left(\frac{dx^0}{ds}\right)^2 + \alpha' e^{2\alpha}\left(\frac{dr}{ds}\right)^2, \quad (37)$$

which reduce to the following equations for $u = u(s)$ and $x^0 = x^0(s)$:

$$\left(\frac{du}{ds}\right)^2 = \frac{\mu^2 E}{4}(e^{-2\eta} - 1),$$
$$\frac{dx^0}{ds} = e^{-2\eta}. \quad (38)$$

From the properties of $e^{-2\eta}$ and E, we easily see that there is no singular behavior for the geodesics $u = u(s)$ and $x^0 = x^0(s)$. The geodesics given by Eq. (38) connect analytically the regions $u > 0$ and $u < 0$. Thus each point of spacetime is specified by u rather than by r. For a given r there correspond two distinct spacetime points, $u > 0$ and $u < 0$. Both $u > 0$ and $u < 0$ spaces are asymptotically flat, and are connected to each other by a Rosen-Einstein bridge of radius r_0. However, unlike the case of the Schwarzschild solution, the region $r < r_0$ is completely disconnected from our space. The geodesics connect the upper $(u > 0)$ space to the lower $(u < 0)$ space, but never penetrate into the region $r < r_0$. Furthermore, the signature of the manifold in this region is zero, so that it does not correspond to the physical spacetime.

On the other hand, we observe that our field solution $y = \sqrt{\rho}\, Z(\rho)$ is a part of an entire function $y^2 = \rho Z^2(\rho)$. The counterpart $y = -\sqrt{\rho}\, Z(\rho)$ is also a solution of the field equations. Taking the branch $y = \sqrt{\rho}\, Z(\rho)$ in one space (say, $u > 0$) and the branch $y = -\sqrt{\rho}\, Z(\rho)$ in the other, we get an analytic solution $y = uZ(u^2)$ defined on the entire space geometry $(-\infty < u < +\infty)$. We thus conclude that $y = uZ(u^2)$ is a natural extension of the usual one-dimensional kink solution in our geometry, connecting the two vacuum states from one flat space $(u \to +\infty)$ to the other $(u \to -\infty)$ through the bridge. The solution $y = uZ(u^2)$ is not only a kink in the usual sense but also "folded" at the edge of the bridge.

It should be emphasized that the apparent singularity in $e^{2\alpha}$ at $x = x_0$ does not imply any singular behavior of the space geometry but is due to the particular topological nature of our space, which is completely nonsingular everywhere. In fact, the curvature invariant[8] is calculated to be

$$I \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 4\left[\left(v - \frac{2}{x}\eta' u^2 E\right)^2 + \frac{2}{x^4}(u^2 E\eta')^2 + f^{-4}\frac{q^2}{x^6} + \frac{1}{2x^2}(E + u^2 E')^2\right], \quad (39)$$

which is nonsingular in the entire space $(-\infty < u < \infty)$. The $x = 0$ singularity never occurs in the physical space. This is the reason why the radius of the Rosen-Einstein bridge is uniquely determined without introducing any arbitrary integral constant when the two parameters μ and f in the original Lagrangian density are specified. The parameter μ^{-1} describes the dimension of the system and f decides the mass of the system except for the scale factor μ^{-1} in Eq. (29). If μ^{-1} is not extremely small $(\mu^{-1} > 10^{-53}$ cm), then the value of f to give the order of elementary-particle masses $(\simeq 10^{-24}$ g) is practically f_0. In an appropriate limit of $\mu^{-1} \to 0$ and $f \to f_0$, our model contains a point particle with an arbitrary mass

M.

Wheeler[9] introduced a similar geometry in the spacetime structure and gave a geometrical interpretation of electric charge. He has shown that the electromagnetic field equations are consistent with such a "wormhole" structure of the spacetime, and the wormhole can be regarded as a source of the electromagnetic field.

In our case, such a wormhole structure is generated automatically by the scalar field S. It is expected that such a geometrical structure of the spacetime together with the kink property of the scalar field would give a geometrical interpretation of, for example, the baryon number.

One of the objections to our model would arise from the fact that only the negative value of the field signature ϵ is permissible. Such a field carries a negative energy density in a flat space and, when quantized, it behaves as a ghost. However, in our model the ghost field S generates a curvature of the spacetime and the total energy of the system recovers the positive value. Furthermore, the scalar field tends rapidly to its vacuum state, having no physical effect outside the particle. Thus from the purely classical viewpoint, the ghost scalar field S does not cause any serious difficulties. It seems that the field S is not observable as a usual particle field but it composes fermions and guarantees their stability. Such a situation is quite analogous to that of Weyl's gauge field studied by Utiyama.[10]

In spite of the difficulty which arises from the simple quantization of the ghost field in a flat space, we may have to wait to decide whether such a ghost field is really unacceptable or not, until a satisfactory quantum field theory in a curved space (or quantized general theory of relativity) is established.

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