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Three-pion states and a new approach to the permutation group S_3. II. Pseudoscalar-meson decays

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Three-pion wave functions with zero spin, and specific isospin and S_3 properties, are constructed from powers of the Dalitz-plot variables. They are then used to study the decays of F, D, and K mesons into three pions. Tests of the selection rules imposed by the standard charm model on F and D decays are discussed, especially in the ratio of rates for the $\pi^+\pi^+\pi^-$ and $\pi^0\pi^0\pi^+$ modes; S_3 properties play an important role in these tests. In K decay quadratic terms are shown to have a likely origin in a state of mixed S_3 symmetry.

I. INTRODUCTION

In the first paper[1] of this series we developed a new approach to the permutation group S_3 which enabled us to construct the most general three-pion wave functions with arbitrary spin and parity. We now employ the spin-zero wave functions to study the decay of various pseudoscalar mesons into three pions.

Our principal interest is to test the isospin selection rules associated with the nonleptonic decays of charmed particles and of strange particles.[2] However, we shall find that S_3 behavior plays an important role in this endeavor because simple statements about isospin do not always lead to simple relations between different decay modes. When the wave functions for a given isospin belong to a single representation of S_3, then we can predict specific values for the ratio of rates for two decay modes, but when the wave functions involve more than one S_3 representation, then the best we can do is to predict bounds for the ratio. In F^+ decay, for example, the three-pion state may be an isovector, but we shall find that the ratio of $\pi^+\pi^+\pi^-$ and $\pi^0\pi^0\pi^+$ decay rates can have any value between 1 and 4. If we specify in addition that the isospin wave function belongs to the 2_M representation of S_3, then we can pin the ratio down to a value of 1; and if we specify the 1_S representation, we obtain a value of 4. Thus, to make specific predictions in cases such as these, we must know the S_3 behavior of the three-pion system as well as its isospin.

Another example of this occurs in the decay $K\to 3\pi$. In K decay, the matrix element is known to be dominated by an energy-independent term, but it does contain a small admixture, roughly of order 10%, of terms which depend on the first and second powers of the Dalitz plot.[3] The energy-independent terms belong to the symmetric 1_S representation of S_3, whereas the linear ones belong to the mixed-symmetry 2_M representation. As a result, the $\Delta T=\frac{1}{2}$ rule leads to one set of numerical relations between decay rates, and to another set of numerical relations for the slopes of different decay modes.[4] Were it not for this sharp difference in strength between energy-dependent and energy-independent terms, the predictions of the $\Delta T=\frac{1}{2}$ rule would be much harder to analyze.

The matter of energy dependence may prove to be a very important difference between $F\to 3\pi$ and $D\to 3\pi$ on the one hand, and $K\to 3\pi$ on the other. At present we do not know whether the charmed-particle decays conform to the K-decay pattern, in which case we can get simple predictions from selection rules, or whether they have a much stronger energy dependence, in which case we may get the much vaguer bounds on ratios mentioned above. Given the large amount of energy released in the charmed-particle case, we might expect significant components of 2π resonances such as ρ and f in the final state. However, we will not be able to settle the question until the decays have been seen in large quantities.

The charm and strangeness selection rules of the standard model[2] are such that $F^+\to 3\pi$ is a Cabibbo-allowed decay, while $D\to 3\pi$ is Cabibbo-forbidden. Thus, the amplitude for F decay is proportional to $G\cos^2\theta$, where θ is the Cabibbo angle, and that for D decay is proportional to $G\cos\theta\sin\theta$. Even though the D decay is suppressed by a factor of $\tan\theta\approx\frac{1}{5}$ in amplitude, it is still important to detect it and to measure its properties as a confirmation of the standard model.[5]

Cabibbo-allowed decays are expected to satisfy the isospin selection rule[2] $\Delta T=1$, and Cabibbo-forbidden ones are an admixture of $\Delta T=\frac{1}{2}$ and $\frac{3}{2}$. Since the F^+ meson is an isosinglet, the $\Delta T=1$

rule requires the three pions to be in a pure $T=1$ state. The D meson is an isospinor and so its three-pion final state will, in general, be an admixture of $T=0$, 1, and 2. Whether one of these states will be dominant over the other two will depend in part on the weak Hamiltonian, and in part on the structure of the final state. If, for example, the $\Delta T=\frac{1}{2}$ part of the Hamiltonian is enhanced over the $\Delta T=\frac{3}{2}$ part, and if the final state is purely S wave, then the three pions will be in a $T=1$ state; higher partial waves could introduce a significant $T=0$ component.

Our interest in K decay in this paper will center upon terms with a quadratic dependence upon the Dalitz-plot variables. We derive the predictions of the $\Delta T=\frac{1}{2}$ rule for quadratic terms, and we emphasize a point discussed in an earlier paper on linear terms[4]: namely that relations between different decay modes of the same K meson depend *only* on the isospin of the final state, but relations between decay modes of different K mesons depend on the isospin selection rule as well.

The paper is organized in the following way. We write down the most general forms of three-pion wave functions with spin zero, definite isospin eigenvalues, and definite S_3 behavior in Sec. II, and we apply them to F decay in Sec. III. Section IV is devoted to the quadratic terms in K decay and Sec. V to D decay. The results are summarized in a brief Conclusion, and potentially useful relations between various decay rates are emphasized.

II. SPIN-ZERO WAVE FUNCTIONS

In this section we describe the most general three-pion wave functions for zero spin in terms of the standard Dalitz-plot variables and the seven independent isospin states.

Each pion is assigned a four-momentum vector k_i $(i=1,2,3)$ and the center-of-mass momentum is

$$K=\sum_{i=1}^{3} k_i\,. \tag{2.1}$$

The Mandelstam variables are

$$s_i=-(K-k_i)^2,\qquad \sum_{i=1}^{3} s_i=3s_0=M^2+\sum_{i=1}^{3} m_i^{\,2}, \tag{2.2}$$

where M is the invariant mass of the three-pion system and m_i is the mass of the ith pion. The standard Dalitz-plot variables are

$$y=(s_0-s_3)=\rho\cos\varphi\,,\qquad x=\frac{1}{\sqrt{3}}\,(s_1-s_2)=\rho\sin\varphi\,, \tag{2.3}$$

and they belong to the 2_M representation of the permutation group S_3,

$$Z_s=\rho e^{-j\varphi}, \tag{2.4}$$

where[1] j is the "permutation complex element":

$$j^2=-1,\qquad j^x=-j. \tag{2.5}$$

It has the same formal properties as the imaginary number i but is entirely distinct from it. As was shown in our previous paper,[1] powers of Z_s form representations of S_3 in the following way:

$$\begin{aligned}
&1_S:\ \rho^2,\ \rho^{3k}\cos 3k\varphi,\\
&1_A:\ \rho^{3k}\sin 3k\varphi\,,\\
&2_M:\ \rho^{3k+1}e^{-j(3k+1)\varphi},\ \rho^{3k+2}e^{+j(3k+2)\varphi}.
\end{aligned} \tag{2.6}$$

There are seven independent isospin wave functions for three pions: one with $T=0$, three with $T=1$, two with $T=2$, and one with $T=3$. The isosinglet state is totally antisymmetric and the $T=3$ state is totally symmetric under all permutations of the three pions. One of the $T=1$ states is totally symmetric and the other two form a mixed symmetry 2_M representation.[6] The two $T=2$ states belong to a 2_M.[6] In a condensed notation we write this classification as

$$\begin{aligned}
&1_S:\ |1\,T_z(1_S)\rangle\rangle,\ |3\,T_z(1_S)\rangle\rangle,\\
&1_A:\ |0\,0(1_A)\rangle\rangle\,,\\
&2_M:\ Z|1\,T_z(2_M)\rangle\rangle\,,\ Z|2\,T_z(2_M)\rangle\rangle\,,
\end{aligned} \tag{2.7}$$

where the first two numbers in the ket denote T and T_z, and the third indicates the representation of S_3. We can form these states by coupling the first two pions, π_1 and π_2, to a resultant T_R, and then by coupling T_R to the third pion π_3 to form the final isospin state $|(T_R)TT_z\rangle$; details are given in the preceding paper,[1] and in the original one by Barton, Kacser, and Rosen.[4]

Since we intend to expand decay amplitues as a power series in the Dalitz-plot variables, we define the basic states in terms of their isospin, their S_3 behavior, and the power of their dependence on Z_s. Using the rules developed in the preceding paper[1] to ensure total symmetry under all permutations of the pions, we obtain the following set of states:

$$|T\,T_z(1_S)3k\rangle=\rho^{3k}\cos 3k\varphi\,|T\,T_z(1_S)\rangle\rangle\,,\quad T=1,3 \tag{2.8}$$

$$|0\,0(1_A)3k\rangle=\rho^{3k}\sin 3k\,\varphi\,|0\,0(1_A)\rangle\rangle\,, \tag{2.9}$$

$$|1\,T_z(2_M)3k+1\rangle=\rho^{3k+1}\cos(3k+1)\varphi \times\left\{\frac{\sqrt{5}}{3}\,|(2)1\,T_z\rangle-\tfrac{2}{3}\,|(0)1\,T_z\rangle\right\} -\rho^{3k+1}\sin(3k+1)\varphi\,|(1)1T_z\rangle\,, \tag{2.10}$$

$$|1\,T_z(2_M)3k+2\rangle=\rho^{3k+2}\cos(3k+2)\varphi \times\left\{\frac{\sqrt{5}}{3}\,|(2)1T_z\rangle-\tfrac{2}{3}\,|(0)1T_z\rangle\right\} +\rho^{3k+2}\sin(3k+2)\varphi\,|(1)1T_z\rangle\,,$$

$$|2\,T_z(2_M)3k+1\rangle=\rho^{3k+1}\cos(3k+1)\varphi\,|(2)2T_z\rangle +\rho^{3k+1}\sin(3k+1)\varphi\,|(1)2T_z\rangle\,, \tag{2.11}$$

$$|2\,T_z(2_M)3k+2\rangle=\rho^{3k+2}\cos(3k+2)\varphi\,|(2)2T_z\rangle -\rho^{3k+2}\sin(3k+2)\varphi\,|(1)2T_z\rangle\,.$$

The states on the right-hand side of Eqs. (2.10) and (2.11) are of the form $|(T_R)T\,T_z\rangle$, while those on the right-hand side of Eqs. (2.8) and (2.9) are as in Eq. (2.7).

III. THE DECAY $F^+\to 3\pi$

According to the standard model,[2] the F^+ will decay into a three-pion state with isospin $T=1$ and z component $T_z=1$. There are two possible charge states $\pi^+\pi^+\pi^-$ and $\pi^0\pi^0\pi^+$, and the isospin of the final state should lead to definite relations between them. Our aim in this section is to find these relations.

In general, the three pions will be in a linear combination of the states $|1\,1(1_S)3k\rangle$, $|1\,1(2_M)\,3k+1\rangle$, and $|1\,1(2_M)3k+2\rangle$ of Eqs. (2.8), (2.9), and (2.10). If we define the amplitudes for decay into these states to be $a(3k)$, $b(3k+1)$, and $c(3k+2)$, respectively, then the matrix elements for decay into specific charge states can easily be shown to be[4]

$$\mathfrak{M}(F^+\to\pi^+\pi^+\pi^-)=\sum_{k=0}^{\infty}\left\{\frac{2}{\sqrt{15}}\,a(3k)\rho^{3k}\cos3k\varphi+\frac{1}{\sqrt3}\,b(3k+1)\rho^{3k+1}\cos(3k+1)\varphi+\frac{1}{\sqrt3}\,c(3k+2)\rho^{3k+2}\cos(3k+2)\varphi\right\} \tag{3.1}$$

and

$$\mathfrak{M}(F^+\to\pi^0\pi^0\pi^+)=\sum_{k=0}^{\infty}\left\{-\frac{1}{\sqrt{15}}\,a(3k)\rho^{3k}\cos3k\varphi+\frac{1}{\sqrt3}b(3k+1)\rho^{3k+1}\cos(3k+1)\varphi+\frac{1}{\sqrt3}\,c(3k+2)\rho^{3k+2}\cos(3k+2)\varphi\right\}. \tag{3.2}$$

In both cases, the rate is given by

$$\Gamma=\frac{1}{2!}\int|\mathfrak{M}|^2d(\text{phase space})\,, \tag{3.3}$$

where the 2! enters because of the two like pions. There are no terms in $\sin(p\varphi)$ in Eqs. (3.1) and (3.2) because π_1 and π_2 are identified with the like pions in the isospin wave functions, and $\sin(p\varphi)$ is always antisymmetric under the permutation of π_1 and π_2. Notice also that the a, b, c amplitudes can be functions of ρ^2 without disturbing the permutation symmetry of the matrix element.

If we make a general cosine series expansion of the matrix element for $F^+\to\pi^\alpha\pi^\beta\pi^\gamma$ in the form

$$\mathfrak{M}(F^+\to\pi^\alpha\pi^\beta\pi^\gamma)=\sum_{n=0}^{\infty}f(\alpha\beta\gamma;n)\rho^n\cos n\varphi\,, \tag{3.4}$$

then Eqs. (3.1) and (3.2) imply that

$$\begin{aligned}&f(++-;3k)=-2f(00+;3k),\\&f(++-;\mathfrak{M})=f(00+;\mathfrak{M})\quad(\mathfrak{M}=3k+1,3k+2).\end{aligned} \tag{3.5}$$

The difference between the two relations in Eq. (3.5) stems from the fact that the terms associated with ρ^{3k} transform according to the 1_S representation of S_3 while those assocated with ρ^{3k+r} $(r=1,2)$ transform according to the 2_M representation. If the symmetric isospin wave function is the dominant one [$a(3k)\gg b(3k+1)$ and $c(3k+1)$ in Eqs. (3.1 and 2)], then the rate for $\pi^+\pi^+\pi^-$ mode will be four times that for $\pi^0\pi^0\pi^+$; on the other hand, if the mixed-symmetry isospin states are dominant, the rates for these decay modes will be equal. In general, for an arbitrary admixture of 1_S and 2_M the ratio of rates will lie somewhere between 4 and 1.[7]

Should the experimental ratio fall outside these limits, it would indicate a clear violation of the $\Delta T=1$ rule. Should it fall within them, it would be consistent with the selection rule and indicate the presence of 1_S and 2_M states. The φ dependence of the Dalitz plot will have much to tell us about the properties of these decays, in this case.

Finally we note that this analysis for $F^+\to3\pi$ is exactly the same as the original analysis by Weinberg[8] of the τ and τ' decay modes of $K^+\to3\pi$.

IV. QUADRATIC TERMS IN $K\to3\pi$

Since the Dalitz-plot densities for $K\to3\pi$ have been shown[3] to contain a small, but significant quadratic dependence upon the kinetic energies of the pions, we use our present techniques to extend an earlier isospin analysis[4] which only went so far as the linear terms. In that analysis we sought to emphasize the point that relations between decay modes of the same K meson depend only on the isospin of the final state, while relations between the decay modes of the charged and neutral kaons depend upon additional constraints imposed by the isospin properties of the weak-interaction Hamiltonian. We make the same point here for the quadratic terms. We also indicate tests for the presence of specific isospin states in the three-pion system and compare them with experiment.

To second order in kinetic energy, the matrix element has the general form

$$\begin{aligned}\mathfrak{M} &= E\{1+\sigma(s_3-s_0)+3\beta(s_3-s_0)^2+\gamma(s_2-s_1)^2\}\\ &= E\{1-\sigma\rho\cos\varphi+\tfrac{3}{2}(\beta-\gamma)\rho^2\cos2\varphi+\tfrac{3}{2}(\beta+\gamma)\rho^2\},\end{aligned} \tag{4.1}$$

where the index 3 is reserved for the unlike pion in K^+ decay and the neutral one in $K_L^0\to\pi^+\pi^-\pi^0$. If mass differences between charged and neutral pions are neglected, the energy variables in Eq. (4.1) can be expressed in terms of the nonrelativistic Dalitz-plot variables X and Y,

$$\begin{aligned}\sqrt{3}\,(s_2-s_1) &= 2M_KQX,\\ 3(s_3-s_0) &= -2M_KQY,\\ Q &= M_K-3m_\pi\end{aligned} \tag{4.2}$$

and the Dalitz-plot density becomes

$$|\mathfrak{M}|^2=|E|^2\left\{1-\frac{4M_KQ}{3}\sigma Y+\frac{8M_K^2Q^2}{3}\left[(\beta+\frac{1}{6}\sigma^2)Y^2+\gamma X^2\right]\right\}. \tag{4.3}$$

Throughout this discussion we neglect final state interactions and CP violation; thus the coefficients in Eq. (4.1) are all real.

In the notation of Eqs. (2.8)–(2.11), the most general form of the final state for $K\to3\pi$ is

$$\sum_{T=1,3}a(T,T_z)|T\,T_z(1_S)\rangle+\sum_{T=1,2}b(T,T_z)|T\,T_z(2_M)1\rangle+\sum_{T=1,2}c(T,T_z)|T\,T_z(2_M)2\rangle, \tag{4.4}$$

where the amplitudes b and c are constant, but the amplitudes a depend linearly on ρ^2,

$$a(T,T_z)=a_0(T,T_z)+a_2(T,T_z)\rho^2. \tag{4.5}$$

If we neglect the effects of CP violation, the final state in K_L^0 decay involves only odd isospins, and so we may set

$$b(2,0)=c(2,0)=0. \tag{4.6}$$

From Eqs. (4.4)–(4.6) we can determine the parameters E, σ, β, and γ of Eq. (4.1) for each decay mode; they are given in Table I. Notice that Bose symmetry rules out a $\cos2\varphi$ term in $K_L^0\to3\pi^0$ and hence[9]

$$\beta^{000}=\gamma^{000}, \tag{4.7}$$

where the superscripts indicate the three-pion charge state.

In general the effective Hamiltonian contains isospins ΔT running from $\frac{1}{2}$ to $\frac{7}{2}$,

$$H=H_{1/2}+H_{3/2}+H_{5/2}+H_{7/2}, \tag{4.8}$$

and the a,b,c amplitudes of Eq. (4.4) can be expressed in terms of the reduced matrix elements of the $H_{n/2}$. For the $T=1$ states we have

$$\left.\begin{aligned}x(1,1)&=\lambda_1(x)-\tfrac{1}{2}\lambda_3(x)\\ x(1,0)&=\lambda_1(x)+\lambda_3(x)\end{aligned}\right\},\quad x\equiv a,b,c, \tag{4.9}$$

where $\lambda_n(x)$ is the reduced matrix element of $H_{n/2}$ between the K meson and the appropriate $T=1$ state. For the $T=3$ state the only relevant amplitudes are $a(3,T_z)$ and they are given by

TABLE I. Amplitudes for $K^\alpha\to\pi^l\pi^m\pi^n$.

Charge state	E	$E\sigma$	$E(\beta+\gamma)$	$E(\beta-\gamma)$
$++-$	$\frac{2}{\sqrt{15}}a_0(1,1)+\frac{1}{\sqrt{15}}a_0(3,1)$	$-\frac{1}{\sqrt3}b(1,1)+\frac{1}{\sqrt3}b(2,1)$	$\frac{2}{\sqrt{15}}a_2(1,1)+\frac{1}{\sqrt{15}}a_2(3,1)$	$-\frac{2}{3\sqrt3}c(1,1)+\frac{2}{3\sqrt3}c(2,1)$
$00+$	$-\frac{1}{\sqrt{15}}a(1,1)+\frac{2}{\sqrt{15}}a(3,1)$	$-\frac{1}{\sqrt3}b(1,1)-\frac{1}{\sqrt3}b(2,1)$	$-\frac{1}{\sqrt{15}}a_2(1,1)+\frac{2}{\sqrt{15}}a_2(1,1)$	$-\frac{2}{3\sqrt3}c(1,1)-\frac{2}{3\sqrt3}c(2,1)$
$+-0$	$\frac{1}{\sqrt{15}}a_0(1,0)+\frac{1}{\sqrt{10}}a_0(3,0)$	$\frac{1}{\sqrt3}b(1,0)$	$\frac{1}{\sqrt{15}}a_2(1,0)+\frac{1}{\sqrt{10}}a_2(3,0)$	$\frac{2}{3\sqrt3}c(1,0)$
000	$-(\frac{3}{5})^{1/2}a_0(1,0)+(\frac{2}{5})^{1/2}a_0(3,0)$	0	$-(\frac{3}{5})^{1/2}a_2(1,0)+(\frac{2}{5})^{1/2}a_2(3,0)$	0

$$a(3,1)=\left(\frac{2}{3}\right)^{1/2}\eta_5-\left(\frac{3}{8}\right)^{1/2}\eta_7\,,$$
$$a(3,0)=\eta_5+\eta_7\,, \qquad (4.10)$$

where η_n are the reduced matrix elements $\langle\,||H_{n/2}||\,\frac{1}{2}\rangle$. Since the $T=2$ state does not occur in $K_L^0\to 3\pi$, there is no need to relate the corresponding amplitudes to reduced matrix elements of $H_{n/2}$.

We can use Eqs. (4.9) and (4.10) and Table I to deduce the consequences of any selection rule for K-meson decay. Suppose, for example, that the Hamiltonian is an admixture of $\Delta T=\frac{1}{2}$ and $\frac{3}{2}$: The final state is then an admixture of $T=1$ and 2, and the $T=3$ amplitudes must vanish,

$$a(3,1)=a(3,0)=0. \qquad (4.11)$$

Substituting Eq. (4.11) into Table I, we find that

$$\beta^{++-}+\gamma^{++-}=\beta^{00+}+\gamma^{00+}, \qquad (4.12a)$$

$$\beta^{+-0}+\gamma^{+-0}=\beta^{000}+\gamma^{000}=2\gamma^{000}. \qquad (4.12b)$$

The additional constraint of a pure $T=1$ final state, i.e.,

$$b(2,1)=c(2,1)=0; \qquad (4.13)$$

gives two more relations,

$$\sigma^{00+}=-2\sigma^{++-}, \qquad (4.14a)$$

$$2(\gamma^{++-}-\beta^{++-})=\beta^{00+}-\gamma^{00+}, \qquad (4.14b)$$

the second of which can be combined with Eq. (4.12a) to yield

$$3\gamma^{++-}-\beta^{++-}=2\beta^{00+}. \qquad (4.15)$$

If we now impose the $\Delta T=\frac{1}{2}$ rule [$\lambda_3(x)=0$ in Eq. (4.9)], we have

$$x(1,1)=x(1,0)\ \text{for}\ x\equiv a,b,c \qquad (4.16)$$

and hence from Table I,

$$\sigma^{+-0}=\sigma^{00+}, \qquad (4.17a)$$

$$\beta^{++-}+\gamma^{++-}=\beta^{+-0}+\gamma^{+-0}, \qquad (4.17b)$$

$$2(\gamma^{++-}-\beta^{++-})=\beta^{+-0}-\gamma^{+-0}. \qquad (4.17c)$$

Equations (4.17b) and (4.17c) combine with Eq. (4.15) to give

$$\beta^{00+}=\beta^{+-0} \qquad (4.18)$$

These results agree with those given by Gaillard,[9] except that Eq. (4.15) is a consequence of a $T=1$ final state rather than the $\Delta T=\frac{1}{2}$ rule. The important feature they bring out is that, as long as the selection rule merely restricts the isospin of the final state, we only obtain relations between different decay modes of the same K meson [Eqs. (4.12), (4.14), and (4.15)]. In order to relate charged kaon decay to neutral kaon decay we must impose a stronger restriction upon the isospin of the Hamiltonian: for example, $\Delta T=\frac{1}{2}$ [$\lambda_3(x)=0$], or $\lambda_3(x)(=-4\lambda_1(x)$, or any other precise relationship between reduced matrix elements.

Just as the experimental values of combinations such as $(\sigma^{+-0}+2\sigma^{++-})$ indicate whether or not the $\Delta T=\frac{1}{2}$ rule is satisfied, so other combinations indicate the presence or absence of specific isospin states. For example, the difference between $(E\sigma)^{++-}$ and $(E\sigma)^{00+}$ is proportional to the amplitude $b(2,1)$ (see Table I), and should its experimental value be different from zero, then $|2T(2_M)1\rangle$ must be present in the final state. Similarly if the combination $(E\beta+E\gamma)$ vanishes for every decay mode, then the totally symmetric isospin states will make no contribution to the quadratic dependence of the three-pion system. In the same way the combination $(E\beta-E\gamma)$ can be used to detect the presence of the quadratic states $|T\,T_z(2_M)2\rangle$ with $T=1,2$, through relations such as (see Table I)

$$(E\beta-E\gamma)^{++-}-(E\beta-E\gamma)^{00+}=-\frac{2}{\sqrt{3}}\,a(2,1). \qquad (4.19)$$

The data on $K_L^0\to\pi^+\pi^-\pi^0$ suggest that

$$\beta^{+-0}+\gamma^{+-0}\approx 0. \qquad (4.20)$$

Therefore, if the final state is pure isovector, then γ^{000} should vanish [Eq. (4.12b)] and the Dalitz-plot density for $K_L\to 3\pi^0$ should be constant (in the quadratic approximation). In addition, if the $\Delta T=\frac{1}{2}$ rule is satisfied then we should also have

$$\beta^{00+}+\gamma^{00+}\approx 0,$$
$$\beta^{++-}+\gamma^{++-}\approx 0, \qquad (4.21)$$
$$2\gamma^{++-}+\gamma^{+-0}\approx 0.$$

The data are not inconsistent with the second and third parts of Eq. (4.21) but the errors are too large to draw a definite conclusion from them. In conjunction with Eq. (4.20), the first two equations of (4.21) imply that the symmetric isospin states $|T\,T_z(1_S)\rangle$ with $T=1,3$ should not contribute significantly to the quadratic dependence of the final state; however, the fact that $(E\beta-E\gamma)^{+-0}$ does not vanish means that $|1\,T_z(2_M)2\rangle$ is present in the decay.

V. THE DECAY $D\to 3\pi$

We turn now to the Cabibbo-forbidden decay of the D meson into three pions. In general the isospin selection rules predicted by the standard model[2] are an admixture of $\Delta T=\frac{1}{2}$ and $\frac{3}{2}$. The $\Delta T=\frac{3}{2}$ component drops out if the interaction is dominated by the 6 representation of SU(3), but the $\Delta T=\frac{1}{2}$ component is present in both the 6 and the 15* representations.

Because the D meson is an isospinor, the three-pion final state will, in general, be an admixture of $T=0$, 1, and 2. Moreover, the $T=0$ and 2 states are expected to contribute to neutral D decays as well as to charged ones: Neutral D mesons, unlike their kaon counterparts, do not appear to split into approximate CP eigenstates with vastly different lifetimes, and hence we cannot apply the usual CP arguments to them.[3,4]

Another possible difference between D and K mesons concerns the energy dependence of the decay amplitudes. As we have discussed in the preceding section, the K-meson amplitudes are largely independent of energy; we do not know, however, whether the same holds true for the D-decay amplitudes, or whether they have a strong dependence on energy. We shall therefore analyze them in the most general way, and wait for experiment to answer the question.

Using the states of Eqs. (2.9)–(2.11), we write the final state of the three pions in the form

$$\sum_{k=0}^{\infty}[a(1\,T_z;3k)\,|1\,T_z(1_S)3k\rangle+d(3k)\,|0\,0(1_A)3k\rangle]$$
$$+\sum_{\substack{k=0\\T=1,2}}[b(T\,T_z;3k+1)\,|T\,T_z(2_M)3k+1\rangle+c(T\,T_z;3k+2)\,|T\,T_z(2_M)3k+2\rangle\,,\qquad(5.1)$$

where the amplitudes a, b, c, and d may be constants or functions of ρ^2. The matrix elements for D^+ decay into specific charge states can be written as a cosine series in φ:

$$\mathfrak{M}^{(+)}(\pi^\alpha\pi^\beta\pi^\gamma)=\sum_{k=0}^{\infty}\{f_{3k}(\alpha\beta\gamma)\rho^{3k}\cos3k\varphi+f_{3k+1}(\alpha\beta\gamma)\rho^{3k+1}\cos(3k+1)\varphi+f_{3k+2}(\alpha\beta\gamma)\cos(3k+2)\varphi\}\,,\qquad(5.2)$$

where

$$\begin{aligned}
&f_{3k}(++-)=-2f_{3k}(00+)=(2/\sqrt{15})a(1\,1;3k),\\
&f_{3k+1}(++-)=(1/\sqrt{3})[b(1,1;3k+1)-b(2\,1;3k+1)],\\
&f_{3k+1}(00+)=(1/\sqrt{3})[b(1\,1;3k+1)+b(2\,1;3k+1)],\\
&f_{3k+2}(++-)=(1/\sqrt{3})[c(1\,1;3k+2)-c(2\,1;3k+2)],\\
&f_{3k+2}(00+)=(1/\sqrt{3})[c(1\,1;3k+2)+c(3\,1;3k+2)].
\end{aligned}\qquad(5.3)$$

Terms in $\sin p\varphi$ do not appear in these expressions because of Bose symmetry for the two like pions. They do, however, appear in the matrix elements for D^0 decay:

$$\mathfrak{M}^{(0)}(\pi^\alpha\pi^\beta\pi^\gamma)=\sum_{k=0}^{\infty}[f_{3k}(\alpha\beta\gamma)\rho^{3k}\cos3k\varphi+f_{3k+1}(\alpha\beta\gamma)\rho^{3k+1}\cos(3k+1)\varphi+f_{3k+2}(\alpha\beta\gamma)\rho^{3k+2}\cos(3k+2)\varphi$$
$$+g_{3k}(\alpha\beta\gamma)\rho^{3k}\sin3k\varphi+g_{3k+1}(\alpha\beta\gamma)\rho^{3k+1}\sin(3k+1)\varphi+g_{3k+2}(\alpha\beta\gamma)\rho^{3k+2}\sin(3k+2)\varphi]\,,\qquad(5.4)$$

where

$$\begin{aligned}
&f_{3k}(+-0)=-\frac{1}{3}f_{3k}(000)=\frac{1}{\sqrt{15}}a(1\,0;3k)\,,\\
&f_{3k+1}(+-0)=-\frac{1}{\sqrt{3}}b(1\,0;3k+1),\quad f_{3k+1}(000)=0,\\
&f_{3k+2}(+-0)=-\frac{1}{\sqrt{3}}c(1\,0;3k+2),\quad f_{3k+2}(000)=0,\\
&g_{3k}(+-0)=-\frac{1}{\sqrt{6}}d(3k),\quad g_{3k}(000)=0,\\
&g_{3k+1}(+-0)=\frac{1}{\sqrt{3}}b(2\,0;3k+1),\quad g_{3k+1}(000)=0,\\
&g_{3k+2}(+-0)=-\frac{1}{\sqrt{3}}c(2\,0;3k+1),\quad g_{3k+2}(000)=0.
\end{aligned}\qquad(5.5)$$

All of these matrix elements are defined so that the decay rate is

$$\Gamma=\frac{1}{n!}\int|\mathfrak{M}|^2d(\text{phase space}),\qquad(5.6)$$

where n is the number of like pions in the final state.

The Hamiltonian in the standard model contains a $\Delta T=\frac{1}{2}$ term and a $\Delta T=\frac{3}{2}$ term but no $\Delta T=\frac{5}{2}$ term:

$$H=H_{1/2}+H_{3/2}\,.\qquad(5.7)$$

Consequently the $T=2$ states in both D^+ and D^0 decays are engendered by a single term, namely $H_{3/2}$, and the corresponding amplitudes are proportional to one another:

$$x(2\,1;3k+r)=(\tfrac{3}{2})^{1/2}\,x(2\,0;3k+r)$$
$$\text{for } x\equiv b,c \text{ and } r=1,2. \qquad (5.8)$$

The $T=1$ states, however, can arise from both parts of the Hamiltonian and there are no simple relations between the amplitudes for D^+ and D^0 decays. In terms of the reduced matrix elements $\lambda_n(3k+r)$ of $H_{n/2}$ between the K meson and the appropriate $T=1$ states we have

$$x(11;3k+r)=\lambda_1(3k+r)-\tfrac{1}{2}\,\lambda_3(3k+r),$$
$$x(10;3k+r)=\frac{1}{\sqrt{2}}\,[\lambda_1(3k+r)+\lambda_3(3k+r)], \qquad (5.9)$$
$$\text{for } x\equiv a,b,c \text{ and } r=0,1,2.$$

It is evident from Eqs. (5.2)–(5.9) that we cannot obtain any relation between the rates for different decay modes without making further assumptions. Some possible assumptions we shall explore include (i) pure $T=1$ final state, (ii) a $\Delta T=\frac{1}{2}$ rule, and (iii) a pure $T=2$ final state. In general, assumptions about the isospin of the final state give relations between different decay modes of the same D meson, but not between decay modes of different D mesons; assumptions about the isospin of the Hamiltonian are needed to obtain relations of the second kind.

If the final state is pure isovector, then $d(3k)$ and all $x(2T_z;3k+r)$ vanish, and exactly as in the case of F^+ decay, we have

$$f_{3k}(++-)=-2f_{3k}(00+);$$
$$f_{3k+r}(++-)=+f_{3k+r}(00+),\quad r=1,2 \qquad (5.10)$$

for D^+ decay [compare with Eq. (3.5)]; and for D^0 we have

$$f_{3k}(+-0)=-\tfrac{1}{3}f_{3k}(000),\quad g_{3k}=g_{3k+1}=g_{3k+2}=0. \qquad (5.11)$$

The difference between the relation for f_{3k} and f_{3k+r} in Eq. (5.10) reflects the difference in the S_3 properties of the isospin wave functions with which they are associated. If we further assume that only the 1_S isospin state is present then all f_{3k+r} amplitudes with $r=1,2$ vanish, and we find that

$$\Gamma(++-)=4\Gamma(00+),\quad \Gamma(000)=\tfrac{3}{2}\,\Gamma(+-0) \qquad (5.12)$$

exactly as in K decay. By contrast, if only the 2_M isospin functions are present, then all f_{3k} amplitudes vanish and

$$\Gamma(++-)=\Gamma(00+),\quad \Gamma(000)=0. \qquad (5.13)$$

An admixture of 1_S and 2_M isospin states yields no simple relations between the rates, but does limit the ratio to lie between 4 and 1.[7]

To relate the D^+ decays to the D^0 ones we now add the $\Delta T=\frac{1}{2}$ rule to our isovector final state assumption. (Notice that $\Delta T=\frac{1}{2}$ by itself does not exclude the $T=0$ state in D^0 decay.) This implies that $\lambda_3(3k+r)$ must vanish in Eq. (5.9) and hence that

$$x(11;3k+r)=\sqrt{2}\,x(1\,0;3k+r)$$
$$\text{for } x=a,b,c \text{ and } r=0,1,2. \qquad (5.14)$$

This then implies an additional relation for the amplitudes, namely

$$f_{3k+r}(00+)=-\sqrt{2}\,f_{3k+r}(+-0)\quad (r=0,1,2). \qquad (5.15)$$

Hence the rules obey

$$\Gamma(00+)=\Gamma(+-0). \qquad (5.16)$$

Notice that this result is valid for all forms of the energy dependence of the amplitudes and it is also independent of whether the D^+ decay modes satisfy either Eq. (5.12) or Eq. (5.13), or neither of them.

An alternative to $\Delta T=\frac{1}{2}$ with a pure isovector final state is to assume a pure isotensor final state. In this case all the amplitudes $x(1\,T_z;3k+r)$ and the $d(3k)$ vanish, and we find that, in addition to Eq. (5.8), the relations

$$\mathfrak{M}^{(+)}(++-)=-\mathfrak{M}^{(+)}(00+),$$
$$\mathfrak{M}^{(0)}(000)=0 \qquad (5.17)$$

also hold. Even though the D^+ amplitudes form a cosine series in φ while the D^0 amplitudes form a sine series, Eqs. (5.8) and (5.16) imply that in the $T=2$ case the rules obey

$$\Gamma(++-)=\Gamma(00+)=\tfrac{3}{4}\,\Gamma(+-0),\quad \Gamma(000)=0. \qquad (5.18)$$

In general both the $\Delta T=\frac{1}{2}$ and the $\Delta T=\frac{3}{2}$ interaction may be equally important in $D\to 3\pi$, and in this case Eq. (5.8) is the only relation that will hold between different amplitudes. The above analysis does show, however, that a comparison of rates for various decay modes can provide clues regarding the isospin of the final state and the selection rules imposed by the Hamiltonian. Detailed information about the Dalitz plot would be very helpful in this regard, but it will probably be a long time in coming.

VI. CONCLUSIONS

We have analyzed the decays of various pseudoscalar mesons into three pions according to the selection rules expected to hold for charmed-particle and strange-particle nonleptonic decays.[2] In the case of K meson decay, we have shown that if the $\Delta T = \frac{1}{2}$ rule is valid, then the quadratic terms in the matrix element come from an isovector state which has mixed symmetry (2_M) and S_3. In F and D decay, the appropriate selection rules make detailed predictions about amplitudes, but it will be some time before we can test them.

There are, however, some crude tests we can apply to the rates for various decay modes. Thus, if the final state in F^+ decay is an isovector, then the ratio

$$R^{(+)} = \frac{\Gamma(\pi^+\pi^+\pi^-)}{\Gamma(\pi^0\pi^0\pi^+)} \tag{6.1}$$

will lie between 1 and 4 (Ref. 7); it will have the lower value if the state is a pure 2_M representation of S_3 and the upper value if it is pure 1_S. Exactly the same holds for $D^+ \to 3\pi$. Thus a measurement of $R^{(+)}$ can be used to decide whether or not the final state is consistent with an assignment of isovector.

In the case of D^0 decay, the isovector final state implies that

$$R^{(0)} = \frac{\Gamma(\pi^0\pi^0\pi^0)}{\Gamma(\pi^+\pi^-\pi^0)} \tag{6.2}$$

lies between 0 and $\frac{3}{2}$; it will be 0 when the final state belongs to the 2_M representation of S_3, $\frac{3}{2}$ when it belongs to 1_S, and something in between for an admixture of S_3 properties. Irrespective of S_3 properties, the isovector final state plus the $\Delta T = \frac{1}{2}$, which is equivalent to 6-dominance,[2] implies that

$$\Gamma(\pi^0\pi^0\pi^+) = \Gamma(\pi^+\pi^-\pi^0). \tag{6.3}$$

These relations are useful in both a positive and a negative sense. If they turn out to be satisfied, they give us some precise information about the final states; if they are not satisfied, they rule out certain simple possibilities for the final state.

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[7] As pointed out by Zemach (Ref. 6), the interference term between wave functions of different symmetry types vanished when integrated over the entire Dalitz plot. Thus even though the $\cos(3k+r)\varphi$ are not orthogonal functions over the Dalitz plot in general [(the boundary is $1 = (1+\alpha)\rho^2 + 2\rho^3 \cos 3\theta$ with $\alpha = 2M(M-3m_\pi)(M+3m_\pi)^{-2} \approx 1.08$ for $M = 2$ GeV)], the part of the wave function associated with the 1_S isospin state does not interfere with the 2_M isospin part. Alternatively, we could use the harmonic functions developed by B. W. Lee, New Phys. (Korean Physical Society) Suppl. 7, 41 (1968); these functions form an orthonormal set over the Dalitz plot and belong to specific representations of S_3.

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