Three-pion states and a new approach to the permutation group S_3. I

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A new method is developed for picking a given irreducible representation of the permutation group S_3 out of the product of many representations. It is then used to construct three-pion wave functions of arbitrary spin and parity. Applications of the method to three-fermion systems are briefly described. The essence of the method is to represent the two-dimensional representation of S_3 as a "complex number" in an Argand diagram; because the action of S_3 involves simple rotations and reflections in the diagram, the behavior of products of many two-dimensional representations is easy to analyze.

I. INTRODUCTION

The application of Bose symmetry to systems of three pions[1-6] has always raised interesting problems in the permutation group of three objects.[7] When the wave function is written as a product of an isospin part and a spatial part, the properties of these two parts must complement each other in such a way as to ensure that the wave function is symmetric under all permutations of the particles. These complementary properties are easily inserted by hand when the spatial part of the wave function is relatively simple; but as the spatial part becomes more complicated, it becomes necessary to use more sophisticated group-theoretical methods. This, in turn, gives rise to the problem of picking out a specific irreducible representation of the permutation group from the product of many representations.

Similar problems arise in the study of three-nucleon[8] and other three-fermion systems.[9] The wave function is usually expressed in terms of products of isospin, intrinsic-spin, orbital-motion, and possibly color wave functions, and from this product it is necessary to pick out the totally antisymmetric combination. Again this can be done by hand in relatively simple cases, but it becomes more difficult as the level of complexity increases.

As is well known, the permutation group of three objects, S_3, has three irreducible representations[7]: a totally symmetric one, 1_S, a totally antisymmetric one, 1_A, and a two-dimensional one of mixed symmetry, 2_M. Products of 1_S and 1_A are easily characterized with respect to their permutation properties, but products of two-dimensional representations, especially products with a large number of factors, are much more difficult to handle. The purpose of this paper is to introduce a simple method for constructing irreducible representations of S_3 from n-fold products of two-dimensional ones.

To illustrate the method, we use it to construct properly symmetrized states of three pions with arbitrary spin and parity, and with various isospins. We also construct antisymmetric states of three nucleons with various isospin, intrinsic-spin, and orbital-motion properties. Many of our results have, of course, been obtained by other means in the past; however, our general method and our results for general orbital motion are new.

Our general method is based upon a very simple device which is described in Sec. II. It is then applied to Dalitz-plot[10] variables, momentum vectors, and isospin and intrinsic-spin variables in Sec. III. The general forms of three-particle wave functions are given in Sec. IV, and some applications are briefly discussed in the Conclusion.

II. THE TWO-DIMENSIONAL REPRESENTATION

Our basic approach to the two-dimensional representation of S_3 is to represent it as a vector in an Argand diagram. The actions of S_3 upon this vector consist of rotations through 120° and reflection in the real axis of this diagram. It is the simplicity of these actions that enables us to classify the n-fold products of two-dimensional representations almost by inspection.

Let us represent the three objects to be permuted as basis vectors e_1, e_2, e_3, in some linear space. The vector

$$E=\tfrac{1}{3}(e_1+e_2+e_3) \tag{2.1}$$

is symmetric under any permutation of the basis vectors, while the two orthogonal vectors

$$A=\tfrac{1}{6}(e_1+e_2-2e_3),\qquad B=\frac{1}{2\sqrt{3}}(e_1-e_2) \tag{2.2}$$

transform into one another. We now construct an

Argand diagram in which A and B are the "real" and "imaginary" parts, respectively, of a "complex" number

$$Z = A - jB, \quad j^2 = -1. \tag{2.3a}$$

The symbol j has exactly the same properties as the imaginary number i, but is entirely distinct from it; here j serves as a basis vector for the 2_M representation. In analogy with complex conjugation, we define the "permutation conjugate" of Z to be

$$Z^x = A + jB. \tag{2.3b}$$

The six elements of S_3 consist of three transpositions (12), (13), (23); two cycles (123), (132); and the identity element (I). It is not difficult to show that the actions of these elements upon Z are

$$\begin{aligned} &(12)\ Z = Z^x, && (I)\ Z = Z \\ &(13)\ Z = Z^x e^{j2\pi/3}, && (132)\ Z = Z e^{j2\pi/3}, \\ &(23)\ Z = Z^x e^{j4\pi/3}, && (123)\ Z = Z e^{j4\pi/3}, \end{aligned} \tag{2.4}$$

where $e^{j\theta}$ has the usual meaning of $\cos\theta + j\sin\theta$. We emphasize that e_1, e_2, e_3, and hence A and B, can represent any objects, be they spatial coordinates, spins, isospins, or anything else. Thus Eq. (2.4) represents the actions of S_3 upon any specific realization of the basis vectors.

Now consider a pair of two-dimensional representations Z_α and Z_β which transform according to Eq. (2.4). The Clebsch-Gordan series for the product is[7]

$$2_M \otimes 2_M = 1_S \oplus 2_M \tag{2.5}$$

and our problem is to pick out the appropriate expressions for the representations on the right-hand side of Eq. (2.5). Since the basic actions of S_3 consist of permutation conjugation and rotations of 120° in the Argand diagram, we begin by considering the product $Z_\alpha Z_\beta^x$; under the first column of Eq. (2.4) it transforms into $Z_\alpha^x Z_\beta$, and under the second column it remains unchanged. Thus the S_3 invariant, or totally symmetric product is the "real" part:

$$1_S \equiv (Z_\alpha Z_\beta^x) + (Z_\alpha^x Z_\beta). \tag{2.6}$$

The totally antisymmetric product changes sign under the first column of Eq. (2.4), but is invariant under the second column. Thus it is the imaginary part:

$$1_A \equiv j(Z_\alpha Z_\beta^x - Z_\alpha^x Z_\beta)\ . \tag{2.7}$$

When Z_α and Z_β are identical, the totally symmetric combination is the "square modulus" of Z, and the totally antisymmetric combination vanishes, as it should.

To pick out the 2_M representation we consider the algebraic product $Z_\alpha Z_\beta$. Under Eq. (2.4) it transforms in almost the same way as Z itself: The only difference is that the phase factors for $Z_\alpha Z_\beta$, i.e., the $e^{j2\pi/3}$ factors, are the permutational conjugates of the corresponding factors for Z. This means that the permutation conjugate of $Z_\alpha Z_\beta$ transforms in exactly the same way as Z; thus

$$2_M \equiv Z_\delta = (Z_\alpha Z_\beta)^x. \tag{2.8}$$

It is easy to extend these results to products of more representations. From Eqs. (2.6), (2.7), and (2.8), for example, we find that for three representations

$$1_S - j1_A \equiv (Z_\gamma Z_\delta^x) = Z_\alpha Z_\beta Z_\gamma. \tag{2.9}$$

For products of an arbitrary number of representations, we define

$$Z_n = Z_\alpha Z_\beta Z_\gamma \cdots Z_\tau \ (n\text{-factors}) \tag{2.10}$$

and find that under Eq. (2.4)

$$\begin{aligned} &(12)\ Z_n = Z_n^x, && (I)\ Z_n = Z_n, \\ &(13)\ Z_n = Z_n^x e^{j2\pi n/3}, && (132)\ Z_n = Z_n e^{j2\pi n/3}, \\ &(23)\ Z_n = Z_n^x e^{j4\pi n/3}, && (123)\ Z_n = Z_n e^{j4\pi n/3}. \end{aligned} \tag{2.11}$$

We now distinguish three cases corresponding to

$$n = 3k + r, \quad r = 0, 1, 2. \tag{2.12}$$

When $r = 0$, the phase factors in Eq. (2.11) are all unity, and we have exactly the same situation as in Eq. (2.9), namely that the "real" part of Z_n is totally symmetric, and the "imaginary" part is totally antisymmetric:

$$Z_n \equiv 1_S - j1_A, \quad n = 3k. \tag{2.13}$$

When $r = 1$, the phase factors in Eq. (2.11) are exactly the same as those for Z in Eq. (2.4), and when $r = 2$ they are the permutation conjugates of the phases for Z. Thus the two-dimensional representations in the product are

$$\begin{aligned} &Z_n \equiv 2_M, \quad n = 3k + 1 \\ &Z_n^x \equiv 2_M, \quad n = 3k + 2. \end{aligned} \tag{2.14}$$

From Eq. (2.13) and the first line of Eq. (2.14), we can show that if C_A is a totally antisymmetric quantity, then the 2_M contained in the product of C_A with any Z is given by

$$C_A jZ \equiv 2_M. \tag{2.15}$$

We can also show this directly from Eq. (2.4).

Given these results, we can determine the Clebsch-Gordan series for any number of two-dimensional representations. It should be emphasized that the various Z factors in Eq. (2.10) need not all refer to the same realization of the basis vectors. Thus some Z factors could refer to spatial coordinates, others to isospin, and still others to color. The following sections will demonstrate this point.

III. CONSTRUCTION OF S_3 REPRESENTATIONS

In this section we construct representations of S_3 from various powers of the coordinates describing a three-particle system. We begin with the Dalitz plot, or energy variables and determine the S_3 properties of arbitrary powers of these variables; then we do the same for the momentum vectors of the particles. Finally we consider spin, or isospin variables: Here the requirements are somewhat different from the energy and momentum variables in that the basic spin, or isospin wave function must always be a product of the wave functions of each particle. Our classification of the representations of S_3 applies to all spins or isospin eigenvalues from $\frac{1}{2}$ to infinity.

A. Dalitz-plot variables

We describe each of the three pions by a four-momentum vector k_i $(i=1,2,3)$ and introduce a center-of-mass momentum

$$K=\sum_1^3 k_i. \tag{3.1}$$

Instead of the energies of the pions, we use the usual Mandelstam variables

$$s_i=-(K-k_i)^2,$$
$$\sum_1^3 s_i=3s_0=M^2+\sum_1^3 m_i^2, \tag{3.2}$$

where M is the invariant mass of the three-pion system and m_i is the mass of the ith pion. The standard Dalitz-plot variables are

$$y=(s_0-s_3)=\rho\cos\varphi,\quad x=\frac{1}{\sqrt{3}}(s_1-s_2)=\rho\sin\varphi. \tag{3.3}$$

A comparison of these expressions with Eqs. (2.1) to (2.3) indicates that s_0 is symmetric under permutations and that y and x belong to the 2_M representation:

$$Z_s=y-jx=\rho e^{-j\varphi}. \tag{3.4}$$

It follows immediately from the results of the preceding section that powers of Z_s can be classified in the following representations of S_3:

$$1_S:\ \rho^{2l},\ \rho^{3k}\cos 3k\varphi,$$
$$1_A:\ \rho^{3k}\sin 3k\varphi, \tag{3.5}$$
$$2_M:\ (Z_s)^{3k+1},\ (Z_s^x)^{3k+2}.$$

These results are well known from studies of $K\to 3\pi$.[1-6]

B. Momentum vectors

The Dalitz-plot variables are scalars under rotations, and the only quantities we can construct from them are scalars. We now turn to quantities from which we can construct vectors and higher tensors under rotation, namely the momentum vectors of the pions.

In exactly the same way as in Eqs. (2.3)–(2.3), they can be written as a totally symmetric vector,

$$\tfrac{1}{3}\vec{K}=\tfrac{1}{3}(\vec{k}_1+\vec{k}_2+\vec{k}_3), \tag{3.6}$$

and as a mixed symmetry pair,

$$\vec{K}_A=\tfrac{1}{6}(\vec{k}_1+\vec{k}_2-2\vec{k}_3),$$
$$\vec{K}_B=\frac{1}{2\sqrt{3}}(\vec{k}_1-\vec{k}_2). \tag{3.7}$$

In the center-of-mass frame $\vec{K}$ vanishes and $\vec{K}_A=-\frac{1}{2}(\vec{k}_3)$. Corresponding components of $\vec{K}_A$ and $\vec{K}_B$ form 2_M representations of S_3,

$$Z_\lambda=K_{A\lambda}-jK_{B\lambda}, \tag{3.8}$$

where λ denotes a Cartesian, or a spherical component of the vectors in Eq. (3.7).

For spherical components, it should be remembered that "permutation" conjugation as defined in Eqs. (2.3a) and (2.3b) sends Z_λ into $K_{A\lambda}+jK_{B\lambda}$, but it does not change a spherical component such as $re^{i\theta}$ with $\lambda=+1$ into $re^{-i\theta}$ with $\lambda=-1$. Only complex conjugation does that.

The product $Z_\lambda Z_\mu^x$ behaves like a linear combination of 1_S and 1_A under S_3 for all choices of λ and μ:

$$Z_\lambda Z_\mu^x+Z_\lambda^x Z_\mu\equiv 1_S,$$
$$j(Z_\lambda Z_\mu^x-Z_\mu Z_\lambda^x)\equiv 1_A. \tag{3.9}$$

Since the 1_S part is also symmetric in λ and μ, it behaves as a combination of a rotational scalar and a tensor of rank 2. Because the identity

$$\sum_{\alpha=1}^3 Z_\alpha Z_\alpha^x\equiv\tfrac{1}{16}Z_sZ_s^x+\tfrac{1}{18}\left(M^2-3\sum m_i^2\right) \tag{3.10}$$

relates the scalar component to products of Z_s, the second rank tensor, namely

$$1_S\equiv Z_\lambda Z_\mu^x+Z_\mu Z_\lambda^x-\tfrac{2}{3}\delta_{\lambda\mu}\left(\sum Z_\alpha Z_\alpha^x\right), \tag{3.11}$$

is the only new term of interest. The 1_A term in Eq. (3.9) is automatically antisymmetric in λ and μ, and so it behaves as a rotational vector

$$Q_\nu\equiv\epsilon_{\lambda\mu\nu}Z_\lambda Z_\mu^x$$
$$\sim\vec{K}_A\times\vec{K}_B \tag{3.12}$$
$$\sim\vec{k}_1\times\vec{k}_2 \text{ in c.m.s. frame.}$$

In the 2_M representation $(Z_\lambda Z_\mu)^x$ we again have a quantity which is symmetric in λ and μ, and which therefore behaves as a combination of scalar and second-rank tensor. The identity

$$\sum_{\alpha=1}^{3} Z_\alpha Z_\alpha \equiv \frac{1}{16M^2} Z_s{}^2 + \tfrac{1}{6} Z_s^x \qquad (3.13)$$

relates the scalar component to known functions of Z_s, and so the second-rank tensor

$$2_M \equiv \left(Z_\lambda Z_\mu - \tfrac{1}{3}\delta_{\lambda\mu} \sum Z_\alpha Z_\alpha \right)^x \qquad (3.14)$$

is again the only new term of interest.

In general, the n-fold product of Z_λ's has permutation properties which depend on n [see Eqs. (2.10)–(2.14)]:

$$Z_{\lambda_1} Z_{\lambda_2} \cdots Z_{\lambda_n} \equiv \begin{cases} 1_S - j 1_A, & n = 3k \\ 2_M, & n = 3k+1 \\ 2_M^x, & n = 3k+2. \end{cases} \qquad (3.15)$$

It is symmetric in the components $\lambda_1, \lambda_2, \ldots, \lambda_n$, and hence it contains an admixture of rotational tensors of ranks n, $(n-2)$, $(n-4), \ldots$ down to 0 (n even) or 1 (n odd). The highest rank tensor and all of its components can be selected by setting $\lambda_1, \lambda_2, \ldots, \lambda_n$ all equal to their maximum value of $(+1)$ (in spherical coordinates) and then operating on this state $2n$ times with the total angular momentum lowering operator L_-; this operator lowers the λ eigenvalue of Z_λ by one unit without changing its S_3 behavior. The next tensor, of rank $(n-2)$, can be selected by contracting two components, say λ_1 and λ_2, and applying the above procedure to the remaining $\lambda_3, \lambda_4, \ldots, \lambda_n$. Because of the identity in Eq. (3.13), however, we might just as well work with an $(n-2)$-fold product of Z_λ in this case, similarly for the $(n-4)$, $(n-6)$, and lower rank tensors.

To construct tensors of a fixed rank n which transform according to each of the S_3 representations, let us set $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$ in Eq. (3.15) and suppose that $n = 3k$. Then $(Z_1)^n$ and its permutational conjugate form the 1_S and 1_A representations. If we replace one Z_1 factor by its permutation conjugate, we do not alter the rotational property of the product but we do change its S_3 behavior. The product now takes a form $(Z_1 Z_1^x)(Z_1)^{3(k-1)+1}$ in which the first factor is totally symmetric and the second behaves like 2_M: Thus the entire product behaves like 2_M.

The cases $n = 3k+1$ and $n = 3k+2$ can be treated in a similar way. Thus for any n, we can construct rotational tensors with rank n and third-component eigenvalues equal to n which transform according to any of the irreducible representations of S_3; they are displayed in Table I. Tensors with smaller third-component eigenvalues can be generated from them by applying the lowering operator L_- a sufficient number of times.

So far the tensors we have constructed correspond to three-pion states with angular momentum n and parity $(-1)^n(-1)^3$ where the first factor comes from the orbital motion and the second from the intrinsic parity of the pions. We can create states of the same angular momentum but opposite parity by replacing one Z_1 factor by Q_1 [see Eq. (3.12)]. This changes the S_3 properties of the particular product in which the substitution is made, but it is still possible to find nth-rank tensors with parity $(-1)^{n+1}(-1)^3$ belonging to each representation of S_3. These tensors are shown in Table II.

C. Isospin vectors

Turning to the isospin vectors π_i $(i = 1, 2, 3)$ of the three pions, we can again form a symmetric linear combination

$$e_\lambda = \tfrac{1}{3}(\pi_1 + \pi_2 + \pi_3)_\lambda \qquad (3.16)$$

TABLE I. Representations of S_3 in terms of spatial wave functions with $J^P = n(-1)^{n+3}$ and an L_3 eigenvalue of n. The parity includes the intrinsic parity of the pions.

J^P \ S_3 representation	$1_S - j 1_A$	2_M
0^-	ρ^2	$\rho^{3k+1} e^{-j(3k+1)\varphi}$
	$\rho^{3k} e^{-j(3k\varphi)}$	$\rho^{3k+2} e^{j(3k+2)\varphi}$
1^+	$Z_1 \rho e^{j\varphi}$	Z_1
2^-	$(Z_1 Z_1^x)$	$(Z_1 Z_1)^x$
$n(-1)^{n+3}$, $n = 3p$	$(Z_1)^{3p}$	$(Z_1)^{3p-1} Z_1^x$
$n(-1)^{n+3}$, $n = 3p+1$	$(Z_1)^{3p-1}(Z_1^x)^2$	$(Z_1)^{3p+1}$
$n(-1)^{n+3}$, $n = 3p+2$	$(Z_1)^{3p+1}(Z_1)^x$	$(Z_1^x)^{3p+2}$

TABLE II. Representations of S_3 in terms of spatial wave functions with $J^P = n(-1)^{n+4}$ and an L_3 eigenvalue of n. The parity includes the pion intrinsic parity. pc denotes permutational conjugate. Q_λ is defined in Eq. (3.12).

J^P \ S_3 representation	1_S	1_A	2_M
1^-	...	Q_1	...
2^+	...	...	$Q_1 Z_1$
3^-	...	$Q_1(Z_1 Z_1^*)$	$Q_1(Z_1^*)^2$
$n(-1)^{n+4}$, $n = 3p+1$	$Q_1(Z_1^{3p} - \mathrm{pc})$	$Q_1(Z_1^{3p} + \mathrm{pc})$	$Q_1 Z_1^*(Z_1)^{3p-1}$
$n(-1)^{n+4}$, $n = 3p+2$	$Q_1[Z_1^{3p-1}(Z_1^*)^2 - \mathrm{pc}]$	$Q_1[Z_1^{3p-1}(Z_1^*)^2 + \mathrm{pc}]$	$Q_1 Z_1^{3p+1}$
$n(-1)^{n+4}$, $n = 3p+3$	$Q_1(Z_1^{3p+1} Z_1^* - \mathrm{pc})$	$Q_1(Z_1^{3p+1} Z_1^* + \mathrm{pc})$	$Q_1(Z_1^*)^{3p+2}$

for each charge component λ, and a pair with mixed symmetry

$$z_\lambda = a_\lambda - ib_\lambda, \qquad a_\lambda = \tfrac{1}{6}(\pi_1 + \pi_2 - 2\pi_3)_\lambda, \qquad b_\lambda = \frac{1}{2\sqrt{3}}(\pi_1 - \pi_2)_\lambda. \tag{3.17a}$$

The isospin wave function, unlike its spatial counterpart, must be constructed from products of the form $\pi_{1\lambda}$, $\pi_{2\mu}$, $\pi_{3\nu}$. Thus we must invert Eqs. (3.16) and (3.17) and decompose the product into irreducible representations.

Inverting the expressions for e_λ and z_λ, we obtain

$$\pi_{1\lambda} = e_\lambda + z_\lambda e^{j\pi/3} + z_\lambda^* e^{-j\pi/3}, \qquad \pi_{2\mu} = e_\mu + z_\mu e^{-j\pi/3} + z_\mu^* e^{j\pi/3}, \qquad \pi_{3\nu} = e_\nu - z_\nu - z_\nu^* . \tag{3.17b}$$

It then turns out that we can decompose the products of the three-pion wave functions into irreducible representations of S_3 by using the permutational symmetry with respect to λ, μ, ν. For the totally symmetric and antisymmetric representations we have

$$1_S \equiv \tfrac{1}{6} \sum_{\substack{\text{all perms}\\ \lambda\mu\nu}} (\pi_{1\lambda}\pi_{2\mu}\pi_{3\nu}) \equiv (e_\lambda e_\mu e_\nu - z_\lambda z_\mu z_\nu - z_\lambda^* z_\mu^* z_\nu^*) - \tfrac{1}{2} \sum_{\substack{\text{all perms}\\ \lambda\mu\nu}} (e_\lambda z_\mu z_\nu^*) \tag{3.18}$$

and

$$1_A \equiv \det(\pi_{1\lambda}\pi_{2\mu}\pi_{3\nu}) \equiv 3\sqrt{3}\, j \det(e_\lambda z_\mu z_\nu^*), \tag{3.19}$$

respectively. The mixed symmetry expression is

$$2_M \equiv 3 \sum_{\substack{\text{cyclic perms}\\ \lambda\mu\nu}} (z_\lambda^* z_\mu^* e_\nu - e_\lambda e_\mu z_\nu - z_\lambda z_\mu z_\nu^*) - 9(z_\mu^* z_\nu^* e_\lambda - e_\mu e_\nu z_\lambda - z_\mu z_\nu z_\lambda^*) \equiv A(\lambda\mu\nu) - jB(\lambda\mu\nu), \tag{3.20}$$

where

$$2A(\lambda\mu\nu) \equiv [(\pi_{1\lambda}\pi_{2\mu} + \pi_{1\mu}\pi_{2\lambda})\pi_{3\nu} + (\pi_{1\nu}\pi_{2\lambda} + \pi_{1\lambda}\pi_{2\nu})\pi_{3\mu} - 2(\pi_{1\mu}\pi_{2\nu} + \pi_{1\nu}\pi_{2\mu})\pi_{3\lambda}],$$

$$2B(\lambda\mu\nu) = \sqrt{3}\,[(\pi_{1\lambda}\pi_{2\mu} - \pi_{1\mu}\pi_{2\lambda})\pi_{3\nu} - (\pi_{1\nu}\pi_{2\lambda} - \pi_{1\lambda}\pi_{2\nu})\pi_{3\mu}]. \tag{3.21}$$

The totally symmetric expression in Eq. (3.18) contains isospins 1 and 3, the antisymmetric expression of Eq. (3.19) has isospin zero, and the mixed symmetry terms of Eq. (3.21) are admixtures of isospin 1 and 2. In the conventional scheme in which π_1 and π_2 are coupled to a resultant isospin T_R by means of Clebsch-Gordan coefficients and this resultant is coupled with π_3 to an overall isospin (T, T_z), we find that the symmetric states are[3]:

$$|3, T_z(1_S)\rangle = |(2)3, T_z\rangle, \qquad |1, T_z(1_S)\rangle = \frac{\sqrt{5}}{3}\,|(0)1, T_z\rangle + \tfrac{2}{3}\,|(2)1, T_z\rangle, \tag{3.22}$$

where the number in parentheses denotes the intermediate resultant T_R. The antisymmetric state is the triple scalar product, or

$$|0(1_A)\rangle = |(1)0\rangle. \tag{3.23}$$

The A and B states of isospin (T, T_z) corresponding to the mixed symmetry expressions of Eqs.

(3.20) and (3.21) are

$$Z\,|2,T_z(2_M)\rangle = |(2)2,T_z\rangle - j\,|(1)2,T_z\rangle \tag{3.24}$$

and

$$Z\,|1,T_z(2_M)\rangle = \frac{\sqrt{5}}{3}\,|(2)1,T_z\rangle - \tfrac{2}{3}\,|(0)1,T_z\rangle + j\,|(1)1,T_z\rangle. \tag{3.25}$$

General expressions for these states $|(T_R)T,T_z\rangle$ can be found in the paper of Barton, Kacser, and Rosen.[3]

D. Generalizations to other spins

Although the results of the two preceding sections have been derived for the specific case of momentum and isospin vectors, they can easily be generalized to other tensors. In Eq. (3.7), for example, the vectors k_i can be replaced by spherical tensors of rank $\mathfrak{M}$, where $\mathfrak{M}$ is arbitrary; the index λ in Eq. (3.8) now runs from $(-\mathfrak{M})$ to $(+\mathfrak{M})$ instead of (-1) to $(+1)$. The rotational properties of the various products of Z_λ discussed in Eqs. (3.9) to (3.17) will now depend on $\mathfrak{M}$, but their permutational properties under S_3 remain the same for all values of $\mathfrak{M}$. Our methods for picking out rotational tensors from n-fold products of Z_λ are readily generalized from the vector to the mth-rank-tensor case.

In a similar way, the results of Sec. III C, for isovectors can be extended to other isospins by allowing the indices λ, μ, ν in Eqs. (3.16) to (3.21) to run from $(-I)$ to $(+I)$ instead of (-1) to $(+1)$; moreover, I can take on integer, or half-integer values. When $I=\frac{1}{2}$, the results of Eqs. (3.18) to (3.21) reproduce well-known results for the isospins of three-nucleon systems: Thus the 1_S state of Eq. (3.18) has a total isospin of $\frac{3}{2}$; the 1_A state of Eq. (3.19) vanishes identically because two of the three indices λ, μ, ν must always be equal for $I=\frac{1}{2}$; and the 2_M states of Eqs. (3.20) and (3.21) reproduce the standard mixed symmetry isospin wave functions for the triton and ^{3}He (Ref. 8) or for the neutron and proton in terms of u and d quarks.[9] For example, define

$$\alpha = \pi_{1/2}, \quad \beta = \pi_{-1/2} \tag{3.26}$$

and set $\lambda=\mu=\frac{1}{2}$, $\nu=-\frac{1}{2}$ in Eq. (3.21); we then obtain the usual expressions with mixed symmetry[8]

$$2A_{1/2,1/2-1/2} = -\left[(\alpha(1)\beta(2)+\alpha(2)\beta(1))\,\alpha(3) - 2\alpha(1)\alpha(2)\beta(3)\right],$$

$$2B_{1/2,1/2-1/2} = -\sqrt{3}\left[(\beta(1)\alpha(2)-\alpha(1)\beta(2))\alpha(3)\right]. \tag{3.27}$$

The application of these results to the intrinsic spin of nucleons and to particles with higher spins is just a matter of interpretation; the α and β states of Eq. (3.27) could just as well be intrinsic spin states as well as isospin ones.

IV. THREE-PARTICLE WAVE FUNCTIONS

Having constructed the representations of S_3 in terms of Dalitz-plot coordinates, momenta, and isospins, we are now in a position to construct three-pion wave functions for any state of angular momentum, parity, and isospin.

Following Zemach[4] we write the basic wave functions as products of three factors

$$\psi(3\pi) = \psi(I)\times\psi(J^P)\times\psi(F), \tag{4.1}$$

where $\psi(I)$ describes the isospin, $\psi(J^P)$ the spin and parity, and $\psi(F)$ is a form factor which behaves as a scalar under spatial rotations and reflections. The overall permutation properties must be symmetric and so the structure of the wave function is as follows for symmetric isospin states:

$$\psi(3\pi) = \begin{cases} 1_S(I)\times 1_S(J^P)\times 1_S(F) \\ 1_S(I)\times 1_A(J^P)\times 1_A(F) \\ 1_S(I)\times Z(J^P)\times Z^x(F) + \text{pc}, \end{cases} \tag{4.2}$$

where pc denotes permutation conjugate. For antisymmetric isospin states, the possibilities are

$$\psi(3\pi) = \begin{cases} 1_A(I)\times 1_A(J^P)\times 1_S(F) \\ 1_A(I)\times 1_S(J^P)\times 1_A(F) \\ 1_A(I)\times [Z(J^P)\times Z^x(F) - \text{pc}], \end{cases} \tag{4.3}$$

and for mixed-symmetry isospin there are five combinations:

$$\psi(3\pi) = \begin{cases} [Z(I)\times Z^x(J^P)+\text{pc}]\times 1_S(F) \\ [Z(I)\times Z^x(J^P)-\text{pc}]\times 1_A(F) \\ [Z(I)\times Z^x(F)+\text{pc}]\times 1_S(J^P) \\ [Z(I)\times Z^x(F)-\text{pc}]\times 1_A(J^P) \\ [Z(I)\times Z(J^P)\times Z(F)+\text{pc}]\,. \end{cases} \tag{4.4}$$

The isospin wave functions classified according to their S_3 properties are given in Eqs. (3.22)–(3.25), the spin-parity wave functions are given in Tables I and II, and the general forms of the form factors are

$$\psi(F,1_S) = \sum_{k=0}^{\infty} a_k\rho^{3k}\cos 3k\varphi,$$

$$\psi(F,1_A) = \sum_{k=1}^{\infty} b_k\rho^{3k}\sin 3k\varphi, \tag{4.5}$$

$$\psi(F,2_M) = \sum_{k=0}^{\infty} (c_k\rho^{3k+1}e^{-j(3k+1)\varphi} + d_k\rho^{3k+2}e^{j(3k+2)\varphi}).$$

Thus given any set of quantum numbers for the three pions, we can construct all possible wave functions from Eqs. (4.2)–(4.4).

For certain quantum numbers we do not need all the formulas in Eqs. (4.2)–(4.4). When $I=0$, for example, the isospin wave function behaves like 1_A and we only need Eq. (4.3); similarly for spin zero, we need only those expressions involving $1_S(J^P)$, and for $J^P=1^-$ (see Table II), we need only the $1_A(J^P)$ expressions. Other properties such as CP invariance may further reduce the number of independent expressions.

Another application of these methods is to the three-fermion problem. The first case we consider is the three-quark wave function that occurs in the nucleon, the Δ resonance, and in their excited states.[9] The color-singlet wave function is totally antisymmetric under permutation of the quarks, and so the product of isospin, spin, and space wave functions must be totally symmetric. We write this product as

$$\psi(3q)=\psi(I)\times\psi(S)\times\psi(r), \qquad (4.6)$$

where $\psi(I)$ is the isospin factor, $\psi(S)$ the spin, and $\psi(\vec{r})$ the spatial wave function. Since the isospin and spin wave functions belong to the $1_S(I=\frac{3}{2})$ and $2_M(I=\frac{1}{2})$ representations, the relevent formulas are contained in Eqs. (4.2) and (4.4) with the obvious substitutions of S and $\vec{r}$ for J^P and F. When the three quarks are in the same shell (that is, when they have the same principal quantum number and the same orbital angular momentum l), then we can regard the $\vec{k}_i$ of Sec. III as representing the appropriate spherical harmonics and we can construct $\psi(r)$ from the corresponding Z_λ and product formulas analogous to Eqs. (3.18)–(3.21). The 2_M spin and isospin functions are as in Eq. (3.27).

Next we consider the three-nucleon problem.[8] In this case there are no color indices to be antisymmetrized, and so the product of isospin, spin, and spatial wave functions must be antisymmetric. Thus

$$\psi(3N)=\begin{cases} 1_S(I)\times 1_S(S)\times 1_A(\vec{r}) \\ 1_S(I)\times[Z(S)\times Z^x(\vec{r})-\text{pc}] \\ Z(I)\times 1_S(S)\times Z^x(\vec{r})-\text{pc} \\ Z(I)\times Z^x(S)\times 1_S(\vec{r})-\text{pc} \\ Z(I)\times Z^x(S)\times 1_A(\vec{r})+\text{pc} \\ Z(I)\times Z(S)\times Z(\vec{r})-\text{pc}. \end{cases} \qquad (4.6)$$

Again the spatial wave function can be constructed in a manner analogous to Z_λ when the nucleons are in the same shell.

Finally we could also consider the three-electron problem. In this case there is no isospin and so the first two lines of Eq. (4.7) contain the relevant formulas.

We shall apply these results to the decays of various pseudoscalar mesons in a subsequent paper. Of particular interest will be the structure of the Dalitz plot, and relationships between different decay modes predicted by various selection rules.

[1]S. Weinberg, Phys. Rev. Lett. 4, 87 (1957).
[2]R. H. Dalitz, Rev. Mod. Phys. 31, 823 (1958).
[3]G. Barton, C. Kacser, and S. P. Rosen, Phys. Rev. 130, 783 (1963).
[4]C. Zemach, Phys. Rev. 133, B1201 (1964).
[5]A. J. Dragt, J. Math. Phys. 6, 533 (1965).
[6]B. W. Lee, New Phys. (Korean Physical Society) Suppl. 7, 41 (1968).
[7]See, for example, M. Hammermesh, *Group Theory* (Addison-Wesley, Reading, Mass., 1962), Chap. 7.
[8]G. Derrick and J. M. Blatt, Nucl. Phys. 8, 310 (1958). The authors are indebeted to Professor A. Tubis for drawing their attention to this reference.
[9]See, for example, D. B. Lichtenberg, *Unitary Symmetry and Elementary Particles* (Academic, New York, 1970).
[10]R. H. Dalitz, Philos. Mag. 44, 1068 (1953); E. Fabri, Nuovo Cimento 11, 479 (1954).