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Soft-photon expansion and soft-photon theorem

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The question of the existence of the soft-photon expansion and its connection with the derivation of the soft-photon theorem is studied. We find that the soft-photon expansion does not necessarily exist for all possible bremsstrahlung cross sections, and if such an expansion does not exist for some types of cross section, then the soft-photon theorem cannot be derived for them. We define two different classes of bremsstrahlung cross section for our study, which we call the H-type cross section and the R-type cross section. We find that the R-type cross section can be expanded in powers of k (photon momentum) with all coefficients of the expansion which are independent of k, while the H-type cross section can only have an expansion whose coefficients are all functions of k. Moreover, we have shown that those coefficients which are functions of k must necessarily contain off-shell parameters. Therefore, we conclude that the soft-photon expansion exists and hence the soft-photon theorem can be derived only for the R-type cross section. This strongly implies that the R-type cross section is to be preferred for the study of bremsstrahlung processes.

I. INTRODUCTION

The soft-photon theorem (or Low's theorem or the low-energy theorem for photons) has played an important role in the study of all bremsstrahlung processes. The derivation of this theorem is based upon the important soft-photon expansion, which gives the bremsstrahlung cross section σ as an expansion in powers of the photon energy k:

$$\sigma=\frac{\sigma_{-1}}{k}+\sigma_0+\sigma_1 k+\cdots , \tag{1a}$$

where

$$\sigma_{-1}=\lim_{k\to 0}(k\sigma) .$$

$$\sigma_0=\lim_{k\to 0}\frac{\partial}{\partial k}(k\sigma)_{x_i}$$

and

$$\sigma_1=\lim_{k\to 0}\frac{\partial^2}{\partial k^2}(k\sigma)_{x_i} .$$

Here the x_i refer to the set of observables which are held constant in carrying out the partial differentiation. We emphasize that all coefficients in this expansion are independent of k. In terms of this expansion, the soft-photon theorem states that σ_{-1} and σ_0 are independent of the off-mass-shell effects (or off-energy-shell effects) and that they can be evaluated from the amplitude of the corresponding nonradiative process (or elastic process) and its derivatives. This theorem thus provides us with an approximate method for calculating bremsstrahlung cross sections. In this approximation, known as the soft-photon approximation (SPA), one calculates the cross sections by retaining only the first two terms of the expansion given by Eq. (1a):

$$\sigma_{\mathrm{SPA}}=\frac{\sigma_{-1}}{k}+\sigma_0 . \tag{1b}$$

When σ_{SPA} is plotted as a function of k, Eq. (1b) yields a family of hyperbolas characterized by two constants, σ_{-1} and σ_0. For a given σ_{-1}, the shape of the hyperbola is determined and the constant σ_0 will shift this hyperbola up or down along the vertical axis without changing this shape. The universal characteristic curves for the SPA are therefore hyperbolas if σ_{SPA} is plotted as a function of k. Those calculations which fail to produce these characteristic curves should not be classified as the SPA.

The importance of this SPA calculation can be understood as follows: As we know, measuring off-mass-shell effects is one of the main reasons given for studying the bremsstrahlung processes. Since the SPA is a model-independent calculation which uses only the on-shell amplitude of the corresponding nonradiative process as an input for calculating radiative cross sections, the cross section σ_{SPA} produced by Eq. (1b) will provide us with an "on-shell model-independent cross section". A comparison of this with experiment can then provide physically meaningful information about the possible off-mass-shell effects (and resonance effects, if any) which can be extracted from the data. Moreover, since all model-dependent calculations, which may contain some approximations and assumptions, should reproduce the result of the SPA in the soft-photon region, the SPA can be used not only to extract the off-mass-shell effects produced from these models but also to check the validity of these model-dependent calculations.[1]

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The soft-photon theorem was first derived by Low[2] and was later generalized and extended by many other authors[3] using the formalism of relativistic quantum field theory. Since bremsstrahlung can also be calculated from a potential theory, several authors have rederived the theorem using the Lippmann-Schwinger formalism.[4] Unfortunately, in all these derivations, the kinematics and dynamics of the bremsstrahlung process were *not* expanded consistently and completely.[5] As a result, the expansion obtained previously was not appropriate to the soft-photon theorem. The incomplete expansion can be written as

$$\sigma = \frac{\tilde{\sigma}_{-1}(k)}{k} + \tilde{\sigma}_0(k) + \tilde{\sigma}_1(k)k + \cdots \tag{2a}$$

Here all coefficients in Eq. (2a) are functions of k, and as we shall show later, although $\tilde{\sigma}_{-1}(k)$ and $\tilde{\sigma}_0(k)$ are independent of off-shell derivatives, they are still functions of off-shell parameters. The cross section calculated from the first two terms of Eq. (2a) has the form

$$\tilde{\sigma} = \frac{\tilde{\sigma}_{-1}(k)}{k} + \tilde{\sigma}_0(k)\,. \tag{2b}$$

Obviously, when $\tilde{\sigma}$ is plotted against k, Eq. (2b) does not represent a family of hyperbolas. Therefore, all calculations based upon Eq. (2b) should not be classified as the SPA. Equation (2b) has been applied to predict the pion-proton bremsstrahlung cross section, $d^5\sigma/d\Omega_\pi d\Omega_\gamma dk$, at 298 MeV for various θ_γ and ϕ_γ. The predicted spectra rise steeply with increasing photon energy above $k = 80$ MeV in complete disagreement with UCLA data.[6,7] The problems arise from $\tilde{\sigma}_{-1}(k)$ and $\tilde{\sigma}_0(k)$ which are not independent of k (or off-shell parameters). If we look at the UCLA data, we find that every bremsstrahlung spectrum obtained by the UCLA group exhibits the shape of hyperbola in agreement with the characteristic curve of the SPA given by Eq. (1b). This observation was supported by a recent calculation. Nutt and I have used Eq. (1b) to predict the coplanar $\pi^\pm p\gamma$ cross section and have obtained results which are in excellent agreement with the UCLA data.[5] These results show that the soft-photon theorem, derived from the correct soft-photon expansion Eq. (1a), works remarkably well for $\pi^\pm p\gamma$ at 298 MeV. However, there is an important problem which has not been studied. This problem is related to the following question: Does a physically meaningful soft-photon expansion, Eq. (1a), exists for all possible bremsstrahlung cross sections?

In Eq. (1a), the independent variables in the expression for σ are not specified. As we know, in all present bremsstrahlung experiments, there are three outgoing particles with nine degrees of freedom. This is reduced to five by the four equations of energy-momentum conservation. σ is therefore a function of five independent kinematical variables. Since the choice of these five independent variables is quite different for different experimental geometries, many different types of cross section can be constructed. Thus, if Eq. (1a) could be written without specifying the explicit expression of σ, it would imply that the soft-photon expansion exists for all the various cross sections which can be constructed. But the question is whether such an expansion exists in terms of physically meaningful quantities for all possible cross sections. This question has not been studied. The importance of the question can be easily understood. The existence of the soft-photon expansion is essential for the derivation of the soft-photon theorem. This means that the theorem can be derived if and only if the expansion given by Eq. (1a) exists. If such an expansion does not exist for some types of cross section, then the theorem cannot be derived for them and the SPA given by Eq. (1b) cannot be applied to predict those cross sections. In the past, the differential cross section implied by the expression for σ in Eq. (1) was never specified because the existence of the soft-photon expansion was tacitly assumed, and the SPA was used to predict many different cross sections without examining its validity. The question of the existence of the soft-photon expansion and its connection with the derivation of the soft-photon theorem was never before carefully studied.

The main purpose of this paper is to study this problem carefully. We have chosen two different classes of bremsstrahlung cross section for our study. These two classes are very general; they represent various standard forms of the cross section used in many different experimental geometries. We call these two classes of cross section the H-type and the R-type cross section. We will show that, to our great surprise, the previous assumption about the existence of the soft-photon expansion for all possible cross sections was wrong. We have found that the soft-photon expansion does not necessarily exist in terms of physically meaningful amplitudes for all possible differential cross sections. We have also found that the R-type cross section does have a natural expansion in powers of k, while the H-type cross section does not. In the latter case, one must violate the fundamental principle of energy-momentum conservation in order to generate nonzero coefficients for the soft-photon expansion since the H-type cross section is zero in the $k \to 0$ limit. This means that the R-type cross section does have an expansion given by Eq. (1a), while the H-type cross section can only have an expansion which is similar to Eq. (2a). Furthermore, we have shown that in any expansion of the form given by Eq. (2a), the coefficients which are functions of k must necessarily contain off-shell parameters. Therefore, the soft-photon theorem can be derived in terms of physical quantities only for the R-type cross section, and hence that is the cross section for which a physically useful SPA exists. This strongly implies that the R-type cross section is to be preferred for the study of bremsstrahlung processes.

II. KINEMATICS AND ITS SOFT-PHOTON EXPANSION

It is obvious that Eq. (1a) cannot be obtained without expanding the kinematics of the bremsstrahlung process. In fact, Eq. (1a) exists if and only if the expansion of the kinematics in powers of k exists. Therefore, the kinematical and dynamical aspects of bremsstrahlung are equally important in the derivation of the soft-photon theorem, and they must be treated consistently in the soft-photon expansion.

We consider photon emission accompanying the scattering of two particles A and B:

$$A(q_i^\mu)+B(p_i^\mu)\to A(q_f^\mu)+B(p_f^\mu)+\gamma(k^\mu). \tag{3}$$

Here $q_i^\mu\,(q_f^\mu)$ and $p_i^\mu\,(p_f^\mu)$ are the initial (final) four-momenta for particles A and B, respectively, and k^μ is the four-momentum for the emitted photon. These five four-momenta are defined in the laboratory frame as

$$q_i^\mu=(E_i,0,0,q_i), \tag{4a}$$

$$p_i^\mu=(M,0,0,0), \tag{4b}$$

$$q_f^\mu=(E_q,q_f\sin\theta_q\cos\phi_q,q_f\sin\theta_q\sin\phi_q,q_f\cos\theta_q), \tag{4c}$$

$$p_f^\mu=(E_p,p_f\sin\theta_p\cos\phi_p,p_f\sin\theta_p\sin\phi_p,p_f\cos\theta_p), \tag{4d}$$

$$k^\mu=(k,k\sin\theta_\gamma\cos\phi_\gamma,k\sin\theta_\gamma\sin\phi_\gamma,k\cos\theta_\gamma), \tag{4e}$$

where

$$E_i=(m^2+\vec{q}_i^{\,2})^{1/2},\quad E_q=(m^2+\vec{q}_f^{\,2})^{1/2},$$

$$E_p=(M^2+\vec{p}_f^{\,2})^{1/2},$$

and m and M are the masses of particles A and B, respectively. The angles θ and ϕ follow the usual convention in spherical coordinates. These four-momenta satisfy energy-momentum conservation:

$$q_i^\mu+p_i^\mu=q_f^\mu+p_f^\mu+k^\mu, \tag{5}$$

or

$$q_f\sin\theta_q\cos\phi_q+p_f\sin\theta_p\cos\phi_p+k\sin\theta_\gamma\cos\phi_\gamma=0, \tag{6a}$$

$$q_f\sin\theta_q\sin\phi_q+p_f\sin\theta_p\sin\phi_p+k\sin\theta_\gamma\sin\phi_\gamma=0, \tag{6b}$$

$$q_f\cos\theta_q+p_f\cos\theta_p+k\cos\theta_\gamma=q_i, \tag{6c}$$

$$E_q+E_p+k=M+E_i. \tag{6d}$$

For a given incident energy E_i, there are nine variables in these equations: q_f, θ_q, ϕ_q, p_f, θ_p, ϕ_p, k, θ_γ, and ϕ_γ. If five of these are chosen to be independent variables, then the other four can be determined in terms of these five independent variables by solving Eqs. (6), and the bremsstrahlung cross section can be expressed as a function of these five variables. The choice of these five independent kinematical variables is strongly influenced by experimental considerations. During the last 15 years, a great number of nucleon-nucleon (NN), nucleon-nucleus (NA), nucleus-nucleus (AA), and pion-nucleon (πN) bremsstrahlung experiments have been performed. The most typical choices of five independent variables in these experiments were the following:

(i) Choosing θ_q, ϕ_q, θ_p, ϕ_p, and θ_γ as independent variables, such that the bremsstrahlung cross section can be expressed in the form $d^5\sigma/d\Omega_q d\Omega_p d\theta_\gamma$. Here $d\Omega_q\equiv\sin\theta_q d\theta_q d\phi_q$ and $d\Omega_p\equiv\sin\theta_p d\theta_p d\phi_p$.

(ii) Choosing θ_q, ϕ_q, θ_p, ϕ_p, and $E_q\equiv(m^2+\vec{q}_f^{\,2})^{1/2}$ as independent variables, such that the bremsstrahlung cross section can be expressed in the form $d^5\sigma/d\Omega_q d\Omega_p dE_q$.

(iii) Choosing θ_q, ϕ_q, θ_p, ϕ_p, and k as independent variables, such that the bremsstrahlung cross section can be expressed in the form $d^5\sigma/d\Omega_q d\Omega_p dk$.

(iv) Choosing θ_q, ϕ_q, θ_γ, ϕ_γ, and k as independent variables, such that the bremsstrahlung cross section can be expressed in the form $d^5\sigma/d\Omega_q d\Omega_\gamma dk$. Here $d\Omega_\gamma\equiv\sin\theta_\gamma d\theta_\gamma d\phi_\gamma$.

These four special cases can be generalized and classified into two classes of bremsstrahlung cross section:

(A) *H-type cross section* $\sigma_H\equiv d^5\sigma/d\Omega_q d\Omega_p dx$. Here x is chosen from one of the following variables: $q_f, E_q, p_f, E_p, k, \theta_\gamma, \psi_\gamma, \phi_\gamma, \ldots$. The independent variables for the H-type cross section are, therefore, θ_q, ϕ_q, θ_p, ϕ_p, and x.

(B) *R-type cross section* $\sigma_R\equiv d^5\sigma/d\Omega_\gamma dk dy dz$. Here y can be either ϕ_q or ϕ_p, and z is chosen from one of the following variables: $q_f, E_q, p_f, E_p, \theta_q, \theta_p, \ldots$. The independent variables are θ_γ, ϕ_γ, k, y, and z.

One should not confuse the H-type (R-type) *cross section* with Harvard (Rochester) *geometry* since the H-type (R-type) cross section can also be measured from Rochester (Harvard) geometry. [Harvard geometry refers to any experimental arrangement in which two final-state particles (A and B) are detected and the photon momentum is calculated from Eqs. (6). Rochester geometry refers to an experimental arrangement in which all three final-state particles (A, B, and γ) are detected.] There are other types of cross section which cannot be classified into these two classes. These types of cross section will not be discussed here, primarily because they have never been studied experimentally or theoretically. As can be seen from our classification, the basic difference between these two classes of cross section lies in their choice of independent kinematical variables. The kinematics associated with them is therefore different. Now, since it is impossible to have a soft-photon expansion for any type of cross section if the kinematics associated with it cannot be expanded in powers of k, the question of the existence of the soft-photon expansion for a cross section can be answered by studying the existence of the soft-photon expansion for the kinematics.

We shall carefully study the above four special cases, (i), (ii), (iii), and (iv), to see if there exist soft-photon expansions for them. The results will then be generalized to more general cases. In each of our four special cases, there are four dependent kinematical variables which are functions of five independent variables. If a soft-photon expansion exists, then the solutions for these four dependent variables can be expanded in powers of the photon energy k and the soft-photon limits for these dependent variables (i.e., the lowest-order solutions for these dependent variables) must exist as $k \to 0$. Thus, what we are trying to determine is whether the lowest-order solutions for these variables exist as $k \to 0$. The existence of such limits means that the kinematics of the bremsstrahlung process (which is three-body kinematics) can be reduced to the kinematics of the corresponding nonradiative process (which is two-body kinematics for elastic scattering) as $k \to 0$. If such limits do not exist, then the expansion given by Eq. (1) is not physically meaningful, even though one might mathematically use analytic continuation to define its existence.

In case (i), the dependent variables are q_f, p_f, k, and ϕ_γ. For a given E_i, they are functions of θ_q, ϕ_q, θ_p, ϕ_p, and θ_γ. Since k is not an independent variable, one cannot let k approach zero arbitrarily (or simply set k equal to zero). Thus, the soft-photon limits $\lim_{k\to 0} q_f$, $\lim_{k\to 0} p_f$, and $\lim_{k\to 0}\phi_\gamma$ do not physically exist under the restriction of energy-momentum conservation. Let us examine what would happen if we let $k=0$ in Eqs. (6). We obtain

$$q_f \sin\theta_q \cos\phi_q + p_f \sin\theta_p \cos\phi_p = 0\,, \tag{7a}$$

$$q_f \sin\theta_q \sin\phi_q + p_f \sin\theta_p \sin\phi_p = 0\,. \tag{7b}$$

$$q_f \cos\theta_q + p_f \cos\theta_p = q_i\,, \tag{7c}$$

$$E_q + E_p = M + E_i\,. \tag{7d}$$

To satisfy Eqs. (7a) and (7b), ϕ_q and ϕ_p must have the relation

$$\phi_q = \phi_p \pm n\pi \quad (n=0,1,2,\ldots)\,, \tag{7e}$$

which shows that one of them must not be an independent variable. That ϕ_q or ϕ_p is not an independent variable is in contradiction to our original assumption. We thus conclude that the limits for q_f, p_f, and ϕ_γ do not exist as $k \to 0$. Since ϕ_q and ϕ_p are the angles used only in the noncoplanar case, one may wonder if such limits exist for the coplanar case where $\phi_q = 0$, $\phi_p = \pi$, and $\phi_\gamma = 0$. The answer is still no. To see this, we combine Eqs. (7a) and (7b) into one single equation by using $\phi_q = 0$ and $\phi_p = \pi$:

$$q_f \sin\theta_q - p_f \sin\theta_p = 0\,. \tag{7f}$$

If Eqs. (7c) and (7f) are solved for q_f and p_f, we get

$$q_f = \frac{q_i \sin\theta_p}{\sin(\theta_q + \theta_p)} \tag{8a}$$

and

$$p_f = \frac{q_i \sin\theta_q}{\sin(\theta_q + \theta_p)}\,. \tag{8b}$$

Inserting these results into Eq. (7d) gives

$$\left[m^2 + \frac{q_i^2 \sin^2\theta_p}{\sin^2(\theta_q+\theta_p)}\right]^{1/2} + \left[M^2 + \frac{q_i^2 \sin^2\theta_q}{\sin^2(\theta_q+\theta_p)}\right]^{1/2} = E_i + M\,, \tag{9}$$

which shows that either θ_q or θ_p is no longer an independent variable, and this again is in contradiction to the original assumption for this case. Therefore, the limits $\lim_{k\to 0} q_f$, $\lim_{k\to 0} p_f$, and $\lim_{k\to 0}\phi_\gamma$ do not exist for either the coplanar or noncoplanar case. This means that energy-momentum conservation would be violated if we set $k=0$ and if we try to expand the dependent variables q_f, p_f, and ϕ_γ in powers of k. Furthermore, since $q_i^\mu + p_i^\mu - q_f^\mu - p_f^\mu - k^\mu \neq 0$ implies that $\delta^4(q_i + p_i - q_f - p_f - k) = 0$, the bremsstrahlung cross section must be zero for $k=0$. We therefore conclude that the expansion in Eq. (1a) does not exist in terms of physical quantities for the cross section $d^5\sigma/d\Omega_q d\Omega_p d\theta_\gamma$, and hence the soft-photon theorem cannot be derived in terms of physical elastic amplitudes for this type of cross section. [It is interesting to note that for $\theta_q = \theta_p = 30°$, $\phi_q = 0$, $\phi_p = \pi$, and $m = M$, k is zero for any value of θ_γ only if $q_f = p_f = (\sqrt{3}/3)q_i = \sqrt{8}M$; that is, the total incident energy E_i must be $5M$.]

In case (ii), the dependent variables are p_f, k, θ_γ, and ϕ_γ. Here again, k is not an independent variable. Therefore, an argument which is similar to the one used in case (i) can be applied to show that the soft-photon limits $\lim_{k\to 0} p_f$, $\lim_{k\to 0}\theta_\gamma$, and $\lim_{k\to 0}\phi_\gamma$ do not exist. Briefly, if $k=0$, we would obtain Eq. (7e) which shows that either ϕ_q or ϕ_p is not an independent variable in contradiction to our assumption. And for the coplanar case, we find the the following contradictions:

$$p_f = q_f\frac{\sin\theta_q}{\sin\theta_p} = (q_i - q_f\cos\theta_q)/\cos\theta_p\,,$$
$$E_p = (M^2 + \vec{\mathrm{P}}_f^{\,2})^{1/2} = M + E_i - E_q\,. \tag{10}$$

Here q_i, m, and M are given, and q_f, θ_q, and θ_p are independent variables. We therefore conclude that the soft-photon expansion given by Eq. (1a) does not exist for the cross section $d^5\sigma/d\Omega_q d\Omega_p dE_q$, and hence the soft-photon theorem cannot be derived for this type of cross section.

In case (iii), the independent variables are θ_q, ϕ_q, θ_p, ϕ_p, and k. This is an interesting case since k is now an independent variable. It seems that the soft-photon limits exist for four dependent variables (q_f, p_f, θ_γ, and ϕ_γ) as $k\to 0$, and they can be expanded in powers of k. But actually the answer is still no. This is because we always obtain Eqs. (7) if $k=0$, and the situation becomes exactly the same as we had in case (i). This means that we would find the same kind of contradictions as we obtained there. Thus, the limits $\lim_{k\to 0} q_f$, $\lim_{k\to 0} p_f$, $\lim_{k\to 0}\theta_\gamma$, and $\lim_{k\to 0}\phi_\gamma$ do not exist, and the dependent variables cannot be expanded in powers of k. We therefore conclude that a physically meaningful soft-photon theorem cannot be derived for the cross section $d^5\sigma/d\Omega_q d\Omega_p dk$.

In case (iv), the independent variables are θ_q, ϕ_q, θ_γ, ϕ_γ, and k, and the cross section is expressed in the form $d^5\sigma/d\Omega_q d\Omega_\gamma dk$. This is the only case in which the soft-photon limits for four dependent variables exist, and these dependent variables can be expanded in powers of k as

$$q_f = \bar{q}_f + \left(\frac{dq_f}{dk}\right)_{k=0} k + \cdots\,, \tag{11a}$$

$$p_f = \bar{p}_f + \left(\frac{dp_f}{dk}\right)_{k=0} k + \cdots\,, \tag{11b}$$

$$\theta_p = \bar{\theta}_p + \left(\frac{d\theta_p}{dk}\right)_{k=0} k + \cdots\,, \tag{11c}$$

$$\phi_p = \bar{\phi}_p + \left(\frac{d\phi_p}{dk}\right)_{k=0} k + \cdots\,, \tag{11d}$$

where

$$\bar{q}_f = \lim_{k\to 0} q_f\,,\quad \bar{p}_f = \lim_{k\to 0} p_f\,,$$
$$\bar{\theta}_p = \lim_{k\to 0}\theta_p\,,\quad \bar{\phi}_p = \lim_{k\to 0}\phi_p\,.$$

Here $\bar{q}_f$, $\bar{p}_f$, $\bar{\theta}_p$, and $\bar{\phi}_p$ are the lowest terms in the expansion, and they satisfy the energy-momentum-conservation equations for the corresponding nonradiative two-body elastic scattering process,

$$\bar{q}_f\sin\theta_q\cos\phi_q + \bar{p}_f\sin\bar{\theta}_p\cos\bar{\phi}_p = 0\,, \tag{12a}$$

$$\bar{q}_f\sin\theta_q\sin\phi_q + \bar{p}_f\sin\bar{\theta}_p\sin\bar{\phi}_p = 0\,, \tag{12b}$$

$$\bar{q}_f\cos\theta_q + \bar{p}_f\cos\bar{\theta}_p = q_i\,, \tag{12c}$$

$$\bar{E}_q + \bar{E}_p = M + E_i\,, \tag{12d}$$

which are obtained from Eqs. (6) by setting $k=0$. These equations will be solved in order to show that the solutions for $\bar{q}_f$, $\bar{p}_f$, $\bar{\theta}_p$, and $\bar{\phi}_p$ exist. Combining Eqs. (12a) and (12b) gives

$$\bar{\phi}_p = \phi_q + \pi \tag{12e}$$

and

$$\bar{q}_f\sin\theta_q - \bar{p}_f\sin\bar{\theta}_p = 0\,. \tag{12f}$$

The result given by Eq. (12e) is a special case of the general solution given by Eq. (7e). Here we have chosen n in such a way that the coplanar case can be defined as $\phi_q = 0$, $\bar{\phi}_p = \pi$, and $\phi_\gamma = 0$. From Eqs. (12c) and (12f), $\bar{q}_f$ and $\bar{p}_f$ can be written as

$$\bar{q}_f = q_i\sin\bar{\theta}_p/\sin(\theta_q + \bar{\theta}_p)\,, \tag{13a}$$

$$\bar{p}_f = q_i\sin\theta_q/\sin(\theta_q + \bar{\theta}_p)\,. \tag{13b}$$

If we substitute these expressions into Eq. (12d), we obtain

$$\left[m^2 + \frac{q_i^2\sin^2\bar{\theta}_p}{\sin^2(\theta_q + \bar{\theta}_p)}\right]^{1/2} + \left[M^2 + \frac{q_i^2\sin^2\theta_q}{\sin^2(\theta_q + \bar{\theta}_p)}\right]^{1/2} = M + (m^2 + q_i^2)^{1/2}\,, \tag{14}$$

which can be solved for $\bar{\theta}_p$. The solution for $\bar{\theta}_p$ is then used in Eqs. (13a) and (13b) to calculate $\bar{q}_f$ and $\bar{p}_f$. Thus, the solutions for $\bar{\phi}_p$, $\bar{\theta}_p$, $\bar{p}_f$, and $\bar{q}_f$ exist. There is no contradiction or ambiguity in obtaining these solutions.

In order to show that q_f, p_f, θ_p, and ϕ_p have the soft-photon expansions given by Eqs. (11), we have to derive the expressions for $(dq_f/dk)_{k=0}$, $(dp_f/dk)_{k=0}$, $(d\theta_p/dk)_{k=0}$, and $(d\phi_p/dk)_{k=0}$. To do this, we first differentiate Eqs. (6) with respect to k. Remembering that the independent variables in Eqs. (6) are θ_q, ϕ_q, θ_γ, ϕ_γ, and k, we obtain

$$\sin\theta_q \cos\phi_q \left(\frac{dq_f}{dk}\right)_{k=0} + \sin\bar\theta_p \cos\bar\phi_p \left(\frac{dp_f}{dk}\right)_{k=0} + \bar p_f \cos\bar\theta_p \cos\bar\phi_p \left(\frac{d\theta_p}{dk}\right)_{k=0} - \bar p_f \sin\bar\theta_p \sin\bar\phi_p \left(\frac{d\phi_p}{dk}\right)_{k=0} + \sin\theta_\gamma \cos\phi_\gamma = 0\,, \tag{15a}$$

$$\sin\theta_q \sin\phi_q \left(\frac{dq_f}{dk}\right)_{k=0} + \sin\bar\theta_p \sin\bar\phi_p \left(\frac{dp_f}{dk}\right)_{k=0} + \bar p_f \cos\bar\theta_p \sin\bar\phi_p \left(\frac{d\theta_p}{dk}\right)_{k=0} + \bar p_f \sin\bar\theta_p \cos\bar\phi_p \left(\frac{d\phi_p}{dk}\right)_{k=0} + \sin\theta_\gamma \sin\phi_\gamma = 0\,, \tag{15b}$$

$$\cos\theta_q \left(\frac{dq_f}{dk}\right)_{k=0} + \cos\bar\theta_p \left(\frac{dp_f}{dk}\right)_{k=0} - \bar p_f \sin\bar\theta_p \left(\frac{d\theta_p}{dk}\right)_{k=0} + \cos\theta_\gamma = 0\,, \tag{15c}$$

$$\bar\beta_q \left(\frac{dq_f}{dk}\right)_{k=0} + \bar\beta_p \left(\frac{dp_f}{dk}\right)_{k=0} + 1 = 0\,, \tag{15d}$$

where

$$\bar\beta_q = \bar q_f/(m^2+\bar q_f^{\,2})^{1/2}\,,$$

$$\bar\beta_p = \bar p_f/(M^2+\bar p_f^{\,2})^{1/2}\,.$$

If we eliminate $(d\phi_p/dk)_{k=0}$ from Eqs. (15a) and (15b), we find

$$-\sin\theta_q \left(\frac{dq_f}{dk}\right)_{k=0} + \sin\bar\theta_p \left(\frac{dp_f}{dk}\right)_{k=0} + \bar p_f \cos\bar\theta_p \left(\frac{d\theta_p}{dk}\right)_{k=0} + \sin\theta_\gamma \cos(\phi_\gamma - \bar\phi_p) = 0\,. \tag{15e}$$

In deriving Eq. (15e), we have used Eq. (12e) to set $\cos(\phi_q - \bar\phi_p) = -1$. We then solve Eqs. (15c), (15d), and (15e) for $(dq_f/dk)_{k=0}$, $(dp_f/dk)_{k=0}$, and $(d\theta_p/dk)_{k=0}$:

$$\left(\frac{dq_f}{dk}\right)_{k=0} = -\left[\frac{1-c_1\bar\beta_p}{\bar\beta_q - \bar\beta_p\cos(\theta_q+\bar\theta_p)}\right], \tag{16a}$$

$$\left(\frac{dp_f}{dk}\right)_{k=0} = \frac{\cos(\theta_q+\bar\theta_p) - c_1\bar\beta_q}{\bar\beta_q - \bar\beta_p\cos(\theta_q+\bar\theta_p)}\,, \tag{16b}$$

$$\left(\frac{d\theta_p}{dk}\right)_{k=0} = -\left\{\frac{\sin(\theta_q+\bar\theta_p) - \bar\beta_p c_2 - \bar\beta_q c_3}{\bar p_f[\bar\beta_q - \bar\beta_p\cos(\theta_q+\bar\theta_p)]}\right\}, \tag{16c}$$

where

$$c_1 = \cos\bar\theta_p\cos\theta_\gamma - \sin\bar\theta_p\sin\theta_\gamma\cos(\phi_\gamma - \phi_q)\,,$$

$$c_2 = \cos\theta_\gamma\sin\theta_q - \sin\theta_\gamma\cos\theta_q\cos(\phi_\gamma - \phi_q)\,,$$

$$c_3 = \sin\bar\theta_p\cos\theta_\gamma + \cos\bar\theta_p\sin\theta_\gamma\cos(\phi_\gamma - \phi_q)\,.$$

Finally, $(d\phi_p/dk)_{k=0}$ can be obtained by substituting Eqs. (16a), (16b), and (16c) into Eqs. (15a) or (15b):

$$\left(\frac{d\phi_p}{dk}\right)_{k=0} = \sin\theta_\gamma\sin(\phi_\gamma - \phi_q)/(\bar p_f\sin\bar\theta_p)\,. \tag{16d}$$

It is obvious that our method can be easily extended to obtain the higher-order terms. Therefore the soft-photon expansions for q_f, p_f, θ_p, and ϕ_p exist. We shall show in the next section that the soft-photon expansion given by Eq. (1a) exists indeed for the cross section $d^5\sigma/d\Omega_q d\Omega_\gamma dk$, and hence the soft-photon theorem can be derived for this type of cross section.

As we have already mentioned, the special cases discussed above can be generalized and classified into two classes of bremsstrahlung cross section. Our generalization and classification are based upon the following arguments: (i) In order to let k approach zero arbitrarily, we must choose k as an independent variable. (ii) If we set $k=0$ in Eqs. (6), we obtain Eqs. (7a), (7b), (7c), and (7d). There are six kinematical variables, q_f, θ_q, ϕ_q, p_f, θ_p, and ϕ_p, in these four equations. To solve these equations without contradictions or ambiguities, we can choose at most two independent variables from q_f, θ_q, ϕ_q, p_f, θ_p, and ϕ_p. This implies that θ_γ and ϕ_γ must also be chosen as independent variables. (iii) We can rearrange Eqs. (7a), (7b), (7c), and (7d) into two groups: Eqs. (7c), (7d), and (7f) together as a group for the kinematical variables q_f, θ_q, p_f, and θ_p, and Eq. (7e) alone for the variables ϕ_q and ϕ_p. Since these two groups are independent and we are allowed to choose only two independent variables from them, we have to choose one independent variable from the first group (i.e., from q_f, θ_q, p_f, and θ_p) and another one from the second group (i.e., from ϕ_q and ϕ_p). Now, if we follow these guidelines for choosing five independent variables, we can construct a class of bremsstrahlung cross sections, which are defined as R-type cross sections in the preceding section. The existence of the soft-photon expansion is guaranteed for R-type cross sections and the soft-photon theorem can be derived for this class of cross sections. There are many other cross sections which can be constructed without following these three quidelines. Since the soft-photon expansion, Eq. (1a), does not physically exist under the restriction of energy-momentum conservation, if these guidelines are violated, a physically meaningful soft-photon theorem cannot be derived for all these cross sections. The soft-photon approximation is therefore *not* valid for them. The best way to calculate these cross sections would be model-dependent calculations. The H-type cross sections which we have defined in the preceding section are just some examples of these cross sections. We have discussed only the H-type cross section here, primarily because more than 90% of the nucleon-nucleon bremsstrahlung cross sections are H-type cross sections.

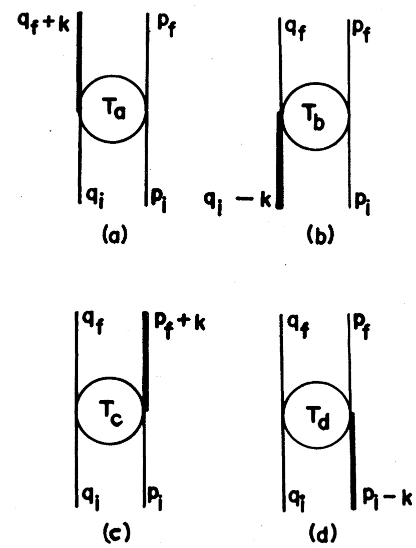

FIG. 1. Four half-off-mass-shell T matrices.

III. BREMSSTRAHLUNG AMPLITUDE

As we have already mentioned in the Introduction, the main purpose of this article is to study the existence of the soft-photon expansion for bremsstrahlung cross sections and its connection with the derivation of the soft-photon theorem. The answer to this question does not depend upon whether the particles (A and B) have spin or not. Therefore, for the sake of simplicity, we shall assume that both A and B have charge e, but they have no spin. The bremsstrahlung amplitude M_μ can be written in a standard way,

$$M_\mu = M_\mu^{(E)} + M_\mu^{(I)}\,, \tag{17}$$

where $M_\mu^{(E)}$ is the sum of those terms which describe photon emission from the external charged lines and $M_\mu^{(I)}$ is the sum of all other terms. In terms of half off-mass-shell T matrices for A-B scattering, the external scattering amplitude $M_\mu^{(E)}$ can be written as

$$M_\mu^{(E)} = \frac{q_{f\mu}}{q_f\cdot k}\,T_a - T_b\frac{q_{i\mu}}{q_i\cdot k} + \frac{p_{f\mu}}{p_f\cdot k}\,T_c - T_d\frac{p_{i\mu}}{p_i\cdot k}\,. \tag{18}$$

Here the half-off-mass-shell T matrices T_a, T_b, T_c, and T_d are given by

$$\begin{aligned} T_a &\equiv T(s_a, t_a, \Delta_a)\,,\\ T_b &\equiv T(s_b, t_b, \Delta_b)\,,\\ T_c &\equiv T(s_c, t_c, \Delta_c)\,, \end{aligned} \tag{19}$$

and

$$T_d \equiv T(s_d, t_d, \Delta_d)\,.$$

In Eq. (19), s_i, t_i, and Δ_i $(i = a, b, c, d)$ are three independent invariants constructed out of the involved four-momenta for diagram (i) of Fig. 1.

There are many ways of choosing these three independent invariants. If we choose s_i to be the average total energy squared, t_i to be the momentum transfer squared, and Δ_i to be the square of the invariant mass of the off-mass-shell leg on which the photon emission occurs, we have

$$\begin{aligned} s_a &= s_c = \tfrac{1}{2}\left[(q_i+p_i)^2 + (q_f+p_f+k)^2\right]\,,\\ s_b &= s_d = \tfrac{1}{2}\left[(q_i+p_i-k)^2 + (q_f+p_f)^2\right]\,,\\ t_a &= t_b = (p_f - p_i)^2\,,\\ t_c &= t_d = (q_f - q_i)^2\,,\\ \Delta_a &= (q_f + k)^2\,,\\ \Delta_b &= (q_i - k)^2\,,\\ \Delta_c &= (p_f + k)^2\,, \end{aligned} \tag{20}$$

and

$$\Delta_d = (p_i - k)^2\,.$$

We have to expand these invariants and the half off-mass-shell T matrices in powers of k in order to obtain the soft-photon expansion for $M_\mu^{(E)}$. Since the existence of such expansion depends upon our choice of five independent variables (or cross sections), two general cases will be discussed.

(*i*) *Bremsstrahlung amplitudes for R-type cross sections.* We have already shown in the preceding section that the soft-photon expansion exists for the kinematics for these cross sections, i.e., four dependent kinematical variables can be expanded in powers of k. This implies that the soft-photon limits for s_i, t_i, and Δ_i exist as $k \to 0$. In the limit of $k = 0$, s_i, t_i, and Δ_i reduce to

$$\lim_{k\to 0} s_i = (q_i+p_i)^2 = (\bar{q}_f+\bar{p}_f)^2 = s\,,$$
$$\lim_{k\to 0} t_i = (\bar{p}_f-p_i)^2 = (\bar{q}_f-q_i)^2 = t\,,$$
$$\lim_{k\to 0}\Delta_a = \lim_{k\to 0}\Delta_b = m^2\,, \tag{21}$$
$$\lim_{k\to 0}\Delta_c = \lim_{k\to 0}\Delta_d = M^2\,,$$

where

$$\bar{q}_f^\mu = \lim_{k\to 0} q_f^\mu\,,$$
$$\bar{p}_f^\mu = \lim_{k\to 0} p_f^\mu\,, \tag{22}$$

which satisfy the equations for energy-momentum conservation in two-body elastic A-B scattering:

$$p_i^\mu + q_i^\mu = \bar{p}_f^\mu + \bar{q}_f^\mu\,. \tag{23}$$

In Eq. (21), s and t are the usual Mandelstam variables for elastic A-B scattering and they are independent of k. We can also expand q_f^μ and p_f^μ in powers of k. To the first order in k, they can be written as

$$q_f^\mu = \bar{q}_f^\mu + R_q^\mu\,, \tag{24a}$$
$$p_f^\mu = \bar{p}_f^\mu + R_p^\mu\,, \tag{24b}$$

where

$$R_q^\mu = \left(\frac{\partial q_f^\mu}{\partial k}\right)_{k=0} k\,,$$
$$R_p^\mu = \left(\frac{\partial p_f^\mu}{\partial k}\right)_{k=0} k\,. \tag{25}$$

These expressions for q_f^μ and p_f^μ can be used to expand s_i, t_i, and Δ_i. If terms of order k^2 are neglected, we find

$$s_a = s_c = s\,,$$
$$s_b = s_d = s - 2(q_i+p_i)\cdot k\,,$$
$$t_a = t_b = t + 2R_p\cdot(\bar{p}_f - p_i)\,,$$
$$t_c = t_d = t + 2R_q\cdot(\bar{q}_f - q_i)\,, \tag{26}$$
$$\Delta_a = m^2 + 2\bar{q}_f\cdot k\,,$$
$$\Delta_b = m^2 - 2q_i\cdot k\,,$$
$$\Delta_c = M^2 + 2\bar{p}_f\cdot k\,,$$

and

$$\Delta_d = M^2 - 2p_i\cdot k\,.$$

Applying these results to expand T_a, T_b, T_c, and T_d, we obtain

$$T_a = T(s,t) + 2R_p\cdot(\bar{p}_f-p_i)\frac{\partial T(s,t)}{\partial t} + 2\bar{q}_f\cdot k\left(\frac{\partial T(s,t,\Delta_a)}{\partial\Delta_a}\right)_{\Delta_a=m^2},$$
$$T_b = T(s,t) - 2(q_i+p_i)\cdot k\frac{\partial T(s,t)}{\partial s} + 2R_p\cdot(\bar{p}_f-p_i)\frac{\partial T(s,t)}{\partial t} - 2q_i\cdot k\left(\frac{\partial T(s,t,\Delta_b)}{\partial\Delta_b}\right)_{\Delta_b=m^2}, \tag{27}$$
$$T_c = T(s,t) + 2R_q\cdot(\bar{q}_f-q_i)\frac{\partial T(s,t)}{\partial t} + 2\bar{p}_f\cdot k\left(\frac{\partial T(s,t,\Delta_c)}{\partial\Delta_c}\right)_{\Delta_c=M^2},$$

and

$$T_d = T(s,t) - 2(q_i+p_i)\cdot k\frac{\partial T(s,t)}{\partial s} + 2R_q\cdot(\bar{q}_f-q_i)\frac{\partial T(s,t)}{\partial t} - 2p_i\cdot k\left(\frac{\partial T(s,t,\Delta_d)}{\partial\Delta_d}\right)_{\Delta_d=M^2}.$$

Here $T(s,t)$ is the elastic scattering amplitude for A-B scattering, and again we have neglected terms of order k^2 and higher. We next expand the propagators and electromagnetic vertices. If we use Eqs. (24), we obtain the following expansions:

$$\frac{q_{f\mu}}{q_f\cdot k} = \frac{\bar{q}_{f\mu}}{\bar{q}_f\cdot k} + \frac{R_{q\mu}}{\bar{q}_f\cdot k} - \frac{(R_q\cdot k)\bar{q}_{f\mu}}{(\bar{q}_f\cdot k)^2}$$

and

$$\frac{p_{f\mu}}{p_f\cdot k} = \frac{\bar{p}_{f\mu}}{\bar{p}_f\cdot k} + \frac{R_{p\mu}}{\bar{p}_f\cdot k} - \frac{(R_p\cdot k)\bar{p}_{f\mu}}{(\bar{p}_f\cdot k)^2}\,. \tag{28}$$

Combining Eqs. (18), (27), and (28), we finally obtain a complete expansion of the external scattering amplitude $M_\mu^{(E)}$:

$$M_\mu^{(E)} = \frac{A_\mu^{(E)}}{k} + B_\mu^{(E)} + c_\mu^{(E)}k + \cdots\,, \tag{29}$$

where

$$A_\mu^{(E)}=\left(\frac{\overline{q}_{f\mu}}{\overline{q}_f\cdot\hat{k}}-\frac{q_{i\mu}}{q_i\cdot\hat{k}}+\frac{\overline{p}_{f\mu}}{\overline{p}_f\cdot\hat{k}}-\frac{p_{i\mu}}{p_i\cdot\hat{k}}\right)T(s,t)\,,$$

$$\begin{aligned}B_\mu^{(E)}=&\,2(q_i+p_i)\cdot\hat{k}\left(\frac{q_{i\mu}}{q_i\cdot\hat{k}}+\frac{p_{i\mu}}{p_i\cdot\hat{k}}\right)\frac{\partial T(s,t)}{\partial s}\\&+\left[2\hat{R}_p\cdot(\overline{p}_f-p_i)\left(\frac{\overline{q}_{f\mu}}{\overline{q}_f\cdot\hat{k}}-\frac{q_{i\mu}}{q_i\cdot\hat{k}}\right)+2\hat{R}_q\cdot(\overline{q}_f-q_i)\left(\frac{\overline{p}_{f\mu}}{\overline{p}_f\cdot\hat{k}}-\frac{p_{i\mu}}{p_i\cdot\hat{k}}\right)\right]\frac{\partial T(s,t)}{\partial t}\\&+\left[\frac{\hat{R}_{q\mu}}{\overline{q}_f\cdot\hat{k}}+\frac{\hat{R}_{p\mu}}{\overline{p}_f\cdot\hat{k}}-\frac{(\hat{R}_q\cdot\hat{k})\overline{q}_{f\mu}}{(\overline{q}_f\cdot\hat{k})^2}-\frac{(\hat{R}_p\cdot\hat{k})\overline{p}_{f\mu}}{(\overline{p}_f\cdot\hat{k})^2}\right]T(s,t)+2\overline{q}_{f\mu}\left(\frac{\partial T(s,t,\Delta_a)}{\partial\Delta_a}\right)_{\Delta_a=m^2}+2q_{i\mu}\left(\frac{\partial T(s,t,\Delta_b)}{\partial\Delta_b}\right)_{\Delta_b=m^2}\\&+2\overline{p}_{f\mu}\left(\frac{\partial T(s,t,\Delta_c)}{\partial\Delta_c}\right)_{\Delta_c=M^2}+2p_{i\mu}\left(\frac{\partial T(s,t,\Delta_d)}{\partial\Delta_d}\right)_{\Delta_d=M^2}\,,\end{aligned}\tag{30}$$

$$\hat{R}_{q\mu}=R_{q\mu}/k\,,$$

$$\hat{R}_{p\mu}=R_{p\mu}/k\,,$$

$$\hat{k}_\mu=k_\mu/k=(1,\,\sin\theta_\gamma\cos\phi_\gamma,\,\sin\theta_\gamma\sin\phi_\gamma,\,\cos\theta_\gamma)\,.$$

It is clear that $A_\mu^{(E)}$ and $B_\mu^{(E)}$ are independent of k.

In the past, the expansions given by Eq. (24) for $q_{f\mu}$ and $p_{f\mu}$ were not introduced. The final expression for the expansion of $M_\mu^{(E)}$ was written in terms of $q_{f\mu}$ and $p_{f\mu}$ without further expansion. Such an expansion is inconsistent, incomplete, and nonunique. Moreover, since $q_{f\mu}$ and $p_{f\mu}$ are still functions of k, all coefficients of the expansion are *not* independent of k. This incomplete expansion can be written as

$$M_\mu^{(E)}=\frac{\tilde{A}_\mu^{(E)}(k)}{k}+\tilde{B}_\mu^{(E)}(k)+\tilde{c}_\mu^{(E)}(k)k+\cdots\,,\tag{31}$$

where

$$\tilde{A}_\mu^{(E)}(k)=\frac{q_{f\mu}}{q_f\cdot\hat{k}}T(s,\tilde{t}_1,\Delta_a=m^2)-\frac{q_{i\mu}}{q_i\cdot\hat{k}}T(s,\tilde{t}_1,\Delta_b=m^2)+\frac{p_{f\mu}}{p_f\cdot\hat{k}}T(s,\tilde{t}_2,\Delta_c=M^2)-\frac{p_{i\mu}}{p_i\cdot\hat{k}}T(s,\tilde{t}_2,\Delta_d=M^2)\,,\tag{32}$$

$$\begin{aligned}\tilde{B}^{(E)}(k)=&\,2(q_i+p_i)\cdot\hat{k}\left[\frac{q_{i\mu}}{q_i\cdot\hat{k}}\frac{\partial T(s,\tilde{t}_1,\Delta_b=m^2)}{\partial s}+\frac{p_{i\mu}}{p_i\cdot\hat{k}}\frac{\partial T(s,\tilde{t}_2,\Delta_d=M^2)}{\partial s}\right]\\&+2q_{f\mu}\left(\frac{\partial T(s,\tilde{t}_1,\Delta_a)}{\partial\Delta_a}\right)_{\Delta_a=m^2}+2q_{i\mu}\left(\frac{\partial T(s,\tilde{t}_1,\Delta_b)}{\partial\Delta_b}\right)_{\Delta_b=m^2}\\&+2p_{f\mu}\left(\frac{\partial T(s,\tilde{t}_2,\Delta_c)}{\partial\Delta_c}\right)_{\Delta_c=M^2}+2p_{i\mu}\left(\frac{\partial T(s,\tilde{t}_2,\Delta_d)}{\partial\Delta_d}\right)_{\Delta_d=M^2}\,,\end{aligned}$$

$$\tilde{t}_1=(p_f-p_i)^2\,,\quad\tilde{t}_2=(q_f-q_i)^2\,.$$

From these expressions for $\tilde{A}_\mu^{(E)}(k)$ and $\tilde{B}_\mu^{(E)}(k)$, we can see that the expansion of $M_\mu^{(E)}$ is expressed in terms of T and its derivatives evaluated at two different angles ($\tilde{t}_1$ and $\tilde{t}_2$). Since $\tilde{t}_1$ and $\tilde{t}_2$ are functions of k, they can be expressed in terms of $p_i\cdot k$, $p_f\cdot k$, $q_i\cdot k$, and $q_f\cdot k$. This means that $\tilde{t}_1$ and $\tilde{t}_2$ are not independent of the off-mass-shell effects. (Note that $p_i\cdot k=\frac{1}{2}M^2-\frac{1}{2}\Delta_d$, $p_f\cdot k=\frac{1}{2}\Delta_c-\frac{1}{2}M^2$, $q_i\cdot k=\frac{1}{2}m^2-\frac{1}{2}\Delta_b$, and $q_f\cdot k=\frac{1}{2}\Delta_a-\frac{1}{2}m^2$.) We can therefore conclude that *all coefficients, including $\tilde{A}_\mu^{(E)}(k)$ and $\tilde{B}_\mu^{(E)}(k)$, which are functions of k must necessarily contain off-mass-shell effects.*

The ambiguity of the expansion given by Eq. (31) arises from the fact that T and its derivatives may be evaluated at many different energies if p_f^μ and q_f^μ are not expanded since we have many ways of choosing s_i ($i=a,b,c,d$). For example, we can use $p_i\cdot q_i+p_f\cdot q_f$ to replace s and obtain different results for $\tilde{A}_\mu^{(E)}(k)$ and $\tilde{B}_\mu^{(E)}(k)$. To see this, we note that $p_i\cdot q_i + p_f\cdot q_f \neq p_i\cdot q_i + \bar{p}_f\cdot\bar{q}_f = s - m^2 - M^2$. This implies that the energy at which T and its derivatives are evaluated will be shifted from s by an amount which depends on k.

(*ii*) *The external amplitude for H-type cross sections.* The independent kinematical variables for this class of cross sections are θ_q, ϕ_q, θ_p, ϕ_p, and X. Here, X can be chosen from $p_f, q_f, \theta_\gamma, \phi_\gamma, k, \ldots$. For simplicity, we shall choose X to be θ_γ for our discussion. In this case, four dependent variables are p_f, q_f, ϕ_γ, and k. As we have already pointed out in the preceding section, the law of conservation of energy-momentum would be violated if we let k approach zero arbitrarily or simply set k equal to zero. Therefore, the soft-photon limits $\lim_{k\to 0} s_j$ $(j=b,d)$, $\lim_{k\to 0} t_i$, and $\lim_{k\to 0}\Delta_i$ $(i=a,b,c,d)$ cannot be obtained, and the soft-photon expansion for $M_\mu^{(E)}$ around $k=0$ is not allowed under the restriction of energy-momentum conservation. In the past, however, an expansion which was similar to Eq. (31) with $\tilde{A}_\mu^{(E)}(k)$ and $\tilde{B}_\mu^{(E)}(k)$ defined by Eq. (32) was obtained. Such expansion is not the soft-photon expansion since all coefficients of the expansion are still functions of k. The dependence of these coefficients on k can be understood from the following argument: All coefficients can be determined by T and its derivatives, which are evaluated either at s and $\tilde{t}_1$ or at s and $\tilde{t}_2$. If we apply energy-momentum conservation, Eq. (5), we obtain the following relations:

$$\begin{aligned}\tilde{t}_1 &= \tilde{t}_2 - 2(q_i - q_f)\cdot k \\ &= \tilde{t}_2 + \Delta_b + \Delta_a - 2m^2 \end{aligned} \tag{33}$$

or

$$\begin{aligned}\tilde{t}_2 &= \tilde{t}_1 - 2(p_i - p_f)\cdot k \\ &= \tilde{t}_1 + \Delta_c + \Delta_d - 2M^2 . \end{aligned}$$

Since $k \neq 0$, Eq. (33) shows that $\tilde{t}_1$ and/or $\tilde{t}_2$ must depend upon k. This implies that all coefficients of the expansion of $M_\mu^{(E)}$ are not independent of k and they must necessarily contain off-mass-shell effects.

The internal amplitude $M_\mu^{(I)}$ can be obtained from $M_\mu^{(E)}$ by imposing the gauge-invariance (current-conservation) condition

$$\begin{aligned} M_\mu k^\mu &= (M_\mu^{(E)} + M_\mu^{(I)})k^\mu \\ &= 0 . \end{aligned} \tag{34}$$

The explicit expression for the leading term of $M_\mu^{(I)}$ can be obtained from Eq. (34). Let us again consider the following two cases:

(*i*) *The internal amplitude for R-type cross sections.* If we assume that M_μ is analytic at $k^\mu = 0$, we can differentiate Eq. (34) with respect to k^μ by keeping all other variables as constants and setting $k^\mu = 0$. We find

$$M_\mu^{(I)} = -\frac{\partial}{\partial k^\mu}(M_\nu^{(E)}k^\nu)\bigg|_{k^\mu=0} + O(k) . \tag{35}$$

Using the external amplitude $M_\mu^{(E)}$ given by Eqs. (29) and (30), we can write $M_\mu^{(I)}$ in the form

$$M_\mu^{(I)} = B_\mu^{(I)} + c_\mu^{(I)}k + \cdots , \tag{36}$$

where

$$\begin{aligned} B_\mu^{(I)} = {} & -4(q_i + p_i)_\mu \frac{\partial T(s,t)}{\partial s} - 2\bar{q}_{f\mu}\left(\frac{\partial T(s,t,\Delta_a)}{\partial \Delta_a}\right)_{\Delta_a = m^2} \\ & - 2q_{i\mu}\left(\frac{\partial T(s,t,\Delta_b)}{\partial \Delta_b}\right)_{\Delta_b = m^2} - 2\bar{p}_{f\mu}\left(\frac{\partial T(s,t,\Delta_c)}{\partial \Delta_c}\right)_{\Delta_c = M^2} - 2p_{i\mu}\left(\frac{\partial T(s,t,\Delta_d)}{\partial \Delta_d}\right)_{\Delta_d = M^2} . \end{aligned} \tag{37}$$

(*ii*) *The internal amplitude for H-type cross sections.* Appropriate care must be taken for this case since $k \neq 0$. Let the range of k be $k_{\min} \leq k < k_{\max}$ with $k_{\min} > 0$, and we assume that M_μ is analytic in this range. Then, from Eq. (34), we have

$$M_\mu^{(I)} = -\frac{\partial}{\partial k^\mu}(M_\nu^{(E)}k^\nu) - \left(\frac{\partial M_\nu^{(I)}}{\partial k^\mu}\right)k^\nu . \tag{38}$$

If Eqs. (31) and (32) are used, we can write $M_\mu^{(I)}$ as

$$M_\mu^{(I)} = \tilde{B}_\mu^{(I)}(k) + \tilde{c}_\mu^{(I)}(k)k + \cdots , \tag{39}$$

where

$$\tilde{B}_\mu^{(I)}(k) = -2(q_i+p_i)_\mu \left[\frac{\partial T(s,\tilde{t}_1,\Delta_b=m^2)}{\partial s} + \frac{\partial T(s,\tilde{t}_2,\Delta_d=M^2)}{\partial s}\right] - 2q_{f\mu}\left(\frac{\partial T(s,\tilde{t}_1,\Delta_a)}{\partial \Delta_a}\right)_{\Delta_a=m^2}$$
$$-2q_{i\mu}\left(\frac{\partial T(s,\tilde{t}_1,\Delta_b)}{\partial \Delta_b}\right)_{\Delta_b=m^2} - 2p_{f\mu}\left(\frac{\partial T(s,\tilde{t}_2,\Delta_c)}{\partial \Delta_c}\right)_{\Delta_c=M^2} - 2p_{i\mu}\left(\frac{\partial T(s,\tilde{t}_2,\Delta_d)}{\partial \Delta_d}\right)_{\Delta_d=M^2}. \tag{40}$$

The results obtained in Eqs. (39) and (40) apply also to R-type cross sections if the expansions for q_f^μ and p_f^μ, Eqs. (24), are not introduced.

Finally, the total bremsstrahlung amplitude M_μ can be obtained from Eq. (17). Let us consider the following cases separately:

(*i*) *The total bremsstrahlung amplitude for R-type cross sections.* The external amplitude $M_\mu^{(E)}$ and the internal amplitude $M_\mu^{(I)}$ are given by Eqs. (29) and (36), respectively. Combining these two equations, we obtain

$$M_\mu = \frac{A_\mu}{k} + B_\mu + c_\mu k + \cdots, \tag{41}$$

where

$$A_\mu = A_\mu^{(E)}$$
$$B_\mu = \left[2(q_i+p_i)\cdot\hat{k}\left(\frac{q_{i\mu}}{q_i\cdot\hat{k}} + \frac{p_{i\mu}}{p_i\cdot\hat{k}}\right) - 4(q_i+p_i)_\mu\right]\frac{\partial T(s,t)}{\partial s}$$
$$+\left[2(\bar{p}_f-p_i)\cdot\hat{R}_p\left(\frac{\bar{q}_{f\mu}}{\bar{q}_f\cdot\hat{k}} - \frac{q_{i\mu}}{q_i\cdot\hat{k}}\right) + 2(\bar{q}_f-q_i)\cdot\hat{R}_q\left(\frac{\bar{p}_{f\mu}}{\bar{p}_f\cdot\hat{k}} - \frac{p_{i\mu}}{p_i\cdot\hat{k}}\right)\right]\frac{\partial T(s,t)}{\partial t}$$
$$+\left[\frac{\hat{R}_{q\mu}}{\bar{q}_f\cdot\hat{k}} + \frac{\hat{R}_{p\mu}}{\bar{p}_f\cdot\hat{k}} - \frac{(\hat{R}_q\cdot\hat{k})\bar{q}_{f\mu}}{(\bar{q}_f\cdot\hat{k})^2} - \frac{(\hat{R}_p\cdot\hat{k})\bar{p}_{f\mu}}{(\bar{p}_f\cdot\hat{k})^2}\right]T(s,t). \tag{42}$$

We see that A_μ and B_μ are independent of the off-mass-shell effects and they can be obtained from the two-body elastic scattering amplitude $T(s,t)$ and its derivatives, $\partial T(s,t)/\partial s$ and $\partial T(s,t)/\partial t$. Therefore, the soft-photon theorem can be derived for R-type cross sections.

(*ii*) *The total bremsstrahlung amplitude for H-type cross sections.* The external amplitude and the internal amplitude are given by Eqs. (31) and (39), respectively. The total amplitude is obtained by combining Eqs. (31) and (39). We have

$$M_\mu = \frac{\tilde{A}_\mu(k)}{k} + \tilde{B}_\mu(k) + \tilde{c}_\mu(k)k + \cdots, \tag{43}$$

where

$$\tilde{A}_\mu(k) = \tilde{A}_\mu^{(E)}(k),$$
$$\tilde{B}_\mu(k) = \left[2(q_i+p_i)\cdot\hat{k}\frac{q_{i\mu}}{q_i\cdot\hat{k}} - 2(q_i+p_i)_\mu\right]\frac{\partial T(s,\tilde{t}_1,\Delta_b=m^2)}{\partial s} + \left[2(q_i+p_i)\cdot\hat{k}\frac{p_{i\mu}}{p_i\cdot\hat{k}} - 2(q_i+p_i)_\mu\right]\frac{\partial T(s,\tilde{t}_2,\Delta_d=M^2)}{\partial s}. \tag{44}$$

From the expressions of $\tilde{A}_\mu(k)$ and $\tilde{B}_\mu(k)$ given by Eq. (44) and the relationship between $\tilde{t}_1$ and $\tilde{t}_2$ given by Eq. (33), we find that although the off-shell derivatives are canceled out precisely, $\tilde{A}_\mu(k)$ and $\tilde{B}_\mu(k)$ are not free of off-shell parameters Δ_a, Δ_b, Δ_c, and Δ_d because of k dependence. Moreover, $\tilde{A}_\mu(k)$ and $\tilde{B}_\mu(k)$ are not obtained from the real two-body elastic scattering amplitude, $T(s,t)$, and its derivatives, $\partial T(s,t)/\partial s$ and $\partial T(s,t)/\partial t$, but rather from two off-shell amplitudes, $T(s,\tilde{t}_1)$ and $T(s,\tilde{t}_2)$, and their derivatives, $\partial T(s,\tilde{t}_1)/\partial s$ and $\partial T(s,\tilde{t}_2)/\partial s$. This means that $\tilde{A}_\mu(k)$ and $\tilde{B}_\mu(k)$ depend on two different scattering angles which are functions of k. We therefore conclude that the soft-photon theorem cannot be derived for H-type cross sections. This conclusion applies also for R-type cross sections if the expansions for $q_{f\mu}$ and $p_{f\mu}$, Eq. (24), are not introduced.

IV. BREMSSTRAHLUNG CROSS SECTION

The bremsstrahlung cross section can be expressed in the form

$$d^9\sigma=\frac{e^2}{4[(p_i\cdot q_i)^2-m^2M^2]^{1/2}}\left[\sum_{\text{pol}}(M_\mu\epsilon^\mu)^\dagger(M_\mu\epsilon^\mu)\right](2\pi)^4\delta^4(p_i+q_i-p_f-q_f-k)\frac{d^3q_f}{(2\pi)^3 2E_q}\frac{d^3p_f}{(2\pi)^3 2E_p}\frac{d^3k}{(2\pi)^3 2k}\ , \quad (45)$$

where ϵ^μ is the photon polarization and the summation sign indicates a sum over the photon polarization.

(i) For R-type cross sections, the five independent kinematical variables are θ_γ, ϕ_γ, k, y, and z. These cross sections can be obtained by integrating Eq. (45) over four dependent variables:

$$\frac{d^5\sigma}{d\Omega_\gamma dk dy dz}=G\int\left[\sum_{\text{pol}}(M_\mu\epsilon^\mu)^\dagger(M_\mu\epsilon^\mu)\right]\delta^4(p_i+q_i-p_f-q_f-k)\left(\frac{k}{E_qE_p}\right)\frac{d^3q_fd^3p_f}{dydz}\ , \quad (46)$$

where

$$G=\frac{e^2}{32[(p_i\cdot q_i)^2-m^2M^2]^{1/2}}\left(\frac{1}{2\pi}\right)^5 .$$

This integral can be carried out to yield nonvanishing cross sections for the whole range of k, $0\leq k\leq k_{\max}$, since solutions for the four dependent variables, which satisfy $p_i^\mu+q_i^\mu-p_f^\mu-q_f^\mu-k^\mu=0$, exist for all k. If the result were expanded in powers of k, we would obtain Eq. (1a), the soft-photon expansion for R-type cross sections.

As an example, let us choose $y=-\cos\theta_q$ and $z=\phi_q$. In this case, Eq. (46) becomes

$$\frac{d^5\sigma}{d\Omega_\gamma d\Omega_q dk}=kGNF \quad (47a)$$

$$=\frac{\sigma_{-1}^R}{k}+\sigma_0^R+\sigma_1^Rk+\cdots . \quad (47b)$$

Here $N=\sum_{\text{pol}}(M_\mu\epsilon^\mu)^\dagger(M_\mu\epsilon^\mu)$ and F is the phase-space factor. If we use the relation

$$\sum_{\text{pol}}(M_\mu\epsilon^\mu)^\dagger(M_\mu\epsilon^\mu)=-M_\mu^\dagger M^\mu \quad (48)$$

and the expression for M_μ given by Eq. (41), we obtain

$$N=-|A|^2/k^2-(A^\dagger\cdot B+B^\dagger\cdot A)/k-(|B|^2+A^\dagger\cdot C+C^\dagger\cdot A)+\cdots . \quad (49)$$

The phase-space factor F is derived in the Appendix. From Eq. (A3) of the Appendix, we have

$$F=F_0+F_1 . \quad (50)$$

The expressions for σ_{-1}^R and σ_0^R can be obtained if we combine Eqs. (47), (49), and (50):

$$\sigma_{-1}^R=-G|A|^2F_0 ,$$
$$\sigma_0^R=-G\left[\frac{|A|^2F_1}{k}+(A^\dagger\cdot B+B^\dagger\cdot A)F_0\right] , \quad (51)$$

which shows that the first two terms of the expansion of R-type cross sections in powers of k are independent of the off-mass-shell effects and that they can be obtained from the amplitude of the corresponding nonradiative process and its derivatives. This confirms our conclusion that the soft-photon theorem can be derived for R-type cross sections.

(ii) For H-type cross sections, the five independent kinematical variables are θ_q, ϕ_q, θ_p, ϕ_p, and x. These cross sections can be written as

$$\frac{d^5\sigma}{d\Omega_qd\Omega_pdx}=G\int\left[\sum_{\text{pol}}(M_\mu\epsilon^\mu)^\dagger(M_\mu\epsilon^\mu)\right]\times\delta^4(p_i+q_i-p_f-q_f-k)\times\left(\frac{q_f^2p_f^2}{E_qE_pk}\right)\frac{dq_fdp_fd^3k}{dx} . \quad (52)$$

We already mentioned several times that energy-momentum conservation would be violated if we were to set k equal to zero for H-type cross sections. This implies that $p_i^\mu+q_i^\mu-p_f^\mu-q_f^\mu-k^\mu\neq0$ for $k=0$. Since $\delta^4(p_i+q_i-p_f-q_f-k)$ must vanish if $p_i^\mu+q_i^\mu-p_f^\mu-q_f^\mu-k^\mu\neq0$, all H-type cross sections must be zero at $k=0$. Thus, in general, if the range of k is $k_{\min}\leq k\leq k_{\max}$ with $k_{\min}>0$, then the cross section will have nonvanishing value only in this range and it should be zero for $0\leq k<k_{\min}$ and $k>k_{\max}$. This is why a physically meaningful soft-photon expansion around $k=0$ for H-type cross sections is not allowed. In the range $k_{\min}\leq k\leq k_{\max}$, however, we could use the expression of M_μ given by Eq. (43) to obtain Eq. (2a) for H-type cross sections. All coefficients of the expansion are still functions of k, i.e., they are not independent of the off-mass-shell effects. This confirms our conclusion that the soft-photon theorem cannot be derived for H-type cross sections.

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APPENDIX A

(1) The expressions for R_q^μ and R_p^μ. We shall derive the expressions for R_q^μ and R_p^μ when independent variables are θ_γ, ϕ_γ, θ_q, ϕ_q, and k. These expressions have already been obtained in Ref. 5 for the coplanar case. They can be written as

$$R_q^\mu \equiv k\left(\frac{dq_f^\mu}{dk}\right)_{k=0} = [m^2 p_i^\mu - (p_i\cdot\bar{q}_f)\bar{q}_f^\mu](\bar{p}_f\cdot k)/N_q\,,$$

$$N_q = (p_i\cdot\bar{q}_f)(\bar{p}_f\cdot\bar{q}_f) - m^2(p_i\cdot\bar{p}_f)\,. \qquad \text{(A1)}$$

$$R_p^\mu \equiv k\left(\frac{dp_f^\mu}{dk}\right)_{k=0} = -R_q^\mu - k^\mu\,.$$

Since these expressions are written in terms of invariants, they are also valid for the noncoplanar case.

(2) *Phase-space factor, F.* F can be written as

$$F = \int \frac{q_f^2}{E_q E_p}\,\delta^4(p_i + q_i - p_f - q_f - k)\,dq_f\,d^3p_f = \frac{[(p_i\cdot q_f)^2 - m^2M^2]^{3/2}}{M^2[(p_i\cdot q_f)(p_f\cdot q_f) - m^2(p_i\cdot p_f)]}\,. \qquad \text{(A2)}$$

If we use $q_f^\mu = \bar{q}_f^\mu + R_q^\mu$ and $p_f^\mu = \bar{p}_f^\mu + R_p^\mu$ to expand F, we obtain

$$F = F_0 + F_1 + O(k^2)\,, \qquad \text{(A3)}$$

where

$$F_0 = \frac{[(p_i\cdot\bar{q}_f)^2 - m^2M^2]^{3/2}}{M^2[(p_i\cdot\bar{q}_f)(\bar{p}_f\cdot\bar{q}_f) - m^2(p_i\cdot\bar{p}_f)]}$$

and

$$F_1 = F_0\left[\frac{3(p_i\cdot\bar{q}_f)(p_i\cdot R_q)}{(p_i\cdot\bar{q}_f)^2 - m^2M^2} - \frac{(p_i\cdot\bar{q}_f)(\bar{q}_f\cdot R_p + \bar{p}_f\cdot R_q) + (\bar{p}_f\cdot\bar{q}_f)(p_i\cdot R_q) - m^2 p_i\cdot R_p}{(p_i\cdot\bar{q}_f)(\bar{p}_f\cdot\bar{q}_f) - m^2(p_i\cdot\bar{p}_f)}\right].$$

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