PHYSICAL REVIEW D VOLUME 18, NUMBER 8 15 OCTOBER 1978

Fisk-Tait equation for spin-3/2 particles

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(Received 13 May 1976; revised manuscript received 22 August 1977)

It is shown that the recently proposed Fisk-Tait spin-3/2 equation which remains causal even when interactions are introduced is a "barnacled" wave equation, and when the barnacles are eliminated the resulting equation is the same as Hurley's "doubled" spin-3/2 theory with $4(2s+1)$ independent components. The Fisk-Tait equation has therefore the undesirable features of Hurley's equation, viz., parity doubling and negative energy.

I. INTRODUCTION

It is well known that the relativistic field description of particles with spin greater than unity in interaction with prescribed external fields is beset with difficulties both in the second-quantized formulation[1] and at the more basic c-number level itself.[2] The pathologies of high-spin theories came to light for the first time in the study of the familiar Rarita-Schwinger equation for spin-$\frac{3}{2}$ particles[3] coupled to external fields: The anticommutators depend on the external field and are not always positive,[1] the propagation of the (c-number) field is noncausal,[2] and in a constant magnetic field there occur "normal modes" whose frequencies cease to be real when the magnitude of the external field exceeds some critical value.[4] One could hope to avoid these pathologies by adopting a different wave equation for the description of spin-$\frac{3}{2}$ particles. The restriction to spin $s=\frac{3}{2}$ in the scheme of Hurley[5] for "doubled" spin-s relativistic wave equations provided one alternative. The word "doubled" refers to the fact that these equations had $4(2s+1)$ independent components as compared to the usual $2(2s+1)$ independent components. However, the Hurley equation of this form avoided the Velo-Zwanziger (VZ) pathology at the cost of having parity doubling and negative energy in the theory.[6] Thus the search for another equation becomes worthwhile. Recently Fisk and Tait proposed a spin-$\frac{3}{2}$ relativistic wave equation which was shown to be free from the VZ pathology[7,8] and had normal modes with real frequencies in a constant magnetic field.[9] Though it has 16 independent components corresponding to two spin-$\frac{3}{2}$ particles, the representation of the Lorentz group under which the 24-component tensor-spinor of the Fisk-Tait equation transformed was different from the representation under which the Hurley equation transformed.

We will not directly show that the Fisk-Tait equation has the same difficulties as the doubled Hurley equation; the statement proven in this paper is much stronger than that. We will show that the Fisk-Tait equation has "barnacles,"[10] and that when these are removed the resulting equation is exactly the Hurley equation for spin $\frac{3}{2}$ with $4(2s+1)=16$ independent components. This means that the two theories are equivalent, and all the difficulties, as well as the advantages, of the Hurley equation will persist in the Fisk-Tait equation. Alternatively the Fisk-Tait equation is a more or less trivial extension of the Hurley equation.

Before we start, a brief word is necessary to give a meaning to the word "barnacled." This structure in the Fisk-Tait equation occurs in a very simple way, so we will give a simple definition of barnacled equations. The most general definition that preserves these features is given elsewhere.[11,12] Suppose we have a relativistic wave equation $(-i\Gamma\cdot\partial+m)\psi=0$ transforming under a representation $T(\Lambda)$ of SL(2, C). Suppose further that $T(\Lambda)$ can be decomposed into two pieces, $T(\Lambda)=T_1(\Lambda)\oplus T_2(\Lambda)$, where both $T_1(\Lambda)$ and $T_2(\Lambda)$ are representations of SL(2, C). If with some such splitting the Γ_μ (or actually only Γ_0) can be rewritten as

$$\begin{array}{cc} & \begin{array}{cc} T_1(\Lambda) & T_2(\Lambda) \end{array} \\ \Gamma_\mu = & \begin{pmatrix} B_\mu & X_\mu \\ Y_\mu & 0 \end{pmatrix} \begin{array}{c} T_1(\Lambda) \\ T_2(\Lambda) \end{array}, \end{array}$$

where either $X_\mu=0$ or $Y_\mu=0$ (or both are zero) then the equation $(-i\Gamma\cdot\partial+m)\psi=0$ is called barnacled of type I or type II (or simultaneous), respectively. Splittings of $T(\Lambda)$ always exist, however, they should exist in such a way as to induce a splitting of Γ_μ, of the above types, before the wave equation can be called barnacled. When this happens the wave equation $(-i\Gamma\cdot\partial+m)\psi=0$ and the smaller equation $(-i\beta\cdot\partial+m)\phi=0$ lead to practically identical theories.

In Sec. II we rewrite the Fisk-Tait equation in the standard form $(-i\Gamma\cdot\partial+m)\psi=0$ and determine the properties of Γ_0. We then show in Sec. III that a first-order wave equation with these properties for Γ_0, and having a wave function ψ transforming as in the Fisk-Tait equation, is a barnacled "doubled" Hurley equation. Thus the Fisk-Tait equation is equivalent to the Hurley equation.

II. FISK-TAIT EQUATION

The wave function is a 24-component antisymmetric tensor-spinor $\phi_\alpha^{\mu\nu} = -\phi_\alpha^{\mu\nu}$ and transforms according to the representation

$$T(\Lambda) = (0, \tfrac{3}{2}) \oplus (1, \tfrac{1}{2}) \oplus (0, \tfrac{1}{2}) \oplus (\tfrac{3}{2}, 0) \oplus (\tfrac{1}{2}, 1) \oplus (\tfrac{1}{2}, 0) \tag{1}$$

of the Lorentz group. The equation of motion is given by

$$-\tfrac{4}{3}\gamma \cdot p\phi^{\mu\nu} - \tfrac{1}{3}\gamma \cdot p(\gamma^\mu\gamma_\rho g_\sigma^\nu - \gamma^\nu\gamma_\rho g_\sigma^\mu)\phi^{\sigma\rho} + \tfrac{1}{3}(\gamma^\mu p_\sigma g_\rho^\nu - \gamma^\nu p_\sigma g_\rho^\mu - p^\mu\gamma_\sigma g_\rho^\nu + p^\nu\gamma_\sigma g_\rho^\mu)\phi^{\sigma\rho} + m\phi^{\mu\nu} = 0\,. \tag{2}$$

Arranging the elements of $\phi^{\mu\nu}$ in a column as

$$\psi = \mathrm{col}(\phi^{01}, \phi^{02}, \phi^{03}, \phi^{23}, \phi^{31}, \phi^{12})\,, \tag{3}$$

it is easily deduced that (2) can be put in the form

$$(\Gamma \cdot p - m)\psi = 0\,, \tag{4a}$$

where Γ_0 is given by

$$\Gamma_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \tag{4b}$$

with

$$A = \begin{bmatrix} \tfrac{2}{3}\gamma^0 & \tfrac{1}{3}\gamma^0\gamma^1\gamma^2 & \tfrac{1}{3}\gamma^0\gamma^1\gamma^3 \\ \tfrac{1}{3}\gamma^0\gamma^2\gamma^1 & \tfrac{2}{3}\gamma^0 & \tfrac{1}{3}\gamma^0\gamma^2\gamma^3 \\ \tfrac{1}{3}\gamma^0\gamma^3\gamma^1 & \tfrac{1}{3}\gamma^0\gamma^3\gamma^2 & \tfrac{2}{3}\gamma^0 \end{bmatrix}\,. \tag{4c}$$

It is immediately obvious that

$$\Gamma_0^\dagger = \Gamma_0\,, \qquad \Gamma_0^3 = \Gamma_0\,, \tag{4d}$$

and Γ_0 has eigenvalues $+1$, -1, and 0 with each occurring eight times. In fact $A = \gamma^0 P_{3/2}$, where $P_{3/2}$ is the projection matrix to the spin-$\frac{3}{2}$ part of the wave function and so the nonzero eigenvalues of Γ_0 correspond to spin-$\frac{3}{2}$ particles.

III. REDUCTION OF THE FISK-TAIT EQUATION

We will now show that any wave equation of the form (4a) consistent with the properties of the Fisk-Tait equation, as expressed and deduced in the last section, will be a barnacled "doubled" Hurley equation for spin-$\frac{3}{2}$. Hence the Fisk-Tait equation will be so related to the Hurley equation.

Consider the most general manifestly Lorentz-covariant wave equation transforming under (1) and satisfying the following two conditions:

(i) The wave equation

$$(-i\Gamma \cdot \partial + m)\psi(x) = 0 \tag{5}$$

describes a unique mass m, and spin $\frac{3}{2}$ with $4(2s+1) = 16$ independent components.

(ii) The Γ_μ satisfy the algebra

$$\sum_{\sigma(\mu\nu\lambda)} (\Gamma_\mu\Gamma_\nu - g_{\mu\nu})\Gamma_\lambda = 0$$

or (6)

$$\Gamma_0(\Gamma_0^2 - I) = 0\,.$$

This is one of the two conditions of (4d). We will consider only the Γ_0 matrix since the Γ_i matrices are completely fixed once Γ_0 and $T(\Lambda)$ are specified[13,14]:

$$\Gamma_0 = \begin{array}{c} \begin{array}{cccccc} (0,\tfrac{3}{2}) & (1,\tfrac{1}{2}) & (0,\tfrac{1}{2}) & (\tfrac{3}{2},0) & (\tfrac{1}{2},1) & (\tfrac{1}{2},0) \end{array} \\ \begin{array}{cccccccc} 3/2 & 3/2 & 1/2 & 1/2 & 3/2 & 3/2 & 1/2 & 1/2 \end{array} \\ \left[\begin{array}{cccccccc} & & & & & a & & \\ & & & & b & c & & \\ & & & & & & -\tfrac{1}{2}c & d \\ & & & & & & g & f \\ & b' & & & & & & \\ a' & c' & & & & & & \\ & & -\tfrac{1}{2}c' & g' & & & & \\ & & d' & f' & & & & \end{array}\right] \end{array} \begin{array}{l} \\ \\ 3/2\,(0,\tfrac{3}{2}) \\ 3/2\,(1,\tfrac{1}{2}) \\ 1/2\,(1,\tfrac{1}{2}) \\ 1/2\,(0,\tfrac{1}{2}) \\ 3/2\,(\tfrac{3}{2},0) \\ 3/2\,(\tfrac{1}{2},1) \\ 1/2\,(\tfrac{1}{2},1) \\ 1/2\,(\tfrac{1}{2},0) \end{array}\,. \tag{7}$$

Equation (7) represents the most general Γ_0 that admits an equation (5) transforming under (1). The complex numbers a, b, c, d, f, and g are not *a priori* related to a', b', c', d', f', and g'.

Now, Γ_0 can be put in another basis and represented as

$$\Gamma_0=\begin{pmatrix}\Gamma_0^{3/2} & \\ & \Gamma_0^{1/2}\end{pmatrix}, \tag{8}$$

where $\Gamma_0^{3/2}$ ($\Gamma_0^{1/2}$) represents the connection of only the spin-$\frac{3}{2}$ ($\frac{1}{2}$) pieces in Γ_0:

$$\Gamma_0^{3/2}=\begin{bmatrix} & & & a\\ & & b & c\\ & b' & & \\ a' & c' & & \end{bmatrix}, \tag{9a}$$

$$\Gamma_0^{1/2}=\begin{bmatrix} & & -\frac{1}{2}c & d\\ & & g & f\\ -\frac{1}{2}c' & g' & & \\ d' & f' & & \end{bmatrix}. \tag{9b}$$

According to Eq. (6), we obtain

$$(\Gamma_0^{3/2})^3=(\Gamma_0^{3/2}), \tag{10a}$$

$$(\Gamma_0^{1/2})[(\Gamma_0^{1/2})^2-I]=0. \tag{10b}$$

In order to have unique spin $\frac{3}{2}$, $\Gamma_0^{1/2}$ has to be made nilpotent, but then

$$\mathrm{Det}[(\Gamma_0^{1/2})^2-I]=\pm 1\neq 0, \tag{11}$$

so $[(\Gamma_0^{1/2})^2-I]$ can be inverted and (10b) implies that $\Gamma_0^{1/2}=0$.

The numbers c, d, g, f, c', d', g', f' are all zero. This means that the $(1,\frac{1}{2})$ and $(\frac{1}{2},1)$ representations decouple and the $(0,\frac{1}{2})\oplus(\frac{1}{2},0)$ appear as simultaneous barnacles,[14] and will therefore not affect the resulting theory in either the free-field case or in standard interactions with an external field.[11]

Since there are $4(2s+1)$ independent components, the numbers a, b, a', b' are all nonzero. Equation (10a) requires that $aa'=bb'=1$. Without any significant loss of generality a, a', b, b' can be taken as real and all equal to 1. Alternatively, one notes that the most general $\Gamma_0^{3/2}$ is

$$\Gamma_0^{3/2}=\begin{bmatrix} 0 & 0 & 0 & a\\ 0 & 0 & b & 0\\ 0 & 1/b & 0 & 0\\ 1/a & 0 & 0 & 0\end{bmatrix}. \tag{12}$$

The Γ_0 of Eq. (7) with these considerations now becomes, by putting the different irreducible representations of SL(2, C) occurring in $T(\Lambda)$ in a slightly different order,

$$\begin{array}{cc} & \begin{array}{cccccccc}\frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{array} \\ \Gamma_0= & \left[\begin{array}{cccccccc} & & & & a & & & \\ & & & b & & & & \\ & & & & & & & \\ & 1/b & & & & & & \\ 1/a & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{array}\right]\begin{array}{l}\frac{3}{2}(0,\frac{3}{2})\\ \frac{3}{2}(1,\frac{1}{2})\\ \frac{1}{2}\\ \frac{3}{2}(\frac{3}{2},0)\\ \frac{3}{2}(\frac{1}{2},1)\\ \frac{1}{2}\\ \frac{1}{2}(0,\frac{1}{2})\\ \frac{1}{2}(\frac{1}{2},0)\end{array}\end{array}. \tag{13}$$

Now, there exists a nonsingular linear transformation V, such that $[V, T(\Lambda)]=0$, for all $\Lambda\in\mathrm{SL}(2,C)$ and

$$V\Gamma_0V^{-1}=\Gamma_0',$$

$$V=\begin{bmatrix}1 & & & & & & & \\ & 1/b & & & & & & \\ & & 1/b & & & & & \\ & & & 1 & & & & \\ & & & & a & & & \\ & & & & & a & & \\ & & & & & & 1 & \\ & & & & & & & 1\end{bmatrix}, \tag{14}$$

$$\Gamma_0'=\begin{bmatrix} & & & & 1\\ & & & 1 & \\ & & & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}, \tag{15}$$

$$\Gamma_0'=\begin{pmatrix}\beta_0 & 0\\ 0 & 0\end{pmatrix}\begin{matrix}T_1(\Lambda)\\ T_2(\Lambda)\end{matrix}, \tag{16}$$

where in the last equation $T_1(\Lambda)$ and $T_2(\Lambda)$ are defined by

$$T_1(\Lambda)=(0,\tfrac{3}{2})\oplus(1,\tfrac{1}{2})\oplus(\tfrac{1}{2},1)\oplus(\tfrac{3}{2},0),$$

$$T_2(\Lambda)=(0,\tfrac{1}{2})\oplus(\tfrac{1}{2},0),$$

$$T(\Lambda)=T_1(\Lambda)\oplus T_2(\Lambda).$$

The existence of such a V establishes that all theories satisfying conditions (i) and (ii) with $T(\Lambda)$ as in (1) can be written as follows, using the Γ_0' of Eq. (15):

$$(-i\Gamma'\cdot\partial+m)\psi(x)=0$$
$$=\left[-i\begin{pmatrix}\beta\cdot\partial & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}m & 0\\ 0 & m\end{pmatrix}\right]\begin{bmatrix}\phi\ (x)\\ \omega\ (x)\end{bmatrix}$$
$$=0 \qquad (17)$$

or

$$(-i\beta\cdot\partial+m)\phi(x)=0\,,\quad \omega(x)=0\,. \qquad (18)$$

The β in Eq. (18) comes from the β_0 in Eq. (16), hence Eq. (18) is the Hurley equation. We are saying that the Fisk-Tait equation can be put into a form where a part $\omega(x)$, of the original wave function $\psi(x)$ is identically zero and the remainder of the wave function satisfies the smaller Hurley equation. This is a Lorentz-covariant decomposition.

In standard external-field interactions this structure will persist, and the two theories will be identical. In general, consider the standard external-field interactions of Eq. (17) as

$$[-i\Gamma'\cdot\partial+m+B(x)]\Psi(x)=0\,,\quad B(x)=B(\Gamma',f) \qquad (19)$$

where $B(\Gamma',f)$ is a notation signifying that $B(x)$ is constructed from products of Γ_μ''s contracted over external potentials f. Due to the nature of Γ_μ' in Eq. (17) one can see that an interaction $B(\Gamma',f)$ of Eq. (19) is

$$[-i\Gamma'\circ\partial+m+B(\Gamma',f)]\Psi(x)=0\,,$$
$$\left[-i\begin{pmatrix}\beta\cdot\partial & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}m & 0\\ 0 & m\end{pmatrix}+\begin{pmatrix}b(\beta,f) & 0\\ 0 & 0\end{pmatrix}\right]\begin{bmatrix}\Psi\ (x)\\ \Omega\ (x)\end{bmatrix}=0\,, \qquad (20)$$

or

$$[-i\beta\circ\partial+m+b(\Gamma',f)]\Phi(x)=0\,,\quad \Omega(x)=0 \qquad (21)$$

where $b(\beta,f)$ is the same type of an external-field interaction of the Hurley equation (21) as $B(\Gamma',f)$ is of the Fisk-Tait equation. As a specific example if $B(\Gamma',f)=e\Gamma_\mu'A^\mu(x)$, say a minimal coupling to an external electromagnetic field for the Fisk-Tait equation, then the form (20) *will* follow and $b(\beta,f)=e\beta_\mu A^\mu(x)$, leading to the equivalence of the minimally coupled Fisk-Tait equation to the minimally coupled Hurley equation. In other words, the extra components of the Fisk-Tait equation $\omega(x)$ never contribute. The range over which such equivalence, Eq. (19) to Eq. (21), exists is therefore vast.[12] The Fisk-Tait equation is the Hurley equation in disguise and as such does not lead to a new theory. This completes our assertion. As already mentioned, the equation of Hurley suffers from difficulties of parity doubling and negative energy and these are therefore unavoidable in the Fisk-Tait formulation also.

ACKNOWLEDGMENTS

We would like to thank Professor E. C. G. Sudarshan and Dr. B. Etemadi for discussions. One of us (M. S.) is grateful to Professor E. C. G. Sudarshan for warm hospitality at the Center for Particle Theory, University of Texas, Austin. This work was supported in part by the U. S. Energy Research and Development Administration under Contract No. E(40-1)3992.

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