Unified treatment of the Cherenkov and Ohmic losses of a relativistic charge in a conducting medium

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Theoretical estimates of the classical part of the stopping power of an Ohmic medium for a fast charged particle are obtained for the two limiting cases of a strongly and a weakly conducting medium. Conforming to the fact that any Cherenkov radiation that is emitted is rapidly absorbed and converted into heat in a strongly conducting medium, there is a single expression for the energy loss, which is Ohmic in nature. On the other hand, in a weakly conducting medium the total energy loss splits up into a Cherenkov loss given by the Frank and Tamm expression and an additional Ohmic loss, whose power spectrum exhibits a marked contrast with it.

I. INTRODUCTION

The Bohr[1] losses of electrically charged particles in cloud chambers, photographic plates, scintillation counters, etc., were summarized by Price,[2] and their Cherenkov losses[3,4] in a variety of media[5] by Jelley.[6] A unified treatment of Bohr and Cherenkov losses was first given by Fermi.[7] The interplay of synchrotron and Cherenkov losses has recently been studied by Schwinger *et al.*,[8] and that of transition and Cherenkov radiation by DeRaad *et al.*[9]

The purpose of the present study is to give a unified account of the Cherenkov and Ohmic losses of a relativistic charged particle moving in a conducting medium. Apart from their intrinsic theoretical interest, studies of radiation in conducting media[10,11] have acquired renewed significance by virtue of their astrophysical and spatial implications.[12,13]

We prefer to work in the rest frame of the particle[14,15] in which the conducting medium flows backwards with a relativistic velocity. By virtue of its motion the constitutive matrix of the medium becomes quite involved,[15] acquiring a uniaxial magnetoelectric character.[16,17] Added to this, the conductivity of the medium worsens the situation, since a conducting medium loses its neutrality and appears to be charged when set in motion in an electric field.[18] All these difficulties are, however, offset by the considerable simplification that the fields become static in the rest frame of the particle, which permits the losses to be derived from only a three-fold Fourier integral.

II. FOURIER SYNTHESIS OF THE FIELD

Let the point charge move with a uniform velocity βc along the x_1^0 axis in a homogeneous isotropic conducting medium. (In what follows, we use the notation of Majumdar and Pal.[15]) The imaginary parts of the permittivity ϵ, permeability μ, and conductivity σ are assumed to be negligible. The energy lost per unit path length of the particle is equal to the retarding force experienced by it: $W = -(eE_1^0)_{\text{charge}}$.[19] Since the longitudinal component of the electric field E_1^0 is unaffected by a special Lorentz transformation along the line of motion of the particle, it proves convenient to work in the rest frame of the particle, in which Maxwell's equations take the form

$$\partial_\mu D_\mu(x_\lambda) = e\delta(x_\lambda)\,, \quad e_{\mu\nu\rho}\partial_\nu E_\rho(x_\lambda) = 0\,,$$
$$\partial_\mu B_\mu(x_\lambda) = 0, \tag{1}$$
$$e_{\mu\nu\rho}\partial_\nu H_\rho(x_\lambda) = \frac{1}{c}[J_\mu(x_\lambda) + \beta_\mu \rho_p(x_\lambda)]\,,$$

where $e_{\mu\nu\rho}$ is the Levi-Civita symbol in three dimensions, $J_\mu(x_\lambda)$ is the conduction current density, and $\beta_\mu \rho_p$ is the convection current density due to the apparent charge density in the relativistically moving neutral medium. The static fields can be resolved into their Fourier components as

$$D_\mu(x_\lambda) = \int D_\mu \exp(ik_\lambda x_\lambda) d^3k\,, \quad \text{etc.} \tag{2}$$

The constitutive relations of the moving medium can be obtained from a Lorentz transformation of the covariant material tensors T_{ijkl} (Ref. 15) and σ_{ijk} defined by

$$H_{ij} = \tfrac{1}{2} T_{ijkl} F_{kl} \quad \text{and} \quad J_i = \tfrac{1}{2}\sigma_{ijk} F_{jk}\,, \tag{3}$$

where H_{ij}, F_{ij}, and J_i are the Fourier components of the induction and field tensors and current-density four-vector. T_{ijkl} is the Tamm magnetoelectric tensor connecting $(\vec{D}, \vec{H})$ with $(\vec{E}, \vec{B})$, and σ_{ijk} is the covariant conductivity tensor connecting $(\vec{J}, \rho)$ with $(\vec{E}, \vec{B})$. For an isotropic medium moving uniformly along the $-x_1$ axis, one then obtains

$$D_1 = \epsilon_1 E_1, \quad D_2 = \epsilon_2 E_2 + \xi B_3, \quad D_3 = \epsilon_2 E_3 - \xi B_2,$$
$$H_1 = \lambda_1 B_1, \quad H_2 = \lambda_2 B_2 + \xi E_3, \quad H_3 = \lambda_2 B_3 - \xi E_2, \tag{4}$$
$$J_1 = \eta E_1, \quad J_2 = \eta E_2 + \Psi B_3, \quad J_3 = \eta E_3 - \Psi B_2,$$

and

$$\rho_p = -\Psi E_1,$$

where

$$\epsilon_1 = \epsilon, \quad \epsilon_2 = \frac{\gamma^2}{\mu}(n^2 - \beta^2),$$
$$\lambda_1 = \frac{1}{\mu}, \quad \lambda_2 = \frac{\gamma^2}{\mu}(1 - \beta^2 n^2),$$
$$\xi = \frac{\beta\gamma^2}{\mu}(n^2 - 1), \quad \eta = \sigma\gamma, \quad \Psi = \sigma\beta\gamma, \quad n^2 = \epsilon\mu .$$

Substituting (2) and (4) in (1), we obtain

$$W = -\frac{ie^2}{8\pi^3} \int_{-\infty}^{\infty} \frac{\alpha^2}{\epsilon} k_1 dk_1 \times \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} \frac{dk_3}{k_3^2 + k_2^2 - \alpha^2 k_1^2 - 2i\chi k_1} \tag{5}$$

where

$$\alpha^2 = \gamma^2(\beta^2 n^2 - 1)$$

and

$$\chi = \frac{\sigma\beta\gamma(n^2 - 1)(1 + \beta^2)}{2\epsilon c(1 - \beta^2)} .$$

Although the integral in (5) is purely imaginary, its structure is such that the sum of the residues at its poles is invariably real. This makes W an essentially real quantity. The integral diverges in a nondispersive medium, but dispersion (which is always present in any real medium) provides the necessary cutoff, as $n \to 1$ for $\omega \to \infty$. By a Lorentz transformation of the wave four-vector it can be seen that the frequency ω in the rest frame of the medium is related to the longitudinal component k_1 of the wave vector in the rest frame of the charge by $\omega = \beta\gamma c k_1$. We assume the medium to be temporally dispersive in its rest frame so that ϵ, μ, and σ are given functions of ω. These then become given functions of k_1 in the rest frame of the charge, in which the moving medium exhibits its spatial dispersion.[20]

III. EVALUATION OF THE INTEGRAL

The position of the poles of the integrand of (5) in the complex k_3 plane depends on the sign of k_1 and the magnitude of σ. For weakly conducting media and velocities of the charge below the Cherenkov threshold ($\alpha^2 < 0$), they always remain

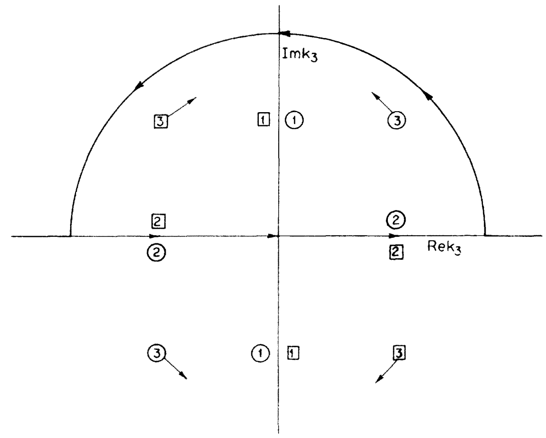

FIG. 1. Configuration of the poles, where ○ $k_1 > 0$, □ $k_1 < 0$: (1) In a weakly conducting medium for all k_2 for velocities of the charge below the Cherenkov threshold, and for $k_2^2 > \alpha^2 k_1^2$ for velocities of the charge above the Cherenkov threshold. (2) In a weakly conducting medium for $k_2^2 < \alpha^2 k_1^2$ for velocities of the charge above the Cherenkov threshold. (3) In a strongly conducting medium for velocities of the charge above the Cherenkov threshold. For $k_2^2 < \alpha^2 k_1^2$, the poles remain stationary, while for $k_2^2 > \alpha^2 k_1^2$ the poles migrate towards the Im k_3 axis as $k_2^2 \to \infty$.

close to the imaginary axis and on either side of it (Fig. 1). In the limit as $\sigma \to 0$, the poles for $k_1 > 0$ and $k_1 < 0$ merge on the imaginary axis. Owing to the presence of k_1 in the numerator of (5), their contributions to the integral cancel one another exactly, which leads to a vanishing absorption and absence of radiation. Two cases arise for velocities of the charge above the Cherenkov threshold. For $k_2^2 < \alpha^2 k_1^2$, the poles are close to the real axis and approach it for small values of σ. Their contributions therefore combine with one another and give rise to a finite energy loss even in a nonabsorbing medium. This loss is identified as Cherenkov emission. For $k_2^2 > \alpha^2 k_1^2$, on the other hand, the poles revert to their positions close to the imaginary axis, wherefore their contribution from this region of k_2 integration is customarily ignored. In the other extreme of a strongly conducting medium, the poles lie midway between the real and imaginary axes for both $k_2^2 \gtrless \alpha^2 k_1^2$ and this demands that both these regions be considered during the evaluation of the stopping power.

The k_3 integral can thus be evaluated by collecting the residues at the appropriate poles. However, the resulting k_2 integrand turns out to be too involved for arbitrary σ, containing a radical within a radical. Such a situation obtains in all studies of the electrodynamics of Ohmic media, as it is inherent in their dispersion relation itself.[21] However, it is possible to extract the asymptotic limits of the stopping power in the two realistic limiting cases of a strongly and a weakly conducting medium. The former would refer to the classical losses of a high-energy proton coursing through a metallic slab, and the latter to its losses in a Cherenkov insulator with conducting impurities.

In the limit of a strongly conducting medium, we obtain

$$W_h = \frac{e^2}{4\pi^2 c^2} \int_0^{\omega_h} \mu \left(1 - \frac{1}{\beta^2 n^2}\right)^{3/2} \left[\frac{2\tau(1-\beta^2)}{(1+\beta^2)(1-1/n^2)}\right]^{1/2} \left(\frac{\omega_h{}^2}{\omega^2} + 1\right)^{1/2} \omega^{3/2} d\omega \,, \tag{6}$$

where τ is the relaxation time[21] of the conducting medium and ω_h is the limiting frequency (usually in the ultraviolet) below which the medium can be classed as strongly conducting. It is important to note that the conductivity enters expression (6) not only through τ but also through ω_h. In (6), there is no longer a neat separation between the Cherenkov and Ohmic losses. This conforms to the fact that any Cherenkov emission that takes place should now be rapidly absorbed and converted into heat within a short distance of the order of the skin depth.[21] Expression (6) gives the total energy loss including this conversion. It is interesting to contrast the Fourier power spectrum in an Ohmic medium with the Cherenkov spectrum in an insulating medium.[4]

In the other extreme of a weakly conducting medium, the energy loss takes the form

$$W_l = \frac{e^2}{4\pi c^2} \left\{ \int_{\beta^2 n^2 > 1} \mu \left(1 - \frac{1}{\beta^2 n^2}\right) \omega \, d\omega + \int_{\omega_l}^{\infty} \frac{\mu(1+\beta^2)}{\pi\tau(1-\beta^2)} \left(1 - \frac{1}{n^2}\right) \left[\left(1 + \frac{\omega^2}{\omega_l{}^2}\right)^{1/2} - 1 \right] d\omega \right\}, \tag{7}$$

where ω_l is the limiting frequency above which the medium can be classed as weakly conducting.[21] The first term of (7) is precisely the Frank and Tamm expression for the Cherenkov losses in an ideal insulator, and the second term can be expected to represent the additional Ohmic losses due to any conducting impurities.

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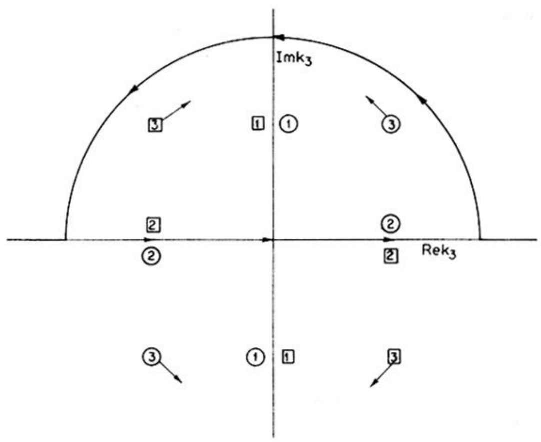

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