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Thermodynamics of the Yang-Mills gas

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The contribution of nonlinear fluctuations (instantons) to the thermodynamics of the Yang-Mills gas at high temperature is estimated.

I. INTRODUCTION

In the canonical approach[1] to the study of the thermodynamics of the Yang-Mills gas, small oscillations about a single ($A_\mu = 0$) vacuum state are considered. However, the existence of a multiply-degenerate vacuum introduces additional quantum fluctuations, the instantons. While the former modes are generally treated as independent linear oscillations in momentum space, the latter modes are highly nonlinear vibrations and are usually expressed in (Euclidean) position space. It has been demonstrated[2] that these nonlinear modes persist at a finite temperature, where they are referred to as "calorons." Despite the well-known difficulties and subtleties of the thermodynamics in a highly nonlinear system,[3] we begin here such a determination. By so doing, we hope also to shed some light on the mechanism whereby quarks may be liberated at high temperature.[4]

We begin in Sec. II with a review of the degeneracies of the caloron solution. Particular emphasis is placed on the scale factor λ, since dilatation invariance is explicitly broken by the temperature T. The physical picture which we shall use is indicated in Fig. 1, with the scale size limited by $\beta = 1/kT$, where k is the Boltzmann constant. As the temperature is raised, the maximum allowable scale size is decreased, i.e., the large-scale calorons are squeezed out of the physical region.[2,5] But, if only small-scale sizes remain, then because of asymptotic freedom the dilute-gas approximation (DGA) can be reliably used, and many of the complications due to the nonlinearities can be overcome.

In Sec. III, we assume that the temperature is sufficiently high that the physical effects of the DGA are calculable.[6] Furthermore, the integration over scale sizes becomes fully specified. For small λ, asymptotic freedom ensures a finite contribution, while for large λ, β acts as an infrared cutoff. That the temperature should act as a disordering effect for long-wavelength correlations is not at all surprising, having been observed in the case of linear fluctuations about the $A_\mu = 0$ vacuum.[7] We show that the nonlinear modes can contribute a significant fraction of the total pressure (or internal energy, specific heat, etc.) in a region where the calculation can be considered reliable ($\mu\beta \sim 0.2$–0.4, where μ is the renormalization mass). For higher temperatures, the calorons are negligible, while for lower temperatures the nonlinear effects cannot be reliably calculated.

In Sec. IV we note that since the temperature range of interest is $\sim 10^{12}$ °K, applications would be limited to theories of the early universe and models of high-energy collisions. A mechanism for quark liberation in this temperature range is postulated.

II. CALORON: ZERO MODES

The functional integral for the partition function can be written as[8]

$$Z = \int [d\phi] \exp\left[\int_0^\beta d\tau \int d^3x\, \mathcal{L}_E(\phi, \partial_\mu \phi)\right], \quad (2.1)$$

where $\mathcal{L}_E(\phi, \partial_\mu\phi)$ is the Euclidean Lagrangian for the generic, periodic field $\phi(\vec{x}, \tau)$,

$$\phi(\vec{x}, \tau) = \phi(\vec{x}, \tau + \beta). \quad (2.2)$$

Since we wish to calculate the physical effects of finite-temperature instantons (calorons), the Lagrangian of interest is that for the non-Abelian SU(3) gauge theory of quarks and gluons called quantum chromodynamics. However, we adopt here the view that the dominant dynamics is determined by the gluon field, whose structure will decide, e.g., whether confinement is present.[9] Of course, the presence of massless quarks could drastically affect this assumption.[6,10] To further simplify the discussion, we shall use initially an SU(2) Yang-Mills theory for the gluons.

We assume that the dominant contributions to

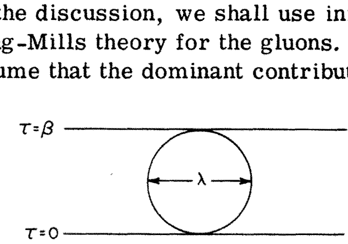

FIG. 1. Temperature (T) limits caloron scale size (λ): $\lambda \lesssim \beta = 1/kT$.

the partition function come from field configurations that minimize the Euclidean action I and obey Eq. (2.2). By expanding around the *trivial* minimum $A_\mu = 0$ and keeping terms up to second order in the fluctuation about this minimum, one obtains an ideal Bose gas for each degree of freedom. Here we shall follow the same procedure, but now expanding about the *nontrivial* minimum,

$$A_\mu^{\rm cl} = i\bar{\sigma}_{\mu\nu}\partial_\nu \ln\phi \,, \tag{2.3}$$

where

$$\phi = 1 + (\pi\lambda^2/\beta\,|\vec{x}-\vec{x}_0|) \times \frac{\sinh(2\pi\beta^{-1}|\vec{x}-\vec{x}_0|)}{\cosh(2\pi\beta^{-1}|\vec{x}-\vec{x}_0|) - \cos 2\pi\beta^{-1}(\tau-\tau_0)} \,. \tag{2.4}$$

[For notation see Ref. 11, while for details of the classical solution, Eq. (2.4), see Ref. 2.] Note that the solution is parametrized by its generalized position $\vec{x}_0, \tau_0$ and by its scale size λ. There are also gauge degrees of freedom $w_j(x)$, where the gauge transformation is effected by the SU(2) matrix $\Omega = \exp[iw_j(x)\sigma_j/2]$, with σ_j the Pauli matrices.

The fact that we no longer have dilatation invariance is manifest in the solution Eq. (2.4). A change of scale $x \to \rho x$ is not equivalent to a change in λ. Rather, the combined transformation $x \to \rho x$, $\beta \to \rho\beta$ is equivalent to $\lambda \to \lambda/\rho$. Though the Euler-Lagrange equations are dilatation invariant, the boundary condition, Eq. (2.2), is not.

Despite the lack of dilatation invariance, there still exists a family of independent solutions, parametrized by λ, which generates a zero mode in the calculation of the quantum fluctuations about the classical solution Eq. (2.3). This can be seen as follows: Let the parameters of the caloron be collectively denoted by

$$X_0 \equiv (\vec{x}_0, \tau_0, \lambda, w_j(x)) \,. \tag{2.5}$$

Then since

$$\left.\frac{\delta I}{\delta A_\mu}\right|_{A_\mu = A_\mu^{\rm cl}(x_0)} = 0 \,,$$

we have

$$\frac{\partial}{\partial x_0}\left.\frac{\delta I}{\delta A_\mu}\right|_{A_\mu^{\rm cl}(x_0)} = \lim_{\Delta x_0 \to 0} \frac{1}{\Delta x_0}\left(\left.\frac{\delta I}{\delta A_\mu}\right|_{A_\mu^{\rm cl}(x_0+\Delta x_0)} - \left.\frac{\delta I}{\delta A_\mu}\right|_{A_\mu^{\rm cl}(x_0)}\right) = 0 \,. \tag{2.6}$$

But

$$\frac{\partial}{\partial x_0}\left.\frac{\delta I}{\delta A_\mu}\right|_{A_\mu^{\rm cl}(x_0)} = \left.\frac{\delta^2 I}{\delta A_\mu \delta A_\nu}\right|_{A^{\rm cl}(x_0)} \frac{\partial A_\nu^{\rm cl}}{\partial x_0} \,, \tag{2.7}$$

so that $\partial A_\nu^{\rm cl}/\partial x_0$ is an eigenfunction with zero eigenvalue of the kernel in the quadratic term of the expansion around the classical solution.

The number and general form of the zero-mode eigenfunctions thus remain unchanged at finite temperature. The similarity with the zero-temperature results is even more striking when one calculates the Jacobians associated with the change to collective coordinates. Typically, one wishes to determine ${J_{\tau_0}}^{1/2}$, where

$$J_{\tau_0} = \int_0^\beta d\tau \int d^3x \left(\frac{\partial A_\mu^{\rm cl}}{\partial \tau_0}\right)^2 \,, \tag{2.8}$$

and a trace over the SU(2) indices is implied. However, by a gauge transformation, we can rewrite Eq. (2.8) as

$$J_{\tau_0} = \int d^4x (F_{0i}^{\rm cl})^2 \,, \tag{2.9}$$

where $\int d^4x$ is used as a shorthand for $\int_0^\beta d\tau \int d^3x$. The self-duality of our solution implies that

$$J_{\tau_0} = \frac{1}{4}\int d^4x (F_{\mu\nu}^{\rm cl})^2 \,, \tag{2.10}$$

so that J_{τ_0} is independent of β (as calculated in Ref. 2):

$$J_{\tau_0} = 4\pi^2 \,. \tag{2.11}$$

The temperature independence of the Jacobian associated with a change in λ is not as obvious. One may in fact show that J_λ is independent of β, where

$$J_\lambda = \int d^4x \left(\frac{\partial A_\mu^{\rm cl}}{\partial \lambda}\right)^2 \,, \tag{2.12}$$

or that $\hat{J}_\lambda = 1$ for all β, where

$$\hat{J}_\lambda = J_\lambda / J_\lambda(\beta \to \infty) \,. \tag{2.13}$$

The usual gauge transformation[12,13] which greatly simplifies J_λ is not periodic in τ and therefore inapplicable here. The denominator of Eq. (2.13) is easily evaluated by using the four-dimensional rotational symmetry of the solution in the zero-temperature limit,

$$J_\lambda(\beta \to \infty) = 16\pi^2 \,. \tag{2.14}$$

After some tedious algebra, we find

$$\hat{J}_\lambda = 3\pi(\lambda/\beta)^2 \int_0^{2\pi} dy \int_0^\infty dx \frac{(\sinh x \sin y)^2 + \{1 - \cosh x \cos y - [(\sinh x)/x](\cosh x - \cos y)\}^2}{[\cosh x - \cos y + 2(\pi\lambda/\beta)^2(\sinh x)/x]^4} \,. \tag{2.15}$$

Though at first glance this appears to be a rather complicated function of λ/β, we have explicitly verified by numerical integration that $\hat{J}_\lambda = 1$, independent of λ/β.

III. THERMODYNAMICS

Assuming that the dilute-gas approximation is valid, the contribution of the calorons to the partition function in an SU(N) theory is determined by[2]

$$\ln Z_c = 2C_N V\beta \int_0^\infty \frac{d\lambda}{\lambda^5}\left[\frac{8\pi^2}{\bar{g}^2(1/\lambda\mu)}\right]^{2N} \times \exp\left(\frac{-8\pi^2}{\bar{g}^2(1/\lambda\mu)}\right) w(\beta,\lambda), \quad (3.1)$$

where several remarks are in order:

(1) The factor of 2 comes from equal caloron and anticaloron contributions.

(2) C_N is a group-theoretical factor, where $C_2 = 0.26$ and $C_3 = 0.1$.[14]

(3) $V\beta$ comes from integrating over the translational zero modes.

(4) The integral over λ reflects the degeneracy in scale sizes of calorons while the λ^{-5} ensures that $\ln Z_c$ is dimensionless (one factor of λ^{-1} for each of the five translation-dilation zero modes).

(5) The factor of $(1/\bar{g})^{4N}$ comes from the normalization of the zero-mode Jacobians, of which there are $4N$-5 from the gauge degrees of freedom.

(6) The exponential factor is the renormalization-group improved result of the lowest-order contribution to the action using A_μ^{cl}.

(7) Since the previous factor contains only the temperature-independent quantum oscillations about the caloron, the remaining contributions are lumped into the weighting function $w(\beta,\lambda)$.

The explicit form of $w(\beta,\lambda)$ is not known, though presumably determinable in a lengthy calculation generalizing to finite temperature the consistent perturbative expansion about an instanton.[12,15] Before embarking on such a project, it seems prudent to obtain an order-of-magnitude estimate of caloron effects. We do this by referring back to the intuitive picture of Fig. 1 (see also Ref. 5) which implies that $w(\beta,\lambda)$ can be approximated by a step function, i.e.,

$$w(\beta,\lambda) \simeq \theta(\beta-\lambda)\,. \quad (3.2)$$

This ansatz also removes a glaring deficiency of Eq. (3.1), namely, that with an appropriate choice of β, the λ integration need not be extended outside the region of applicability of asymptotic freedom. More specifically, we have[6]

$$8\pi^2/\bar{g}^2(1/\lambda\mu) \xrightarrow[\lambda\mu\ll 1]{} (11N/3)\ln(1/\lambda\mu) \quad (3.3)$$

for all λ in the region of integration ($\lambda \le \beta$).

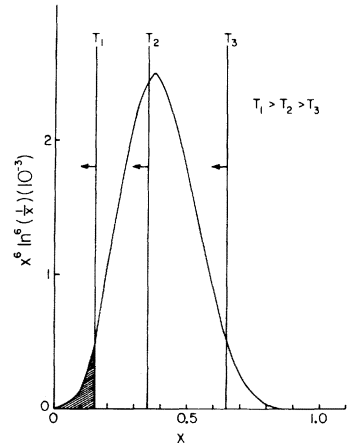

FIG. 2. Integrand of Eq. (3.4) vs $x = \lambda\mu$, showing that increasing temperature decreases the caloron's contribution (available area under the curve) to the thermodynamics.

Specializing to the case of SU(3), we now have

$$\ln Z_c = 2(0.1)(11)^6 V\beta\mu^4 \int_0^{\mu\beta} dx\, x^6 \ln^6(1/x)\,. \quad (3.4)$$

The integrand in Eq. (3.4) is shown in Fig. 2. Note that as the temperature is raised, the caloron contribution (area under the curve) is decreased. At very high temperatures ($\mu\beta \ll 1$), degrees of freedom are being "boiled away". Also note that, according to Callan, Dashen and Gross,[6] the DGA should not be trusted above $\mu\beta \sim 0.2$–0.4. Keeping the leading term in Eq. (3.4), we find that

$$\ln Z_c = aV\mu^3(\mu\beta)^8 \ln^6(1/\mu\beta)\,, \quad (3.5)$$

where $a \simeq 5.1\times 10^4$. The neglected terms would slightly increase the caloron contribution.

From Eq. (3.5) we can immediately obtain the thermodynamic functions, such as the free energy

$$F = -\beta^{-1}\ln Z\,, \quad (3.6a)$$

the internal energy

$$U = -\frac{\partial}{\partial\beta}\ln Z\,, \quad (3.6b)$$

the pressure

$$P = \beta^{-1}\frac{\partial}{\partial V}\ln Z\,, \quad (3.6c)$$

and the specific heat

$$C = -\beta^2\frac{\partial U}{\partial\beta}\,. \quad (3.6d)$$

Our results are summarized in Table I, where they are compared to the corresponding ideal SU(3) Bose gas contributions.

For illustrative purposes, the ratio of the pres-

TABLE I. Contributions of calorons and ideal bosons to the internal energy U, the pressure P, and the specific heat C. Here $a \sim 5.1\times10^4$, $\mu\sim 200$ MeV, $\beta = 1/kT$.

	SU(3): Caloron gas	SU(3): Ideal Bose gas
U	$-8aV\mu^4(\mu\beta)^7 ln^6(1/\mu\beta)$	$(8\pi^2/15)V\beta^{-4}$
P	$a\mu^4(\mu\beta)^7 ln^6(1/\mu\beta)$	$(8\pi^2/45)\beta^{-4}$
C	$56aV\mu^3(\mu\beta)^8 ln^6(1/\mu\beta)$	$(32\pi^2/15)V\beta^{-3}$

sure due to the caloron P_c to that due to the ideal Bose gas P_i is displayed in Fig. 3 [for the SU(3) case]. Note that

$$P_c/P_i = (2.9\times10^4)(\mu\beta)^{11}\ln^6(1/\mu\beta)\,. \tag{3.7}$$

Again, one should keep in mind that $\mu\beta \sim 0.4$ is to be considered an upper limit on the validity of the calculation, so that the peak which occurs at $\mu\beta = 0.58$ is not to be taken very seriously. It is interesting to note that the peak occurs at precisely the same point for SU(2) though the ratio rises to only 1.1 for SU(2) [as opposed to 1.9 for SU(3)]. Of course, the numerical values given by Eq. (3.7) should only be regarded as qualitative estimates, since the cutoff function, Eq. (3.2), has not been rigorously determined.

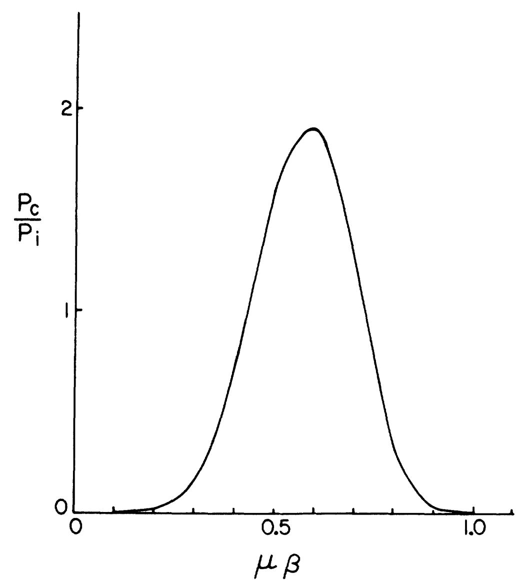

FIG. 3. Ratio of pressures due to calorons and ideal bosons as a function of inverse temperature β (μ, the renormalization mass, is fixed~ 200 MeV).

IV. CONCLUSION

We have shown that for $\mu\beta \ll 1$, the degrees of freedom represented by the caloron (finite-temperature instanton) are not effectively activated. However, when $\mu\beta\sim 0.2$–0.4, the caloron's contribution to thermodynamic functions cannot be neglected. Throughout this entire range, the scale-size λ is so bounded by β that the requisite scale size for the disassociation into merons is not reached. It has been argued that merons, resulting from a phase transition of large instantons, are responsible for quark confinement.[6] If β determines the maximum scale size, then for high enough temperature (β small), the instanton-meron gas will be suppressed and cannot therefore confine quarks.

Of course, these results are only tentative. We have not included the effects of fermions on the gluons, we have not precisely determined the weighting function in Eq. (3.1) and it is not yet generally accepted that merons are the culprits for confinement.

However, the above picture suggests a new look at theories of the very early universe for temperatures such that $\mu\beta \sim 0.2$–0.4, i.e., for $T\sim 10^{12}$ °K, assuming $\mu \sim 200$ MeV. Unfortunately, for $T > 10^{12}$ °K, large numbers of strongly interacting particles are present, vastly complicating life in the first 0.0001 sec. Density effects would also be serious since the mean distance between particles would be less than a Compton wavelength.[16] However, for $T \gg 10^{12}$ °K, it appears that calorons would be of no consequence and the corresponding densities would be such as to ensure the validity of ordinary perturbation theory of quarks and gluons.[17]

A more promising observational effect may occur in high-energy hadron-hadron collisions.[18] Here the physics can possibly be described in terms of the heating of interpenetrating gluon gases which may lead to excitation of caloron modes. Much work has been done on statistical and hydrodynamical models of high-energy collisions.[19] For example, it is well known that the transverse momentum (P_T) distribution of a particle with mass m and center-of-mass rapidity $y = 0$ produced from a projectile scattered off a nucleon target is of the form[19]

$$(d\sigma/dp_T^{\,2}) \propto \exp[-\beta(p_T^{\,2}+m^2)^{1/2}]\,. \tag{5.1}$$

Here $\beta^{-1}\sim 110$–130 MeV, independent of the projectile, the particle with mass m, and the beam energy. In this temperature region, we expect the calorons to be important, though we are in danger of losing the DGA as a guide. In this regard, current efforts[20] to go beyond the DGA are welcomed.

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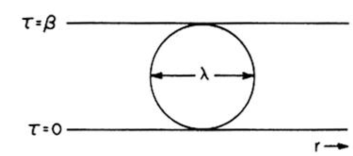
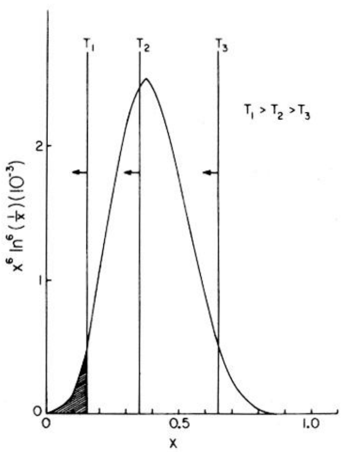


FIG. 1. Temperature (T) limits caloron scale size (λ): $\lambda \lesssim \beta = 1/kT$.



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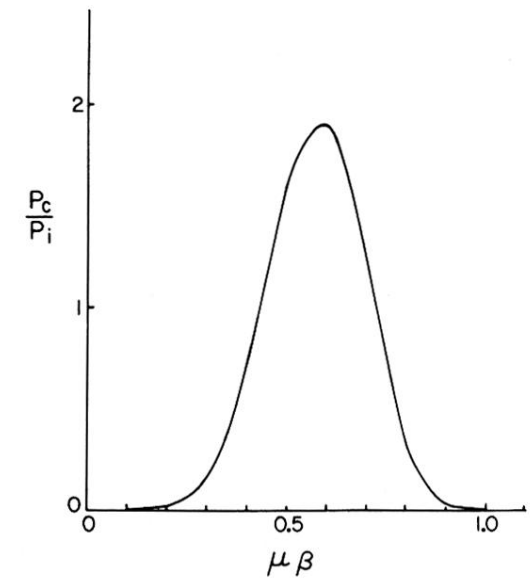

FIG. 3. Ratio of pressures due to calorons and ideal bosons as a function of inverse temperature β (μ, the renormalization mass, is fixed ~ 200 MeV).