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Nonlinear contributions to the partition function of a double-well oscillator

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The role that instantons play in the determination of the partition function for a double-well anharmonic oscillator is studied.

I. INTRODUCTION

The discovery of nontrivial minima of the Euclidean action (instantons) for the SU(2) Yang-Mills field theory[1] has stimulated an incredible amount of activity in theoretical physics over the past few years. However, the experimental implications of these instantons remain somewhat elusive. Whether such modes necessitate the axion[2] and whether they (or their immediate descendants, merons) give rise to quark confinement[3] are, obviously, exciting possibilities.

Most of the previous theoretical work has concentrated on the role that instantons play in elucidating the vacuum structure. However, if instantons are to be interpreted as maximal tunneling amplitudes between otherwise degenerate vacuums,[4] then they should affect more than just the ground state. By considering an ensemble of Yang-Mills particles at a nonzero temperature (T), we can possibly learn about the level structure in the presence of instantons. Ultimately, the extra freedom embodied in the instanton should become manifest in the thermodynamics of the Yang-Mills gas.

A first step in this program was realized in the proof that finite-temperature instantons (called "calorons") do exist in the SU(2) theory.[5] An explicit realization was found and a first estimate (equivalent to the dilute-gas approximation) of the partition function in the presence of calorons was made.[6] However, to reassure ourselves and to better understand the role of calorons, we undertake here a study of a much simplified theory, that of a single scalar field in one time and zero space dimensions. The Langrangian is

$$\mathcal{L} = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 - \frac{\lambda}{4}\left(q^2 - \frac{\mu^2}{\lambda}\right)^2 , \tag{1.1}$$

where μ^2, $\lambda > 0$. The double-well, anharmonic oscillator possesses the dual feature of having an instanton, the kink that connects the two wells of Fig. 1,

$$q_k(t) = \left(\frac{\mu^2}{\lambda}\right)^{1/2} \tanh \mu t/\sqrt{2} , \tag{1.2}$$

and of having a known partition function in the low-temperature ($\beta = 1/kT \to \infty$) limit,[7]

$$Z = e^{-\beta(E_0 - \Delta E/2)} + e^{-\beta(E_0 + \Delta E/2)} , \tag{1.3a}$$

where

$$E_0 = \sqrt{2}\hbar\,\mu/2 , \tag{1.3b}$$

$$\Delta E \simeq \sqrt{2}\hbar\,\mu(16\sqrt{2}\mu^3/\pi\lambda\hbar)^{1/2} \exp(-2\sqrt{2}\mu^3/3\lambda\hbar) . \tag{1.3c}$$

We shall reproduce these results by use of a dilute gas of calorons in the low-temperature limit. What is novel about our approach is the delay in the $\beta \to \infty$ limit. We work with modes [the analog of Eq. (1.2)] that are explicitly temperature dependent so that our methodology is applicable away from zero temperature. Additional advantages include:

(1) There is no need to patch together a kink with an antikink[8] since we are working with periodic solutions.

(2) Translational invariance is explicitly maintained so we avoid the use of a quasizero eigenvalue.[7]

(3) The breaking of dilatation invariance is

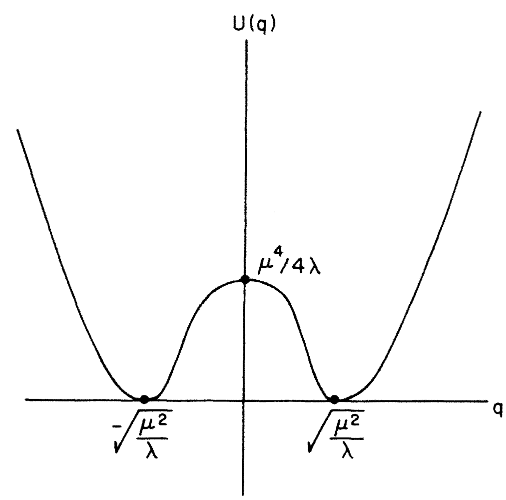

FIG. 1. Potential for the double-well anharmonic oscillator.

elucidated.

(4) The approximate instability of the basic nonlinear solution at any nonzero temperature is made manifest.

(5) A limiting temperature, above which no periodic solution exists, becomes evident.

We begin our presentation in Sec. II with an explication of the methodology to be employed in the determination of the quantum partition function. In particular, we emphasize the natural appearance of Euclidean functional integrals and the advantage of including the full phase-space measure. Section III on the harmonic oscillator illustrates the latter point as well as prepares us for the more involved calculation of Sec. IV. Here, the double-well anharmonic oscillator is considered in detail and the partition function is calculated in the low-temperature limit. Finally, in Sec. V, we consider some of the unsolved problems and directions for future research.

II. QUANTUM PARTITION FUNCTION

As is well known,[9] the partition function for a scalar field $\phi(\vec{x}, t)$ described by the Largrangian $\mathcal{L}(\phi, \phi_t)$ is given by

$$Z = N\oint d[\phi]\exp\left[\int_0^\beta d\tau \int d\vec{x}\mathcal{L}_{\text{eff}}(\phi, \phi_\tau)\right], \quad (2.1)$$

where N is a β-dependent normalizing factor and $\mathcal{L}_{\text{eff}}$ contains $\mathcal{L}$ and any necessary ghost and gauge-fixing terms. Also,

$$\phi_\tau = i\partial\phi(\vec{x}, \tau)/\partial\tau \quad (2.2)$$

while $\phi(\vec{x}, \tau)$ obeys the periodicity condition

$$\phi(\vec{x}, \tau) = \phi(\vec{x}, \tau+\beta). \quad (2.3)$$

To determine the partition function we look for modes which dominate the Euclidean functional integral, Eq. (2.1). Such τ-dependent modes which obey Eq. (2.3) and reduce to instantons as $\beta \to \infty$ are known as calorons. In the dilute-gas approximation, one assumes that the various τ-dependent and τ-independent modes contribute nonoverlapping contributions to Z.

If the normalizing factor in Eq. (2.1) is ignored, then even in the simplest cases, the partition function is ill defined. By a judicious redefinition of the functional integral (known as the ζ-function regularization[10]), one can circumvent this difficulty. Here, however, we prefer to evaluate N which will obviate the necessity of introducing a regulating procedure. To this end, we follow Schwinger[11] in writing the partition function for a single quantum degree of freedom,

$$Z = \oint d[q,p]\exp(-w[q,p]), \quad (2.4)$$

where

$$w[q,p] = \int_0^\beta d\tau\left[\tfrac{1}{2}p^2 + \tfrac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + U(q)\right] \quad (2.5)$$

and

$$\int d[q,p] = \prod_{n=-\infty}^{+\infty} \int \frac{dq_n dp_n}{2\pi}, \quad (2.6)$$

while $q(\tau)$ and $p(\tau)$ each have period β:

$$q(\tau) = \frac{1}{\sqrt{\beta}}\left\{q_0 + \sqrt{2}\sum_{n=1}^{\infty}\left[q_n \sin(2\pi n\tau/\beta) - q_{-n}\cos(2\pi n\tau/\beta)\right]\right\}, \quad (2.7)$$

$$p(\tau) = \sqrt{\beta}\left\{p_0 + \sqrt{2}\sum_{n=1}^{\infty}\left[p_n\cos(2\pi n\tau/\beta) + p_{-n}\sin(2\pi n\tau/\beta)\right]/2\pi n\right\}. \quad (2.8)$$

The coefficients in the expansions of Eqs. (2.7) and (2.8) are chosen so that the action operator for special canonical transformations have a simplified form[11] and so that, with

$$q(\tau) = \sum_{n=-\infty}^{+\infty} q_n q_n(\tau), \quad (2.9)$$

we have

$$\int_0^\beta d\tau\, q_n(\tau)q_m(\tau) = \delta_{nm}. \quad (2.10)$$

It should be noted that the τ dependence of $q(\tau)$ and $p(\tau)$ is a particularly quantum effect. This can be seen by defining γ,

$$\gamma = \tau/\beta, \quad (2.11)$$

and looking at Eqs. (2.4) and (2.5) in the high-temperature ($\beta \to 0$) limit. First, we have

$$w[q,p] = \int_0^1 d\gamma\left[\tfrac{1}{2}\beta p^2 + \left(\frac{dq}{d\gamma}\right)^2\Big/2\beta + \beta U(q)\right], \quad (2.12)$$

and then when $\beta \to 0$, any γ dependence in q will lead to a large $w[q,p]$, i.e., from Eq. (2.5) a small contribution to the partition function. Thus, the classical limit involves just the q_0 and p_0 modes.

However, we can easily go beyond this classical approximation and exactly include all the momentum modes. Performing the Gaussian integrations yields

$$Z = N(\beta)\oint d[q]\exp\{-I[q]\}, \quad (2.13)$$

where

$$N(\beta)=\frac{1}{\beta}\prod_{n=1}^{\infty}(2\pi n/\beta)^2 \qquad (2.14)$$

and

$$I[q]=\int_0^\beta d\tau\left[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2+U(q)\right]. \qquad (2.15)$$

The exact evaluation of Eq. (2.15) for the case of a harmonic oscillator is briefly considered in the following section while an approximate evaluation of Eq. (2.15) for the case of a double-well anharmonic oscillator is considered in Sec. IV.

III. HARMONIC OSCILLATOR

When

$$U(q)=\tfrac{1}{2}\omega^2q^2, \qquad (3.1)$$

we can immediately diagonalize the quadratic form in Eq. (2.15) so that

$$Z=\frac{1}{\beta}\prod_{n=1}^{\infty}(2\pi n/\beta)^2\,\frac{1}{\omega}\prod_{n=1}^{\infty}[(2\pi n/\beta)^2+\omega^2]^{-1}. \qquad (3.2)$$

The factors $1/\beta$ and $1/\omega$ arise from the p_0 and q_0 integrations, respectively, and together form the classical limit. The quantum oscillations embodied in the infinite product can be evaluated in closed form to give the canonical result

$$Z=[2\sinh\beta\omega/2]^{-1}=\sum_{n=0}^{\infty}e^{-\beta\omega(n+1/2)}. \qquad (3.3)$$

For the sake of fellow pedants, let us consider another, more generally applicable, approach to this problem. We begin by functionally expanding $I[q]$ about a typical periodic minima $q_c(\tau)$:

$$\left.\frac{\delta I[q]}{\delta q(\tau)}\right|_{q_c(\tau)}=0, \qquad (3.4a)$$

i.e.,

$$\ddot{q}_c(\tau)=U'(q_c), \qquad (3.4b)$$

where the dots refer to derivatives with respect to τ and the prime refers to a derivative with respect to q_c. The "classical field" $q_c(\tau)$ is a misnomer when it is actually τ dependent.

Next, we keep only terms to second order in the Volterra expansion (equivalent to the one-loop approximation), i.e., we say

$$Z\simeq N(\beta)e^{-I[q_c]}\oint d[Q]e^{-I_2[Q]}, \qquad (3.5)$$

where

$$I_2[Q]=\tfrac{1}{2}\int_0^\beta d\tau d\tau' d\tau'' \times\left.\frac{\delta^2 I[q]}{\delta q(\tau')\delta q(\tau'')}\right|_{q_c(\tau)}Q(\tau')Q(\tau''), \qquad (3.6a)$$

i.e.,

$$I_2[Q]=\tfrac{1}{2}\int_0^\beta d\tau\, Q(\tau)\left[-\frac{d^2}{d\tau^2}+U''(q_c)\right]Q(\tau), \qquad (3.6b)$$

and where

$$Q(\tau)=q(\tau)-q_c(\tau). \qquad (3.7)$$

We are implicitly assuming a single solution to Eq. (3.4b). If there are many solutions, then, in general, it is quite difficult to estimate the overlap in phase space of the contributions that each gives to the partition function. The dilute-gas approximation applies when the overlap can be ignored.

The quadratic form in Eq. (3.6b) is diagonalized by letting

$$Q(\tau)=\sum_n c_nQ_n(\tau), \qquad (3.8)$$

where

$$\left[-\frac{d^2}{d\tau^2}+U''(q_c)\right]Q_n(\tau)=\omega_n^{\,2}Q_n(\tau). \qquad (3.9)$$

Momentarily assuming that $\omega_n^{\,2}>0$, we find for the partition function

$$Z=N(\beta)e^{-I[q_c]}\frac{1}{\Pi_n\omega_n}. \qquad (3.10)$$

For the harmonic oscillator, the Volterra series expansion vanishes beyond I_2 so that Eq. (3.10) is exact. In this case, Eq. (3.4b) reads

$$\ddot{q}_c=\omega^2q_c, \qquad (3.11)$$

which has one periodic solution, namely the trivial solution $q_c=0$. For this solution the action likewise vanishes. Finally, the eigenvalue equation corresponding to Eq. (3.9) reads

$$\left(-\frac{d^2}{d\tau^2}+\omega^2\right)Q_n(\tau)=\omega_n^{\,2}Q_n(\tau). \qquad (3.12)$$

The eignefunctions $Q_n(\tau)$ are just trigonometric functions with eigenvalues,

$$\omega_n^{\,2}=\omega^2+(2\pi n/\beta)^2. \qquad (3.13)$$

This immediately leads us back to Eq. (3.3).

IV. DOUBLE-WELL ANHARMONIC OSCILLATOR

The methodology presented at the end of the preceding section can now be applied to the potential of Fig. 1, i.e., to

$$U(q)=\frac{\lambda}{4}(q^2-\mu^2/\lambda)^2. \qquad (4.1)$$

To clarify the nature of the weak-coupling ansatz that we shall make, let us rescale the position,

$$\phi \equiv q\Big/\left(\frac{\mu^2}{\lambda}\right)^{1/2}, \tag{4.2a}$$

with a corresponding scale change of the momentum,

$$\pi \equiv p\left(\frac{\mu^2}{\lambda}\right)^{1/2}, \tag{4.2b}$$

and also let

$$t \equiv \mu\tau . \tag{4.2c}$$

Then the partition function becomes

$$Z = \mathfrak{N}\oint d[\phi]\exp\left(-\frac{1}{g^2}I[\phi]\right), \tag{4.3}$$

where

$$\mathfrak{N} = \frac{1}{g\bar{\mu}}\prod_{n=1}^{\infty}(2\pi n/g\bar{\mu})^2 \tag{4.4}$$

and

$$\bar{\mu} = \mu\beta , \tag{4.5}$$

while the "coupling" g is given by

$$g = (\lambda/\mu^3)^{1/2}, \tag{4.6}$$

and the action $I[\phi]$ is simply

$$I[\phi] = \int_0^{\bar{\mu}} dt\,[\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{4}(\phi^2-1)^2]. \tag{4.7}$$

Here the dot refers to a derivative with respect to "t". The rescaling has emphasized [through Eq. (4.3)] that the approximation of dominating the partition function with modes which minimize the action works best when the coupling "g" is small.

The field configurations that dominate the action in Eq. (4.7) obey

$$\ddot{\phi}_c = \phi_c({\phi_c}^2 - 1). \tag{4.8}$$

There are two, trivial, periodic solutions of Eq. (4.8), namely,

$$\phi_\pm(t) = 1 , \tag{4.9}$$

for which the action vanishes. These configurations will give separable contributions to the partition function provided the temperature β^{-1} is small compared to the energy barrier $\mu^4/4\lambda$ (see Fig. 1), i.e., when

$$\bar{\mu}/4g^2 \gg 1 . \tag{4.10}$$

Expanding the action about either of these solutions and retaining only quadratic contributions leads to the eigenvalue equation [using Eq. (3.9)]

$$\left(-\frac{d^2}{dt^2}+2\right)Q_n(t) = (\omega_n^\pm)^2 Q_n(t). \tag{4.11}$$

As in Eq. (3.13), we have for the eigenvalues,

$$(\omega_n^\pm)^2 = 2 + (2\pi n/\bar{\mu})^2 \tag{4.12}$$

since the period of the $Q_n(t)$ is $\bar{\mu}$. These simple harmonic oscillators contribute $Z_\pm$, respectively, to the partition function, where

$$Z_\pm = (2\sinh\bar{\mu}/\sqrt{2})^{-1}. \tag{4.13}$$

In addition to the trivial solutions $\phi_\pm = 1$, there are other, explicitly τ dependent, quantum modes which satisfy Eq. (4.8). The most general such solution is parametrized by the two constants α and t_0, and can be expressed as $\pm\phi_c(t)$, where

$$\phi_c(t) = [2(1-\alpha^2)]^{1/2}\,\mathrm{sn}[\alpha(t-t_0)] \tag{4.14}$$

and where "α" is related to the modulus "k" of the elliptic function,

$$\alpha^2 = (1+k^2)^{-1}. \tag{4.15}$$

Thus Eq. (4.14) can also be written as

$$\phi_c(t) = \left(\frac{2k^2}{1+k^2}\right)^{1/2}\mathrm{sn}[(t-t_0)/(1+k^2)^{1/2}]. \tag{4.16}$$

The constants α and t_0 represent a modified scale invariance (the amplitude changes under a dilatation) and translational invariance, respectively.

The elliptic function $\mathrm{sn}\,x$ is odd in x and oscillates between ± 1 (as does $\sin x$, the limit of $\mathrm{sn}\,x$ as $k \to 0$) with a period of $4K$, where

$$K(k^2) = \int_0^{\pi/2}\frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}} \tag{4.17}$$

is the complete elliptic integral of the first kind. Since the period in t of our solution is $\bar{\mu}$, we have

$$\bar{\mu} = 4K(1+k^2)^{1/2}, \tag{4.18}$$

i.e., the modulus of the elliptic function is set by the temperature. Thus, the temperature removes the remaining modified scale invariance.

So as to better understand the periodic nature of this solution, let us consider the first integral of Eq. (4.8):

$$\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{4}(\phi^2-1)^2 = \epsilon , \tag{4.19}$$

where periodic motion will ensue provided that the integration constant ϵ is bounded (see Fig. 2),

$$-\tfrac{1}{4} \leq \epsilon \leq 0 . \tag{4.20}$$

The temperature fixes not only k^2 but also ϵ, for

$$k^2 = (1-2|\epsilon|^{1/2})/(1+2|\epsilon|^{1/2}). \tag{4.21}$$

It is evident from Fig. 2 that ϵ determines the amplitude of $\phi_c(t)$, i.e., unlike linear oscillations the amplitude and period are linked.

Two interesting temperature limits occur for the extremes of ϵ. When $\epsilon \to 0$, $k^2 \to 1$ so that $K \to \infty$, i.e., from Eq. (4.18), $\beta \to \infty$. On the other hand, when $\epsilon \to -\frac{1}{4}$, $k^2 \to 0$ and $K \to \pi/2$ so that β

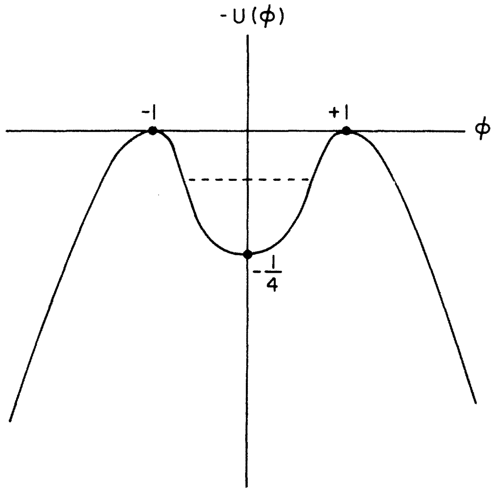

FIG. 2. Inverted double-well potential. The dashed line represents a fixed "energy" in this well.

$\to \beta_{\min} = 2\pi/\mu$. In the first case, as we approach $\phi = \pm 1$, both $\dot\phi$ and $\ddot\phi$ approach zero [see Eqs. (4.8) and (4.19)] so that the period is extended. Indeed, in the limit, we spend all our available "time" in going from one peak of Fig. 2 to the other, i.e.,

$$\phi_c(t) \xrightarrow[\beta\to\infty]{} \tanh[(t-t_0)/\sqrt{2}\,], \tag{4.22}$$

which is just the zero-temperature-kink solution of Eq. (1.2). In the sense that $\phi_c(t)$ for β finite traverses a path from negative to positive values and back again, it consists of a kink-antikink pair.

In the second limiting case, the amplitude for oscillations vanishes when the temperature reaches a maximum value, i.e.,

$$kT_{\max} = \mu/2\pi\,. \tag{4.23}$$

As we approach this temperature from below, $\operatorname{sn} x \to \sin x$ and the amplitude in Eq. (4.16) becomes small, i.e., we have linear oscillations about the bottom of the well in Fig. 2. At the moment, this limiting temperature will not be an obstacle for we are primarily concerned with the calculation of the partition function in the low-temperature limit.

To this end, we wish to determine the contribution Z_c that $\phi_c(t)$ makes to the partition function. In particular, by rewriting Eq. (4.4) as

$$\mathfrak{N} = \frac{1}{g\bar\mu}\left[\det{}'\,\frac{1}{g^2}\left(-\frac{d^2}{dt^2}\right)\right]^{1/2}, \tag{4.24}$$

where the prime refers to the deletion of the zero eigenvalue in the determinant, we can say that

$$Z_c = \frac{1}{g\bar\mu}\,\frac{[\det{}'(1/g^2)(-d^2/dt^2)]^{1/2}}{\{\det(1/g^2)[-d^2/dt^2+U''(\phi_c)]\}^{1/2}} \times \exp\left(\frac{-1}{g^2}\,I[\phi_c]\right). \tag{4.25}$$

Thus the partition function is given by

$$Z = \frac{2}{g\bar\mu}\,\frac{[\det{}'(1/g^2)(-d^2/dt^2)]^{1/2}}{\{\det(1/g^2)[-d^2/dt^2+U''(\phi_+)]\}^{1/2}} \times\left\{1+\frac{\{\det(1/g^2)[-d^2/dt^2+U''(\phi_+)]\}^{1/2}}{\{\det(1/g^2)[-d^2/dt^2+U''(\phi_c)]\}^{1/2}} \times\exp\left(\frac{-I[\phi_c]}{g^2}\right)\right\}. \tag{4.26}$$

The coefficient of the large curly brackets has been evaluated as $Z_+ + Z_-$ [see Eq. (4.13)]. We now calculate the remaining term.

First, the action is given by

$$I[\phi_c] = T + V\,, \tag{4.27}$$

where

$$T = \tfrac{1}{2}\int_0^{\bar\mu} dt\,\dot\phi_c^{\,2} \tag{4.28}$$

and

$$V = \tfrac{1}{4}\int_0^{\bar\mu} dt(\phi_c^{\,2}-1)^2\,. \tag{4.29}$$

Explicitly evaluating these quantities yields

$$T = \frac{4}{3(1+k^2)^{3/2}}\,[-K(1-k^2)+E(1+k^2)] \tag{4.30}$$

and

$$V = \frac{4}{3(1+k^2)^{3/2}} \times\left[-\frac{K}{4}\,(1+2k^2-3k^4)+E(1+k^2)\right], \tag{4.31}$$

where E is the complete elliptic integral of the second kind. In the zero-temperature limit, we have the equipartition result

$$\beta\to\infty:\; T = V = \tfrac{1}{2}I = 2\sqrt{2}/3\,. \tag{4.32}$$

Next, we determine the second ratio of determinants in Eq. (4.26). Note that the eigenvalue equation (3.9) becomes, for ϕ_c,

$$\left\{-\frac{d^2}{dt^2}+\frac{6k^2}{1+k^2}\operatorname{sn}^2[t/(1+k^2)^{1/2}]-1\right\}Q_n(t) = \omega_n^{\,2}Q_n(t)\,, \tag{4.33}$$

where $\tau_0 = 0$ has been chosen for convenience. Letting

$$s = t/(1+k^2)^{1/2} \tag{4.34}$$

gives

$$\left[\frac{d^2}{ds^2}+(1+k^2)(1+\omega_n{}^2)-6k^2\,\mathrm{sn}^2 s\right]Q_n(s)=0\,, \tag{4.35}$$

which is just the Jacobian form of Lamé's equation[12]

$$\frac{d^2Q}{ds^2}+[h-i(i+1)(k\,\mathrm{sn}s)^2]\,Q=0\,, \tag{4.36}$$

for $i=2$. The five smallest eigenvalues of this equation are known, as well as the corresponding eigenfunctions.[13] The latter are quadratic polynomials in the elliptic functions (see Table I). The remaining eigenvalues occur in ascending pairs and their eigenfunctions are transcendental in the elliptic functions.

The first two eigenvalues present difficulties in the calculation of the determinant for they do not satisfy the restriction $\omega_n{}^2>0$. Let us first concentrate on $\omega_0{}^2$.

The vanishing of $\omega_0{}^2$ is the familiar zero-mode problem, a reflection of translational invariance. Indeed, the eigenfunction $Q_0(s)$ is proportional to the derivative of ϕ_c with respect to s since

$$\frac{d}{ds}\,\mathrm{sn}s=\mathrm{cn}s\,\mathrm{dn}s\,. \tag{4.37}$$

In the form of Eq. (3.10), the zero mode is a problem. However, if one keeps all terms, including the quartic contribution, in the expansion of $I[\phi]$, the functional integral over this mode is well defined. Proceeding in this way, though, would miss the central point that all translates of $\phi_c(t)$ should be included on an equal footing in the functional integral. To accomplish this, we employ the Faddeev-Popov technique[14] at finite temperature,[7,15] resulting in

$$g/\omega_0 \underset{\text{all }\beta}{\longrightarrow} \bar{\mu}(2\pi)^{-1/2}(2T)^{1/2}\,, \tag{4.38}$$

where the first factor is the length of the integration region, the second a reflection of an unused Gaussian integral, and the third the length of the zero-mode eigenvector.

We are now in a position to understand the negative eigenvalue $\omega_{0'}{}^2$. Unlike the kink, our basic solution $\phi_c(t)$ is not a monotonically increasing function of t. Rather, the periodicity condition produces maxima and/or minima. Thus the derivative of $\phi_c(t)$ possesses zero (s), i.e., the zero-mode eigenfunction has node (s). There must then exist an eigenfunction with a lower (i.e., negative) eigenvalue. This negative eigenvalue represents the instability of the kink-antikink solution at finite temperature. As in the case of the zero-mode problem, the difficulty arises in truncating the expansion of the action at the quadratic order. The positivity of the quartic term ensures that the integrals are well defined. However, to use this term as a damping factor in the low-temperature limit would be deceiving. Indeed, as $\beta\to\infty$, we note that both $Q_{0'}$ and Q_0 approach $\mathrm{sech}^2 s$ while $\omega_{0'}\to\omega_0=0$. This indicates that the appropriate evaluation in the $\beta\to\infty$ limit would be

$$g/\omega_{0'} \underset{\beta\to\infty}{\longrightarrow} \bar{\mu}(2\pi)^{-1/2}(2T)^{1/2}\,. \tag{4.39}$$

To complete the evaluation of the partition function in the zero-temperature limit, we evaluate the ratio R of the remaining eigenvalues,

$$R=\left(\prod_{n=2}^{\infty} g/\omega_n\right)\Bigg/\left(\prod_{n} g/\omega_n^+\right), \tag{4.40}$$

as $\beta\to\infty$. Writing

$$\omega_n{}^2=\epsilon_n{}^2+2\,, \tag{4.41a}$$

$$(\omega_n^+)^2=(\epsilon_n^+)^2+2\,, \tag{4.14b}$$

where $\epsilon_n{}^2$ and $(\epsilon_n^+)^2$ satisfy, respectively,

$$\left[-\frac{d^2}{dt^2}-3\,\mathrm{sech}^2(t/\sqrt{2})\right]Q_n(t)=\epsilon_n{}^2Q_n(t)\,, \tag{4.42a}$$

$$\left[-\frac{d^2}{dt^2}\right]Q_n^+(t)=(\epsilon_n^+)^2Q_n^+(t)\,, \tag{4.42b}$$

we have

TABLE I. The five smallest eigenvalues of the Lamé equation, with the corresponding un-normalized eigenfunctions.

n	$Q_n(s)$	h	$\omega_n{}^2$	$\omega_n{}^2(k=1)$
0′	$1-[1+k^2-(1-k^2+k^4)^{1/2}]\mathrm{sn}^2 s$	$2[1+k^2-(1-k^2+k^4)^{1/2}]$	$1-\{1+3[(1-k^2)/(1+k^2)]^2\}^{1/2}$	0
0	$\mathrm{cn}s\,\mathrm{dn}s$	$1+k^2$	0	0
1	$\mathrm{sn}s\,\mathrm{dn}s$	$1+4k^2$	$3k^2/(1+k^2)$	$\frac{3}{2}$
1′	$\mathrm{sn}s\,\mathrm{cn}s$	$4+k^2$	$3/(1+k^2)$	$\frac{3}{2}$
2	$1-[1+k^2+(1-k^2+k^4)^{1/2}]\mathrm{sn}^2 s$	$2[1+k^2+(1-k^2+k^4)^{1/2}]$	$1+\{1+3[(1-k^2)/(1+k^2)]^2\}^{1/2}$	2

$$R=\exp\left\{\tfrac{1}{2}\int_0^\infty d\epsilon'[\rho^+(\epsilon')-\rho(\epsilon')]\ln g^{-2}(\epsilon'+2)\right\}, \tag{4.43}$$

where $\rho^+(\epsilon')$ and $\rho(\epsilon')$ are the energy-level densities for the Schrödinger-type equations (4.42b) and (4.42a), respectively. In the zero-temperature limit, we isolate the eigenvalues $\omega_{0'}$, ω_0, ω_1, and $\omega_{1'}$ for separate evaluation (while ω_2 is at the onset of the continuum) so that

$$\int_0^\infty d\epsilon'[\rho^+(\epsilon')-\rho(\epsilon')]=4\,. \tag{4.44}$$

Thus

$$R=4g^{-4}c\,, \tag{4.45}$$

where the constant c,

$$c=\exp\left\{\tfrac{1}{2}\int_0^\infty d\epsilon'[\rho^+(\epsilon')-\rho(\epsilon')]\ln(1+\epsilon'/2g^2)\right\}, \tag{4.46}$$

has been evaluated by Langer[15] to be ln6. Gathering these results together gives (for $\beta\to\infty$)

$$Z=(\sinh\bar\mu/\sqrt{2})^{-1}\times\left\{1+\frac{[\bar\mu(T/\pi)^{1/2}]^2}{(\sqrt{\tfrac{3}{2}}g^{-1})^2}(4g^{-4}\ln 6)\exp(-2T/g^2)\right\}. \tag{4.47}$$

The second term in the braces can be interpreted as the first contribution of the calorons to the partition function. Considering a dilute gas of such calorons implies that what we have in braces in Eq. (4.47) is just the even part of an exponential expansion (see Refs. 7, 8, and 15 for further details). Thus the partition function is given as the zero-temperature limit of

$$Z=\frac{\cosh\left[\tfrac{4}{3}(2\sqrt{2}\ln 6/\pi)^{1/2}\frac{\bar\mu}{g}\exp(-2\sqrt{2}/3g^2)\right]}{\sinh(\bar\mu/\sqrt{2})}, \tag{4.48}$$

i.e.,

$$Z=e^{-\beta(E_0-\Delta E/2)}+e^{-\beta(E_0+\Delta E/2)}, \tag{4.49a}$$

where

$$E_0=\mu/\sqrt{2} \tag{4.49b}$$

and

$$\Delta E=\tfrac{8}{3}(2\sqrt{2}\ln 6/\pi)^{1/2}\mu\left(\frac{\mu^3}{\lambda}\right)^{1/2}\exp(-2\sqrt{2}\mu^3/3\lambda)\,. \tag{4.49c}$$

The numerical coefficient in Eq. (4.49c) is 3.4 compared to 3.8 from Eq. (1.3c).

V. DISCUSSION

By explicitly calculating (in the dilute-gas approximation) the quantum partition function for a nonlinear Lagrangian, we have encountered some novel features whose generality deserves further study. For example, there exists a limiting temperature in the double-well anharmonic oscillator, above which instantons do not exist. This is in contrast to theories (such as quantum chromodynamics) where classical scale invariance exists and instantons are only gradually squeezed out of the physical region.[6,8] In the former case, the limiting temperature is of the order of the mass term in the Lagrangian [see Eq. (4.23)] while in the latter case, the renormalization mass sets the scale.[6] In either situation, for a sufficiently high temperature, the degrees of freedom represented by the instanton are not effectively activated.

We have also seen that the kink-antikink mode has a negative eigenvalue, seemingly implying its instability at finite temperature. Much remains to be done before this instability is understood. For example, do other instanton–anti-instanton pairs display this behavior in one or more dimensions? Can one reasonably calculate an imaginary part of the partition function and associate this with a complex energy eigenvalue? Will this imaginary part be related to the lifetime of the instanton–anti-instanton pair and, from this, can we learn about their interactions? We leave such questions for future consideration.

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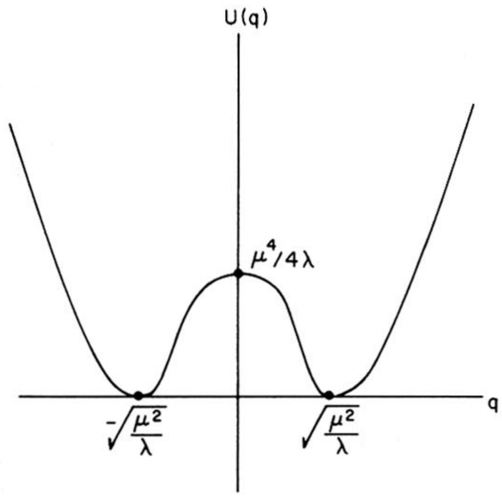

FIG. 1. Potential for the double-well anharmonic oscillator.

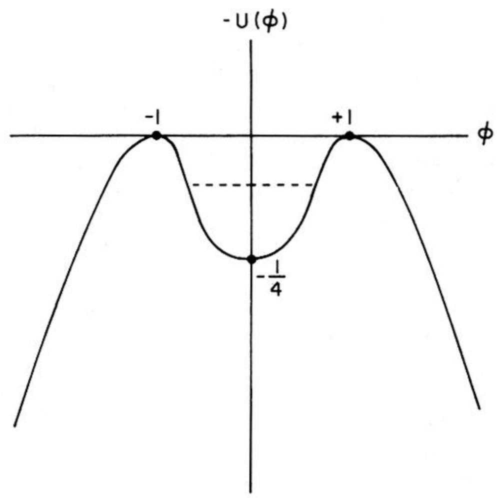

FIG. 2. Inverted double-well potential. The dashed line represents a fixed "energy" in this well.