PHYSICAL REVIEW D VOLUME 18, NUMBER 8 15 OCTOBER 1978

Closed vortices and the Hopf index in classical field theory

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(Received 29 November 1977)

Closed-vortex configurations in classical field theory are investigated here. It is shown in detail that in the Abelian Higgs model such configurations are unstable by collapse. Closed-ring configurations having unit Hopf index and that explicitly exhibit the features of a twisted vortex are constructed for theories where a three-component scalar field is present. However, it is shown that in renormalizable theories the Hopf charge does not ensure the existence of stable solutions. It is proved that a nonlinear σ model where an interaction which has fourth-power field derivatives is present has a twisted-ring solution. A lower bound for the mass and estimates for the radius and mass are given.

I. INTRODUCTION

In the last few years there has been wide interest in localized solutions of field theories such as the Higgs and chiral models,[1-4] which become states bearing particle properties in the corresponding quantum field theory.

Nielsen and Olesen[2] pointed out that vortex solutions of the Higgs model behave classically as Nambu strings in the strong-coupling limit (see Ref. 5). Nambu and Mandelstram[6] proposed models for hadrons based on finite-energy field configurations built with vortex lines and monopoles. In this context, a closed vortex can be interpreted as a Pomeron. Another interpretation for a ring vortex has been proposed in Ref. 7.

In this paper we study closed-vortex configurations in Abelian and non-Abelian Higgs models and in the nonlinear σ model. We also study the possibility of the existence of solutions of that type.

The Abelian Higgs model is defined, as usual, by the Lagrangian

$$\mathcal{L}(x) = -\tfrac{1}{4}F_{\mu\nu}{}^2 + |(\partial_\mu + ieA_\mu)\Phi|^2 - \frac{\lambda}{2}\left(|\Phi|^2 - \frac{\mu^2}{2\lambda}\right)^2, \tag{1.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this theory closed-vortex configurations have no topological charge associated with them. We give an explicit ring-vortex configuration [Eq. (2.2)] which, for a large radius, is like a Nielsen-Olsen vortex curved and closed by its ends. However, it goes smoothly into the vacuum of the theory for vanishing radius. The energy of this static configuration is finite for all radii. We also performed a variational calculation of the energy under the constraint of fixed radius for the ring vortex. The results are plotted in Fig. 1 showing that in Abelian models closed vortices will indeed collapse classically, because the energy decreases monotonically when the radius decreases.

In Sec. II we also discuss multiring configurations which have the same instability properties. Rotating field configurations are considered in Sec. II and Appendix B. We show that the energy is bounded from below by $|\vec{L}|^2$. However, this does not ensure stability because angular momentum as well as energy can be radiated classically in a continuous fashion.

In field theories where a three-component scalar field is present, there can be, in principle, field configurations with a nonzero Hopf index.[8] This is the case for nonlinear σ model with the Lagrangian.

$$\mathcal{L}_\sigma(x) = \frac{\mu^2}{2g^2}(\partial_\mu\sigma_a)^2 + \Lambda(x)(\sigma_a{}^2 - 1) \tag{1.2}$$

[where μ is a mass parameter and $\Lambda(x)$ a Lagrange

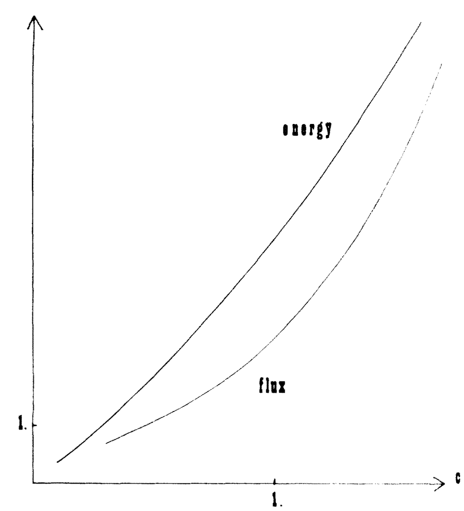

FIG. 1. Energy and magnetic flux of a ring-vortex configuration [Eq. (2.2)] as a function of the filament radius c.

multiplier] and the SU(2) Higgs model (the Georgi-Glashow model)

$$\mathcal{L}(x) = \tfrac{1}{2}(D_\mu Q_a)^2 - \tfrac{1}{4}(F^a_{\mu\nu})^2 + \frac{\lambda}{8}\left(Q_a^{\,2} - \frac{\mu^2}{2\lambda}\right)^2 , \tag{1.3}$$

where

$$D_\mu Q_a \equiv \partial_\mu Q_a + g\epsilon_{abc} A^b_\mu Q_c$$

and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc} A^b_\mu A^c_\nu \ .$$

As we shall see in Sec. III, a configuration with a unit Hopf index (Q_H) is a closed vortex with "twist" one.[4] It has the desired toroidal symmetry. We analyze the properties of this type of vortices, and we show explicitly that configurations with $Q_H = 1$ and *finite* energy exist. Although these vortices have a nonzero topological charge they are not stable against collapse in the σ model and in the Georgi-Glashow model with zero gauge field. Their energy is proportional to the ring radius. The field configuration goes smoothly into vacuum when c vanishes in the sense of distribution theory. On the contrary, the Hopf charge density (ρ_H) gives a Dirac δ function in the same limit. This is due to the fact that $\rho_H(\vec{r})$ contains a product of the field and its derivatives taken at the same point. As is well known this sort of product is not always well defined.

In the Georgi-Glashow model, finite-energy vortex configurations with nontrivial Hopf charge and nonzero gauge field are possible. However, topological considerations and scale arguments do not ensure nor exclude the existence of closed-vortex solutions. We only prove that if a classical solution with $Q_H = 1$ exists in that model, it becomes an unstable particle in the quantum theory because the "potential barrier" between it and the vacuum may be only of finite height. We conclude that the Hopf index does not provide topologically static solutions in renormalizable field theories.

There exist closed-vortex solutions in the σ model modified by adding to the Lagrangian (1.2) fourth-order terms like

$$\mathcal{L}_I(x) = -g_1[(\partial_\mu\sigma_a)^2]^2 - g_2(\partial_\mu\sigma_a\partial_\mu\sigma_b)^2 \ , \tag{1.4}$$

where we assume $g_1 > 0$, $g_1 + g_2 \geq 0$. In Sec. III we prove the following lower bound for the mass of a twisted-vortex configuration:

$$E(\hat{\phi}) \geq \frac{32\pi^2 g_1}{I(Q_H)} \frac{Q_H^{\,2}}{c} + \frac{8\pi\mu^2}{N(Q_H)} c Q_H g^{-2} , \tag{1.5}$$

where I and N remain bounded for c going to zero and infinity, respectively. An estimate on the mass and radius of the solution gives for $Q_H = 1$

$$c = k_1 g(g_1)^{1/2}\mu^{-1} \ \text{ and } \ E = \mu k_2 \frac{(g_1)^{1/2}}{g} , \tag{1.6}$$

where k_1 and k_2 are numerical coefficients. Thus the mass of the toroidal soliton in units of μ may be small or large depending on the ratio $(g_1)^{1/2}/g$.

At the quantum level it is possible that time-dependent classical vortices correspond to particles or resonances, even if the classical configurations are not stable. Vortices oscillating radially seem good candidates for breather-type solutions.[9] The monotonic increase of the vortex energy with its length provides an inward force preventing the spreading of the field configuration. In this scheme, it seems possible that for each radial mode there exist a set of rotating modes with increasing angular momentum and energy [see Eqs. (2.5) and (B4)].

II. ABELIAN HIGGS MODEL

A ring-vortex configuration of large radius (c) can be easily constructed by taking an infinite-vortex solution of Nielsen-Olesen type, curving and closing it. Thus the magnetic field will be localized in a ring of cross-section $O(m^{-2})$ (where $m \equiv e\mu/\sqrt{\lambda}$), and the Higgs field will differ appreciably from its vacuum value only in a core of section $O(\mu^{-2})$. Furthermore, for $mc \gg 1$, a magnetic flux ($2\pi/e$) will flow around the ring. For all nonzero c, this configuration is topologically different from the vacuum because the Higgs field vanishes on the center of the core (the "filament"). Because the vacuum manifold is U(1), the Higgs field configurations that vanish on a closed line (not interesting with itself) are classified by the homotopy classes $\pi_1(T^2) = Z \times Z$.[10] Here T^2 stands for the two-dimensional torus. This implies that the more general closed-vortex configuration is characterized by two integers in this model. They give the phase change of the Higgs field (in units of 2π) when one describes a closed path on a toroidal surface that surrounds the filament. The first integer corresponds to a curve like A in Fig. 2 and the second one (m) to a path like B in Fig. 2. For $mc \gg 1$, the values of n and m also give the magnetic flux (in units of $2\pi/e$) flowing around the filament and

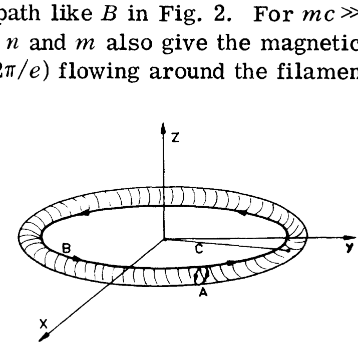

FIG. 2. A ring-vortex configuration.

through the hole encircled by it, respectively. If $m \neq 0$, the Higgs field must also vanish on a line passing through the hole, otherwise it would be multivalued there. In other words, a configuration with $n \neq 0 \neq m$ corresponds to two vortices of types $(n, 0)$ and $(m, 0)$ interlocked. For a $(1, 0)$ configuration, the boundary conditions read as follows, in toroidal coordinates (see Appendix A):

$$\vec{A}(\vec{r}) = \frac{\hat{e}_\beta}{ec}(\cosh\eta - \cos\beta) + O\left(\exp\left[\frac{-mc\sqrt{2}}{(\beta^2+\eta^2)^{1/2}}\right]\right),$$
$$\Phi(\vec{r}) = \frac{\mu}{(2\lambda)^{1/2}} e^{i\beta}\left\{1 + O\left(\exp\left[-\frac{\mu c\sqrt{2}}{(\beta^2+\eta^2)^{1/2}}\right]\right)\right\}, \tag{2.1}$$

and

$$\lim_{\eta\to\infty}\vec{A}(\vec{r}) = 0 \ , \quad \lim_{\eta\to\infty} e^{\eta}\Phi(\vec{r}) = \text{const} < \infty \ .$$

These are necessary conditions to get a finite energy. Although the vortex discussed above is topologically different from the vacuum for $c > 0$, there is no topological charge associated with it. There is no obstacle to continuously deform (varying c) a closed-vortex configuration into the vacuum, as is shown by the following field configuration:

$$\Phi_c(\vec{r}) = e^{i\beta}\left[1 - \exp\left(-B\xi - \frac{1}{c^2}\frac{D\xi}{1+E\xi}\right)\right],$$
$$\vec{A}_c(\vec{r}) = \frac{1}{e}\vec{\nabla}\bar{\beta}\left[1 - \exp\left(-\frac{\xi^2}{1+E\xi}\right)\right], \tag{2.2}$$

where

$$\xi \equiv m(r^2 + c^2 - 2rc\sin\theta)^{1/2},$$
$$\bar{\beta} \equiv 2\cos\theta\left[\tan^{-1}\left(\frac{r}{c}\right) - \frac{\pi}{2}\right] + \tan^{-1}\left(\frac{2rc\cos\theta}{r^2 - c^2}\right). \tag{2.3}$$

Here (r, θ, φ) are spherical coordinates, and B, D, E are positive constants. For all nonzero c, (2.1) describes a vortex of type $(1, 0)$ and finite energy. The magnetic flux through the half-plane $y > 0$ (or $x > 0$) varies continuously from $(2\pi/e)$ to zero when c goes from ∞ to zero. The energy also vanishes in the $c \to 0+$ limit.

We also performed a variational minimization of the energy by constraining the vortex to have a toroidal symmetry and a filament of fixed radius c. The results are plotted in Fig. 1 showing that the collapse of the vortex ring is indeed energetically favorable. Actually, this collapse has been experimentally observed in type-II superconductors.[11] However, it must be noted that the time evolution of such vortices is not described by the relativistic Higgs model.[12]

All the above discussions about Abelian vortices can be repeated without essential changes for a closed-vortex configuration in an SU(2) Higgs model obtained by closing the infinite-vortex solution described in Ref. 13.

For a rotating (i.e., nonstatic) field configuration a lower bound to the energy can be found. We take the z axis in the direction of the rotation axis. Then the orbital angular momentum of the gauge field is

$$L = \int d^3x\,\rho(E_z H_\rho - E_\rho H_z) \ ,$$

where

$$\rho = (x^2+y^2)^{1/2}, \quad E_i = F^{0i}, \quad H_i = \tfrac{1}{2}\epsilon_{ijk}F^{jk} \ . \tag{2.4}$$

Then using

$$\mathcal{H}_M = \tfrac{1}{2}(E_i^{\,2} + H_i^{\,2}) \geq |E_z H_\rho - E_\rho H_z|$$

and the Schwarz inequality for the functions $(\mathcal{H}_M)^{1/2}$ and $\rho|E_z H_\rho - E_\rho H_z|^{1/2}$, it follows that

$$E_M \equiv \int \mathcal{H}_M(\vec{x})\,d^3x \geq \frac{L^2}{\int d^3x\,\rho^2\mathcal{H}_M} \ . \tag{2.5}$$

The integral on the right-hand side is convergent because of (2.1). In fact, lower bounds for the energy of rotating configurations can be found in any field theory (see Appendix B).

For rotating-vortex configurations, the bound (2.5) does not ensure their stability (classically) because they could, in principle, radiate their energy as well as their angular momentum until they collapse.

III. CLOSED VORTICES AND THE HOPF CHARGE

In this section we present and discuss field configurations bearing nonzero topological charge and having the structure of closed vortices. They exist in theories where a scalar field with three components in internal space is present. The topological charge is given by the Hopf index.[8]

The Hopf index classifies mappings from S^3 to S^2 (more generally from S^{2n-1} to S^n, n even). The field on S^2 will be denoted by

$$\hat{\phi}(\vec{r}) = (\hat{\phi}_1(\vec{r}), \hat{\phi}_2(\vec{r}), \hat{\phi}_3(\vec{r})) \ , \tag{3.1}$$

with

$$\sum_{a=1}^{3}[\hat{\phi}_a(\vec{r})]^2 = 1 \ .$$

Here $\vec{r} \in \mathbf{R}^3$, and we assume that the limit of $\phi_a(\vec{r})$ as $r \to \infty$ is independent of $\hat{r}$. There are several mathematically equivalent definitions of the Hopf index. It can be introduced in the following way: If $\hat{\phi}^0$ is some fixed point on S^2, then $\hat{\phi}(\vec{r}) = \hat{\phi}^0$ is the equation of a closed curve Z^0 on S^3. If Σ^0 is some two-dimensional closed connected surface on S^3 having Z^0 as border, then $\hat{\phi}(\vec{r})$ maps Σ^0 on the whole sphere S^2. Then the Hopf index Q_H of $\hat{\phi}(\vec{r})$ can be defined as the number of times $\hat{\phi}(\vec{r})$ maps Σ^0 onto S^2. More precisely, Q_H is the degree of the mapping $\hat{\phi}(\vec{r})$, restricted to Σ^0, from Σ^0 to S^2. Hence Q_H is an integer. It can be shown that Q_H is independent of the point $\hat{\phi}^0 \in S^2$ chosen. The following integral also gives Q_H:

$$Q_H = \int_{\Sigma^0} \vec{h} \cdot d\vec{\sigma} = \oint_{Z^0} \vec{a} \cdot d\vec{l} \ , \tag{3.2}$$

where $\vec{h}$, the "flux density," is given by

$$\vec{h} = \frac{1}{8\pi} \epsilon_{abc} \hat{\phi}_a \vec{\nabla}\hat{\phi}_b \times \vec{\nabla}\hat{\phi}_c \tag{3.3}$$

and

$$\vec{h} = \text{curl}\,\vec{a}(\vec{r}) \ .$$

$\vec{h} \cdot d\vec{\sigma}$ is the normalized surface elements of S^2 mapped by $\hat{\phi}(\vec{r})$ to S^3. This shows immediately that (3.2) indeed gives the Hopf index as defined above. A topological charge density can be introduced as

$$\rho_H(\vec{r}) = -\vec{a}(\vec{r}) \cdot \vec{h}(\vec{r}) \ , \tag{3.4}$$

then[8]

$$Q_H = \int \rho_H(\vec{r}) d^3r \ . \tag{3.5}$$

The normalized field $\hat{\phi}(\vec{r})$ can be identified with the σ field in the nonlinear σ model (1.2). In the Georgi-Glashow model one can take $\hat{\phi}_a \equiv Q_a/(Q_b{}^2)^{1/2}$, where Q_a is the Higgs field in the isovector representation. In the latter model $\vec{h}$ and $\vec{a}$ are not gauge invariant, but the Hopf index is. This can be explicitly shown by first recalling that $\vec{h}$ is related to the 't Hooft magnetic field[16]

$$\vec{B} = \hat{\phi}_a \vec{H}_a - \frac{1}{2g} \epsilon_{abc} \hat{\phi}_a \vec{D}\hat{\phi}_b \times \vec{D}\hat{\phi}_c \tag{3.6}$$

through

$$\begin{aligned} \vec{B} &= \text{curl}(\hat{\phi}_a \vec{A}_a) + \frac{4\pi}{g} \vec{h} \\ &= \text{curl}\left(\hat{\phi}_a \vec{A}_a + \frac{4\pi}{g} \vec{a}\right) \ . \end{aligned} \tag{3.7}$$

Then we can write the Hopf index as

$$Q_H = -\frac{g}{4\pi} \hat{\phi}_a^0 \int_{Z^0} \vec{A}_a \cdot d\vec{l} + \frac{g}{4\pi} \int_{\Sigma^0} \vec{B} \cdot d\vec{\sigma} \ . \tag{3.8}$$

Under an infinitesimal gauge transformation the second term is manifestly invariant and in the first one we get an exact differential which vanishes upon integration on Z^0.

A field configuration with unit Hopf index is given by

$$\begin{aligned} &\hat{\phi}_1^c = \cos\theta(\eta) \ , \\ &\hat{\phi}_2^c(\vec{r}) + i\hat{\phi}_3^c(\vec{r}) = \sin\theta(\eta) e^{i(\beta - \varphi)} \ , \end{aligned} \tag{3.9}$$

where (η, β, φ) are toroidal coordinates of radius c (see Appendix A), and $\theta(\eta)$ is a differentiable function of η with the boundary of

$$\theta(0) = \pi \ , \quad \theta(\infty) = 0 \ . \tag{3.10}$$

The flux density, the potential, and the Hopf charge density for the field (3.9) are given by

$$\vec{h} = \frac{1}{4\pi c^2} \frac{(\cosh\eta - \cos\beta)^2}{\sinh\eta} [\hat{e}_\beta + \sinh\eta\, \hat{e}_\varphi] \sin\theta\, \dot{\theta}(\eta) \ , \tag{3.11}$$

$$\vec{a} = \frac{1}{4\pi c} \frac{\cosh\eta - \cos\beta}{\sinh\eta} \{[1 + \cos\theta(\eta)]\hat{e}_\varphi + \sinh\eta[1 - \cos\theta(\eta)]\hat{e}_\beta\} \ , \tag{3.12}$$

$$\rho_H = -\frac{1}{8\pi^2\sqrt{g}} \frac{d}{d\eta} \cos\theta(\eta) \ . \tag{3.13}$$

The field (3.9) has toroidal symmetry. The points ∞ in S^3 and the filament $\rho = c$ (i.e., $\eta = \infty$) are mapped into the south and north poles of S^2, respectively. Moreover, it is a twisted vortex because the components $\hat{\phi}_2$ and $\hat{\phi}_3$ turn around the $\hat{\phi}_1$ axis when one describes a path like A or like B (see Fig. 2) around the filament. The flux density $\vec{h}$ is analogous to the ordinary magnetic field in superconductor vortices. Here, $\vec{h}$ flows not only in the $\hat{e}_\varphi$ direction but also in the $\hat{e}_\beta$ direction precisely because of the twisting. It is also interesting to note that $\vec{h}$ coincides with the 't Hooft magnetic field $\vec{B}$, at least for pure gauge fields.

The configuration given by Eq. (3.9) can be obtained in the following way: If w and z are two complex variables constrained to satisfy

$$|w|^2 + |z|^2 = 1 \ ,$$

then $f(w,z) = w/z$ gives a mapping with unit Hopf index.[8] By stereographic projection from C to S^2 and from S^3 to R^3 one arrives at Eq. (3.9) with the particular choice $\cos\theta(\eta) = 1 - 2\,\text{sech}^2\eta$. Clearly, the Hopf index is the same for all continuous functions $\theta(\eta)$ such that Eq. (3.10) holds.

The field (3.9) gives a finite-energy configuration in the nonlinear σ model and in the SU(2) Higgs model if we assume that

$$\begin{aligned} &\theta(\eta) = O(e^{-\eta}) \quad \text{for } \eta \to \infty, \\ &\theta(\eta) - \pi = O(\eta^\alpha) \quad \text{with } \alpha > 1 \quad \text{for } \eta \to 0+ . \end{aligned} \tag{3.14}$$

This last condition also ensures that $\vec{\nabla}\hat{\phi}(\vec{r})$ is a single-valued function. For the first model we find from (1.2) and (3.9) that

$$E(\hat{\phi}) = \left(\frac{2\pi}{g}\right)^2 \frac{\mu^2 c}{2} \int_0^\infty d\eta \{\dot{\theta}(\eta)^2 + [\coth\eta \sin\theta(\eta)]^2\} \ , \tag{3.15}$$

and for the second model from (1.3) and (3.9), we find that

$$E_G(\hat\phi)=\int d^3x\left[\tfrac{1}{2}(\vec\nabla Q)^2+\frac{Q^2}{2}(\vec\nabla\hat\phi_a)^2+\frac{\lambda}{8}\left(Q^2-\frac{\mu^2}{2\lambda}\right)^2\right]. \tag{3.16}$$

Here we have $Q_a\equiv\hat\phi_a(\vec r)Q(\vec r)$. We assume that $Q(\infty)=\mu/(2\lambda)^{1/2}$ and zero magnetic charge, i.e., $Q(\vec r)$, never vanishes. It is clear that the Hopf charge is one and E_G is finite for many choices of Q, for example, $Q(\vec r)=\mu/(2\lambda)^{1/2}$. It must be noted that the Euler-Lagrange equations associated with the one-dimensional "action" (3.15) have no solution of finite action fulfilling the boundary conditions (3.14). It is clear from Eqs. (3.15) and (3.16) that the minimun of the energy is obtained by letting the ring collapse, i.e., $c\to 0$. This is true for any *finite* energy configuration by scale arguments applied to the Lagrangian (1.2).[14]

When c vanishes, the field configuration (3.9) goes smoothly to the vacuum in the sense of distribution theory and not any configuration of zero energy (i.e., mass) and nonzero tcpological charge.[15] (Moreover, a static solution with zero mass is hard to understand as a particle in a Poincaré-invariant sense.)

The corresponding Hopf current density and charge density tend to distributions concentrated at the origin:

$$\lim_{c\to 0+}\rho_H(\vec r)=\delta(\vec r)\,,\qquad \lim_{c\to 0+}\vec h(\vec r)=-\frac{1}{\pi}\frac{\delta(r)}{r}\hat e_\varphi\,,\qquad \lim_{c\to 0+}\vec a(\vec r)=-\delta(z)\hat e_z\,, \tag{3.17}$$

whereas

$$\lim_{c\to 0+}\hat\phi_a(\vec r)=-\delta_{a1}\,,$$

and (3.18)

$$\lim_{c\to 0+}\vec a(\vec r)^2=0\,.$$

In other words, we find that

$$\lim_{c\to 0}\rho_H(\hat\phi_a(\vec r))\neq\rho_H(-\delta_{a1})=0\,.$$

This is not a contradiction because the product of distributions at the same point is not well defined in general.

If the gauge field is not zero, standard scale arguments[14] do not forbid the existence of stable static solutions. Then let us consider a finite-energy configuration in the SU(2) Higgs model, where $\hat\phi$ has unit Hopf index, and $\vec A_a(\vec r)$ is not pure gauge. We can take the gauge field

$$\vec A_a(\lambda,\vec r)\equiv\lambda\vec A_a(\vec r)\,, \tag{3.19}$$

and let λ vary from one to zero. In this process Q_H does not change and $E_G(\lambda)$ is finite, as can be seen from Eqs. (1.3), (3.9), (3.14), and the fact that $E_G(1)$ was finite. We return in this way to the former classical unstable configuration.

If $E_G(\lambda)$ increases in some interval when λ decreases from one to zero, the field configuration for $\lambda=1$ may be classically stable (metastable quantum mechanically).

On the contrary, this argument does not apply to the 't Hooft–Polyakov[16] monopole because a pure Higgs configuration with nonzero Brouwer degree has infinite energy.

There is a stable-ring solution if we add to the σ-model Lagrangian (1.2) a nonrenormalizable term like Eq. (1.4). Now scale arguments do not prevent the existence of nontrivial static solutions. Moreover, the energy of a configuration with nonzero Q_H is bounded from below for all c. The proof goes as follows: Let us consider the tensors

$$r_a^i=\epsilon^{ijk}\epsilon_{abc}\partial_j\hat\phi_b\partial_k\hat\phi_c\,,\qquad s_a^i=a_i\hat\phi_a\,,\qquad W_a^i=\epsilon_{abc}\epsilon^{ijk}a_j\hat\phi_b\partial_k\hat\phi_c\,,$$

and the positive-definite scalar product between this type of tensor,

$$(r,s)\equiv\int d^3x\, r_a^i s_a^i\,.$$

Then the Schwarz inequality for this scalar product applied to r_a^i and s_a^i together with Eqs. (3.3)–(3.5) gives the lower bound

$$E_I(\hat\phi)=\int d^3x\{g_1[(\vec\nabla\hat\phi_a)^2]^2+g_2(\vec\nabla\hat\phi_b)^2\}\geqslant\frac{32\pi^2 g_1}{Ic}Q_H^{\,2}\,, \tag{3.22}$$

where

$$I\equiv\frac{1}{c}\int d^3x\,\vec a^2\,. \tag{3.23}$$

From Eqs. (3.12) and (3.14) it follows that the integral in Eq. (3.23) converges for all $c<\infty$. By dimensional arguments I is of order c^0 and in fact I_c vanishes in the $c\to 0+$ limit [see Eq. (3.18)].

A lower bound for

$$E_\sigma(\hat\phi)=\frac{\mu^2}{2g^2}\int d^3x(\vec\nabla\hat\phi_a)^2 \tag{3.24}$$

follows from the relation

$$\vec a^2[(\vec\nabla\hat\phi_a)^2]^2\geqslant(W_a^i)^2(\partial_j\hat\phi_b)^2\geqslant 64\pi^2\rho_H^{\,2}\,.$$

Then upon integration we have

$$E_\sigma\geqslant 8\pi\int d^3x\frac{|\rho_H(\vec r)|}{|\vec a|}\geqslant\frac{4\pi c}{N(Q_H)}Q_H\mu^2 g^{-2}\,, \tag{3.25}$$

where

$$N \equiv c \max_{\vec{r} \in R^3} |\vec{a}(\vec{r})| \tag{3.26}$$

is a finite number for $0 \leqslant c \leqslant \infty$. Then (1.5) follows from (3.22) and (3.25). On the other hand, Eq. (3.9) provides a finite upper bound to $E(\hat{\phi}) = E_\sigma(\hat{\phi}) + E_I(\hat{\phi})$. This completes the proof. By extremizing (1.5), the order of magnitudes of the exact solution parameters as given in Eq. (1.6) is obtained.

It seems a difficult task to compute the explicit exact solution because in toroidal coordinates the equations of motion are, *a priori*, not separable in the SU(2) Higgs model and in the σ model.

Finally, it can be pointed out that, in principle, continuous field configurations with Hopf charge higher than one can also be constructed.[8] It has also been shown that the Hopf charges are additive.[17]

IV. FINAL REMARKS

(1) The fact that the integral I converges follows directly from the finitness of E_σ and dimensional arguments without using the explicit forms given in Eqs. (3.12)–(3.14). A necessary condition for $E_\sigma < \infty$ is that $\vec{\nabla}\hat{\phi}_a = o(r^{-3/2})$ for $r \to \infty$. Thus from Eq. (3.3) $\vec{h} = o(r^{-3})$ and $\vec{a} = o(r^{-2})$ for large r, which ensure the convergence of I.

(2) It is known that the σ model restricted to static fields can describe an isotropic ferromagnet in the Landau approach. $\hat{\phi}_a$ becomes the direction of magnetization and $-\int(\mathcal{L}_\sigma + \mathcal{L}_I)d^3x$ the free energy. In this context our field solutions with nonzero Hopf charge describe metastable defects in the body of a three-dimensional ferromagnet.

(3) Field configurations with finite energy and $|Q_H| > 1$ can be simply obtained by taking $\theta(\eta)$ in Eq. (3.4) such that $\theta(\infty) - \theta(0) = n\pi$, where n is an integer.

APPENDIX A

The toroidal coordinates (η, β, φ) are related to Cartesian coordinates (x, y, z) through

$$x + iy = \frac{c \sinh\eta \, e^{i\varphi}}{\cosh\eta - \cos\beta}\ , \quad z = \frac{c \sin\beta}{\cosh\eta - \cos\beta}\ , \tag{A1}$$

where $0 \leqslant \eta < \infty$, $0 \leqslant \beta, \varphi < 2\pi$, and c is a parameter. The surfaces $\eta = \text{const}$ are toroids, $\beta = \text{const}$ are spherical bowls, and $\varphi = \text{const}$ are planes. $\eta = \infty$ corresponds to the closed curve (the filament) $x^2 + y^2 = c^2$, $z = 0$. $\eta = 0$ gives the z axis. Large values of $r = (x^2 + y^2 + z^2)^{1/2}$ correspond to small values of both β and η. Asymptotically, we have

$$r = \frac{\sqrt{2}c}{(\beta^2 + \eta^2)^{1/2}}[1 + O(\beta^2 + \eta^2)]\ . \tag{A2}$$

Near the filament, we have $\eta \gg 1$ and

$$(x^2 + y^2)^{1/2} - c = 2ce^{-\eta}\cos\beta[1 + O(e^{-\eta})]\ ,$$

$$z = 2ce^{-\eta}\sin\beta[1 + O(e^{-\eta})]\ ;$$

the metric determinant is given by

$$\sqrt{g} = c^3 \sinh\eta(\cosh\eta - \cos\beta)^{-3}.$$

APPENDIX B

In this appendix we give a short proof of a lower bound for the energy of a rotating solution of a field theory with Hamiltonian

$$\mathcal{H}(x) = \tfrac{1}{2}\sum_{i=1}^{N}[\Pi_i{}^2 + (\vec{\nabla}\psi_i)^2] + V(\psi_1, \ldots, \psi_N)\ , \tag{B1}$$

where $V \geqslant 0$. The proof goes as follows: The orbital angular momentum is given by

$$\vec{L} = \int d^3x \sum_{i=1}^{N} \Pi_i \vec{r} \times \vec{\nabla}\psi_i\ . \tag{B2}$$

We also have

$$\begin{aligned}\mathcal{H}(\vec{r}) &\geqslant \tfrac{1}{2}\sum_i [\Pi_i{}^2 + (\hat{r} \times \vec{\nabla}\psi_i)^2] \\ &\geqslant \sum_i |\Pi_i|\,|\hat{r} \times \vec{\nabla}\psi_i|\end{aligned} \tag{B3}$$

Then using the Schwarz inequality for the functions

$$f \equiv [\mathcal{H}(\vec{r})]^{1/2}$$

and

$$g \equiv \frac{r}{\sqrt{\mathcal{H}}}\sum_i |\Pi_i|\,|\hat{r} \times \vec{\nabla}\psi_i|^{1/2} \leqslant r\sqrt{\mathcal{H}}\ ,$$

it follows that

$$E \equiv \int \mathcal{H}(\vec{r})d^3x \geqslant \frac{(\vec{L})^2}{\int d^3x\, r^2 \mathcal{H}(\vec{r})}\ . \tag{B4}$$

This gives a nontrivial lower bound for E if the trace of the inertia tensor is finite:

$$\int r^2 \mathcal{H} d^3x < \infty\ .$$

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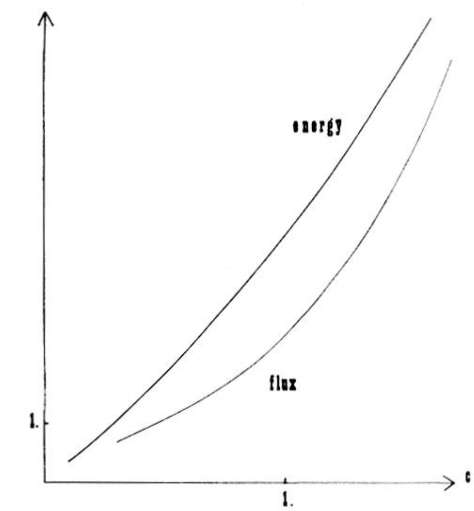

FIG. 1. Energy and magnetic flux of a ring-vortex configuration [Eq. (2.2)] as a function of the filament radius c.

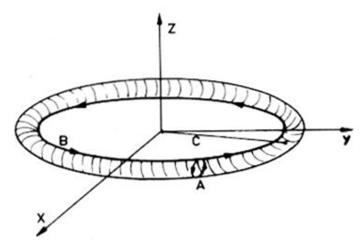

FIG. 2. A ring-vortex configuration.