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General internal gauge symmetry

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A theory is outlined in which n scalar field interact in a way that is invariant under all real, nonsingular local linear transformations of the mesons among themselves. The energy of the system is positive. The symmetry spontaneously breaks down to a compact subgroup of GL(n,R) and the gauge mesons of the broken symmetry become massive. Their longitudinal components are supplied by the derivatives of an internal metric tensor with which no physical particles are associated.

This paper is about a theory that possesses maximal internal symmetry. It is a gauge theory of n real fields ϕ_i whose Lagrangian is invariant under the local action

$$\phi_i'(x) = a_{ij}(x)\phi_j(x) \tag{1}$$

of the general linear group of all real, nonsingular $n\times n$ matrices, GL(n,R). Such a theory is more symmetrical and less arbitrary than one whose internal-symmetry group is a compact subgroup of GL(n,R), as is usually assumed.[1,2]

In what follows a suitable Lagrangian will be proposed and the equations of motion and conserved currents that follow from it will be derived. It will be shown that the Hamiltonian is non-negative.

The theory exhibits an interesting kind of Higgs mechanism. The vacuum cannot be symmetric and the symmetry group GL(n,R) breaks down spontaneously to a subgroup that is similar to SO(n). The gauge mesons associated with the noncompact part of GL(n,R) become massive. A symmetric internal metric tensor supplies the needed longitudinal components.

In order to make objects that are invariant under the general linear transformation (1), it is necessary to introduce a metric tensor $g_{ij}(x)$ that is symmetric and positive and that transforms as

$$g_{ij}'(x) = [a^{-1}(x)]_{ki}\, g_{kl}(x)[a^{-1}(x)]_{lj}\,. \tag{2}$$

In matrix notation Eqs. (1) and (2) become

$$\phi'(x) = a(x)\phi(x) \tag{3}$$

and

$$g'(x) = a^{-1T}(x)g(x)a^{-1}(x), \tag{4}$$

where the T means transpose. Evidently the form $\phi^T(x)g(x)\phi(x)$ is invariant. The tensor g_{ij} plays a somewhat similar role to that of the metric tensor in general relativity and contributes to the field equations terms not present when the gauge group is compact.

A suitable covariant derivative for ϕ is

$$\phi(x)_{;\mu} = \phi(x)_{,\mu} - A_\mu(x)\phi(x), \tag{5}$$

where the subscript comma mu means $\partial/\partial x^\mu$ and the $n\times n$ matrix of Yang-Mills fields $A_\mu(x)$ transforms as

$$A_\mu'(x) = a(x)A_\mu(x)a^{-1}(x) + a(x)_{,\mu}a^{-1}(x). \tag{6}$$

This transformation law for the gauge fields A_μ ensures that the covariant derivative $\phi_{;\mu}$ transforms like ϕ,

$$[\phi(x)_{;\mu}]' = a(x)\phi(x)_{;\mu}. \tag{7}$$

The curvature tensor $F_{\mu\nu}(x)$ is

$$F_{\mu\nu}(x) = A_\mu(x)_{,\nu} - A_\nu(x)_{,\mu} + [A_\mu(x), A_\nu(x)] \tag{8}$$

and transforms as

$$F_{\mu\nu}'(x) = a(x)F_{\mu\nu}(x)a^{-1}(x). \tag{9}$$

A suitable covariant derivative of the metric tensor g is

$$g(x)_{;\mu} = g(x)_{,\mu} + g(x)A_\mu(x) + A_\mu^T(x)g(x), \tag{10}$$

which implies that $g_{;\mu}$ transforms like g,

$$[g(x)_{;\mu}]' = a^{-1T}(x)g(x)_{;\mu}a^{-1}(x). \tag{11}$$

Similarly the covariant derivatives of $\phi^{;\nu}$, $F^{\sigma\tau}$, and $g^{;\nu}$ are

$$\phi^{;\nu}_{;\mu} = \phi^{;\nu}_{,\mu} - A_\mu\phi^{;\nu}, \tag{12}$$

$$F^{\sigma\tau}_{;\mu} = F^{\sigma\tau}_{,\mu} + [F^{\sigma\tau}, A_\mu], \tag{13}$$

$$g^{;\nu}_{;\mu} = g^{;\nu}_{,\mu} + g^{;\nu}A_\mu + A_\mu^T g^{;\nu}. \tag{14}$$

The Lagrange density

$$L = -(2e)^{-2}\,\mathrm{tr}(F_{\mu\nu}^T g F^{\mu\nu} g^{-1}) + (2f)^{-2}\,\mathrm{tr}(g_{;\mu}g^{-1}g^{;\mu}g^{-1}) + \tfrac{1}{2}\phi_{;\mu}^T g\phi^{;\mu} - \tfrac{1}{2}V(\phi^T g\phi) \tag{15}$$

is invariant under the gauge transformation (1). The numbers e and f are independent coupling constants. The variational equations of its integral over space-time are

$$\phi^{;\mu}_{;\mu}+g^{-1}g_{;\mu}\phi^{;\mu}+V'\phi=0, \tag{16}$$

$$F^{\mu\nu}_{;\nu}=[F^{\mu\nu},g^{-1}g_{;\nu}]+e^2\phi^{;\mu}\phi^Tg-(e^2/f^2)g^{-1}g^{;\mu}, \tag{17}$$

$$g^{;\mu}_{;\mu}=g_{;\mu}g^{-1}g^{;\mu}+(f^2/2e^2)g[g^{-1}F^T_{\mu\nu}g,F^{\mu\nu}] +f^2g\phi^{;\mu}\phi^T_{;\mu}g-f^2g\phi V'\phi^Tg, \tag{18}$$

where V' is the derivative of V. The $\mu=0$ component of the equation for $F^{\mu\nu}$ is a constraint, which may be called Gauss's law.

In the usual way, the antisymmetry of $F^{\mu\nu}$ implies that the matrix of n^2 currents

$$J^{\mu}=[F^{\mu\nu},g^{-1}g_{;\nu}-A_\nu]+e^2\phi^{;\mu}\phi^Tg -(e^2/f^2)g^{-1}g^{;\mu} \tag{19}$$

is conserved, $J^{\mu}_{,\mu}=0$.

By using Gauss's law, one may write the component T^{00} of the canonical stress-energy tensor as the sum of

$$H=(2e)^{-2}\,\mathrm{tr}(F^T_{\mu\nu}gF_{\mu\nu}g^{-1})+(2f)^{-2}\,\mathrm{tr}(g_{;\mu}g^{-1}g_{;\mu}g^{-1}) +\tfrac{1}{2}\phi^T_{;\mu}g\phi_{;\mu}+\tfrac{1}{2}V \tag{20}$$

and the total divergence

$$D=e^{-2}\,\mathrm{tr}(g^{-1}F^T_{0i}gA^0)^{,i}. \tag{21}$$

Thus apart from surface terms, the Hamiltonian may be taken as the integral of the density H. Now H is non-negative as long as g is symmetric and of non-negative signature. [The latter constraint may be enforced by the device of writing $g=h^Th$. The field equations for h, which may be chosen to be symmetric, are those that result from the substitution of h^Th for g in (18).] The energy is therefore non-negative.

The metric tensor g_{ij} participates in an interesting variation of the Higgs mechanism.[3] If the potential V assumes its minimum value at $\phi^Tg\phi=0$, then in the lowest approximation the (super-) vacuum expectation values of the fields ϕ and A_μ vanish, while that of the metric g is a positive symmetric matrix g_0. By expanding the fields about these vacuum values with

$$g=(1-\epsilon^T)g_0(1-\epsilon), \tag{22}$$

one may identify the quadratic part of the Lagrangian (15) as

$$L_2=-(2e)^{-2}[\mathrm{tr}(E_{\mu\nu}E^{\mu\nu})+\mathrm{tr}(G^T_{\mu\nu}G^{\mu\nu})] +f^{-2}\,\mathrm{tr}(W_\mu W^\mu)+\tfrac{1}{2}\phi^T_{,\mu}g_0\phi^{,\mu} -\tfrac{1}{2}V'(0)\phi^Tg_0\phi, \tag{23}$$

where $E_{\mu\nu}$ is the curl $W_{\mu,\nu}-W_{\nu,\mu}$ of the symmetric combination

$$W_\mu=\tfrac{1}{2}[g_0^{1/2}(A_\mu-\epsilon_{,\mu})g_0^{-1/2} +g_0^{-1/2}(A^T_\mu-\epsilon^T_{,\mu})g_0^{1/2}] \tag{24}$$

while $G_{\mu\nu}$ is the curl $C_{\mu,\nu}-C_{\nu,\mu}$ of the antisymmetric combination

$$C_\mu=\tfrac{1}{2}(g_0^{1/2}A_\mu g_0^{-1/2}-g_0^{-1/2}A^T_\mu g_0^{1/2}). \tag{25}$$

The mass spectrum of physical particles is clear from the structure of L_2. There are $n(n+1)/2$ vector mesons W_μ of mass $M=\sqrt{2}\,(e/f)$. The longitudinal components of W_μ are contributed by the $n(n+1)/2$ components of the symmetric metric tensor g. There are $n(n-1)/2$ massless vector mesons C_μ. There are n scalar mesons of mass $\mu=[V'(0)]^{1/2}$. There are no physical particles corresponding to the metric g.

If the potential V assumes its minimum value at $\phi^Tg\phi>0$, then the usual Higgs mechanism comes into play as well. The particle spectrum becomes $\frac{1}{2}n(n+3)-1$ massive vector mesons, $\frac{1}{2}n(n-3)+1$ massless vector mesons, and 1 massive scalar meson. Thus for $n=2$ there are no massless particles, while for $n=3$ only one gauge meson is massless.

It is perhaps worth emphasizing that the gauge mesons W_μ are massive even in the absence of the scalar mesons ϕ. They generate the noncompact part of $\mathrm{GL}(n,R)$, while the gauge mesons C_μ generate the compact subgroup $\mathrm{SO}(n)$.

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[1]Complex groups such as SU(n) here are thought of as subgroups of GL($2n$, R).

[2]Noncompact internal-symmetry groups have been considered by various authors, e.g., P. Nath and R. Arnowitt, Phys. Lett. 56B, 177 (1975); R. Arnowitt, P. Nath, and B. Zumino, *ibid.* 56B, 81 (1975); and T. T. Wu and C. N. Yang, Phys. Rev. D 13, 3233 (1976).

[3]A somewhat similar mechanism has been observed by R. Arnowitt and P. Nath, Phys. Rev. Lett. 36, 1526 (1976).