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Time-independent Yang-Mills statics

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We investigate classical Yang-Mills theory with sources in the limit of no time dependence and find a class of "Abelian" solutions with nonvanishing magnetic field, even without spatial source-current density. The field is due to nontrivial point magnetic monopoles of topogical origin.

I. INTRODUCTION

It is usually possible to obtain insight regarding many phenomena by considering the physical systems of interest in a limiting or extreme case. Sometimes the resulting equations are easy to interpret and occasionally completely solvable. In this paper we obtain a limiting situation by making a compound ansatz for the Yang-Mills system with source-current density; we construct and treat a particular "static" limit of the equations of motion.

We take our cue from elementary texts on electrodynamics, the opening chapters of which contain expositions of so-called electrostatics.[1] Two features appear: first, *all* the quantities under consideration have no time dependence,[2] hence "statics"; second, for simplicity, it is usually assumed that magnetic phenomena are not present. Technically, a steady-state current density does give rise to a "static" magnetic field, and because the equations are linear, magnetostatics can easily be incorporated.

The general Yang-Mills system displays many similarities (and differences) to electrodynamics, the Abelian version. We propose to study the system when the non-Abelian symmetry group is SO(3), and retain the assumption which eliminates all time dependence. However, because of the nonlinearities associated with SO(3) Yang-Mills theory, the separation of the magnetic part as in electrostatics cannot be consistently accomplished. A less stringent ansatz will be made, which can be implemented, and which leads to a nontrivial construction.

We use the three-vector representation for elements of SO(3) to show that certain distributions of the Yang-Mills source lead, via the fields, to topologically conserved quantities. These charges are seen to exist only at discrete points in space, and play the role of magnetic monopoles in the resulting theory.

The next section shows how we use the compound ansatz to derive a non-Abelian Poisson equation for the static potential, as well as a subsidiary condition which has the Meissner effect as its Abelian analog.

The third section contains the solution of the equations and the discovery of the monopoles.

Section IV has some interesting examples, while Sec. V is a summary and list of conclusions.

II. CLASSICAL YANG-MILLS STATICS

The dynamical equation of motion from which we will extract our results are the Yang-Mills equations with source current density[3]:

$$\begin{aligned} &D^\nu \vec{F}_{\mu\nu} = \vec{J}_\mu \quad (\mu, \nu = 0,1,2,3)\,, \\ &\vec{F}_{\mu\nu} \equiv \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \vec{A}_\mu \times \vec{A}_\nu\,, \\ &\vec{D}^\mu \equiv \partial^\mu + \vec{A}^\mu \times\,. \end{aligned} \tag{1}$$

The current density $\vec{J}_\mu$ is assumed given, while $\vec{F}_{\mu\nu}$ is the field strength in terms of the gauge potential $\vec{A}_\mu$. The operator $\vec{D}^\mu$ is the covariant derivative for the gauge group SO(3) in the adjoint representation. We have set the coupling constant equal to unity. Finally, $\vec{J}_\mu$ must be covariantly conserved:

$$D^\mu \vec{J}_\mu = 0\,. \tag{2}$$

The preceding set of equations defines a theory whose solution we will give in an interesting static limit. The solution will be closely related to the kind of static theory presented in the opening chapters of elementary texts on electrodynamics.

Our ansatz consists of assuming

$$\frac{d}{dt} = 0\,, \tag{3a}$$

and

$$D^j \vec{F}_{ij} = \vec{J}_i \quad (i,j = 1,2,3)\,. \tag{3b}$$

The first part states that all quantities in the theory have no time dependence; this includes gauge transformations. Thus, no gauge transformation can take us out of the static regime, but the gauge group is now defined on ordinary three-dimensional Euclidean space. The second part of the ansatz consists of isolating the "magnetic" part and treating it separately. In ordinary electrostatics this separation is accomplished by setting the magentic field and spatial current density to zero. The general Yang-Mills case does not permit this additional simplification because of the nonlinearity of the equations of motion. Therefore, we allow more general configurations given by Eq. (3b) and treat them consistently. Under the time-independent gauge transformations, this part of the ansatz is "geometrical," since both sides transform like three-vectors in group space (and three-vectors in ordinary space).

If we now take the temporal and spatial components of the dynamical equations of motion in this static limit, we obtain

$$D^i\vec{F}_{0i}=\vec{j}_0\equiv\vec{\rho}\,, \tag{4a}$$

$$\vec{A}_0\times\vec{F}_{0i}=0\,, \tag{4b}$$

with

$$\vec{F}_{0i}=-D_i\vec{A}_0\,. \tag{4c}$$

The last equation states that the electric field is the negative spatial, covariant derivative of the temporal components of the gauge potential: the generalization of ordinary electrostatics. Substituting for $\vec{F}_{0i}$ in Eqs. (4a) and (4b) from (4c) gives the system

$$D^iD_i\vec{A}_0=\vec{\rho}\,, \tag{5a}$$

$$\vec{A}_0\times D_i\vec{A}_0=0\,. \tag{5b}$$

Here the first relation is seen as a non-Abelian generalization of Poisson's equation. The second result has an Abelian analog also. It is the Meissner effect of superconductivity in a three-dimensional Euclidean theory, if $\vec{A}_0$ is interpreted as a Higgs field.[4] $\vec{A}_0$ transforms properly for this.

The Eqs. (5) are invariant under time-independent gauge transformations parametrized by, say, $\vec{\lambda}(x_i)$. However, while

$$\delta\vec{A}_i=\partial_i\vec{\lambda}+\vec{A}_i\times\vec{\lambda}\,, \tag{6}$$

the temporal piece of the gauge potential transforms linearly,

$$\delta\vec{A}_0=\vec{A}_0\times\vec{\lambda}\,, \tag{7}$$

because of (3a). The quantities $\vec{\rho}$, $D_i\vec{A}_0$, and $\vec{F}_{ij}$ transform like $\vec{A}_0$. In addition, we see that the solution of Eqs. (5) will give consistency conditions on $\vec{A}_i$ which will in turn affect the second part of the ansatz.

III. SOLUTION OF EQUATIONS

In order to solve Eqs. (5), we define the unit vector

$$\hat{A}_0\equiv\vec{A}_0/A_0,\quad A_0\equiv(\vec{A}_0\cdot\vec{A}_0)^{1/2}\,.$$

This quantity has meaning if $A_0\neq 0$. We will see, however, that points where A_0 vanishes are also of physical interest.

Consider now the Meissner effect (5b). By taking its covariant divergence and using (5a), we find that

$$\vec{A}_0\times\vec{\rho}=0\,. \tag{8}$$

Thus Eq. (2) implies that

$$D^i\vec{j}_i=0\,, \tag{9}$$

consistent with the second part of the ansatz.

The Meissner effect and its covariant divergence (8) are completely solved by

$$D_i\vec{A}_0=\alpha_i\vec{A}_0 \tag{10a}$$

and

$$\vec{\rho}=\alpha\vec{A}_0\,, \tag{10b}$$

respecitvely. The proportionality factors α_i and α are related functions of space. These factors are determined by substituting Eqs. (10a) and (10b) into both sides of Eq. (5a). Thus

$$-\alpha=\partial^i\alpha_i+\alpha^i\alpha_i,\quad \text{if } A_0\neq 0\,, \tag{11}$$

and no condition if $A_0=0$.

By taking the inner product of Eq. (10a) with $\hat{A}_0$ one shows with the help of Eq. (5b) that

$$\alpha_i=\partial_i\ln A_0\,. \tag{12}$$

Thus

$$A_0=\exp[\xi(x,x_0)]\,,$$

where

$$\xi\equiv\int_{\substack{x_0\\ \zeta}}^{x}dx^i\alpha_i\,. \tag{13}$$

The curve ζ is arbitrary, but begins at x_0 and ends at x.

Considering again spatial points for which $A_0\neq 0$, we may take the magnitude of Eq. (10b) to write

$$\begin{aligned}|\alpha|&=\rho/A_0\\ &=\rho e^{-\xi},\quad \rho\equiv(\vec{\rho}\cdot\vec{\rho})^{1/2}\geq 0\,.\end{aligned} \tag{14}$$

Substituting into Eq. (11) gives

$$\begin{aligned}&\partial^i\alpha_i+\alpha^i\alpha_i=-\text{sgn}(\alpha)\rho e^{-\xi}\,,\\ &\text{sgn}(\alpha)\equiv\alpha/|\alpha|\,.\end{aligned} \tag{15}$$

This integro-differential equation reduces to[5]

$$\partial^i\partial_i e^t = -\mathrm{sgn}(\alpha)\rho\,, \tag{16a}$$

or

$$\partial^i\partial_i A_0 = -\mathrm{sgn}(\alpha)\rho\,. \tag{16b}$$

Hence

$$A_0 = -\frac{1}{4\pi}\int d^3x'\,\frac{\mathrm{sgn}(\alpha')\rho'}{|x-x'|} \tag{17a}$$

and

$$\hat{A}_0 = \mathrm{sgn}(\alpha)\hat{\rho}\,. \tag{17b}$$

The sign of α is determined by the sense of $\vec{A}_0$ relative to $\vec{\rho}$. If $\vec{A}_0$ and $\vec{\rho}$ are parallel (antiparallel), then $\mathrm{sgn}(\alpha)$ is positive (negative). This extra problem is a vestige of the original nonlinear structure, and the determination must be done consistently. For example, if A_0 is zero at a point in space, then by Eq. (17a) we determine that $\mathrm{sgn}(\alpha)$ cannot be constant—otherwise Eq. (17a) evaluated at the point could not vanish. Exploration of this mechanism will be performed in specific calculations presented in a later section of this paper. Also note that, for consistency, $A_0 \geq 0$.

Let us continue. Since it contains a term dependent on $\vec{A}_i \times \vec{A}_0$, we see that Eq. (5b) includes information about $\vec{A}_i$. Taking the inner product of $\vec{A}_0$ and Eq. (5b) gives nothing. However, the cross product of these vectors enables us to write

$$\begin{aligned} \vec{A}_i &= -\hat{A}_0 \times \partial_i \hat{A}_0 + a_i \hat{A}_0\,,\\ a_i &\equiv \hat{A}_0 \cdot \vec{A}_i\,. \end{aligned} \tag{18}$$

The a_i, coefficients of $\hat{A}_0$, are arbitrary as expected. However, the leading term in Eq. (18) is of particular interest since it is the first quantity that we have encountered which is not parallel to $\vec{A}_0$.[6] The significance of this development is understood by constructing $\vec{F}_{ij}$ from its definition in Eq. (1) and Eq. (18):

$$\begin{aligned} \vec{F}_{ij} &= F_{ij}\hat{A}_0\,,\\ F_{ij} &\equiv f_{ij} - \hat{A}_0 \cdot (\partial_i \hat{A}_0 \times \partial_j \hat{A}_0)\,,\\ f_{ij} &\equiv \partial_i a_j - \partial_j a_i\,. \end{aligned} \tag{19}$$

We follow other authors[7] and interpret F_{ij} as the magnetic part of an invariant field strength; thus

$$B^k \equiv \tfrac{1}{2}\epsilon^{kij}F_{ij} \tag{20}$$

is the magnetic field. Calculation of the divergence gives

$$\partial_k B^k = -\tfrac{1}{2}\epsilon^{kij}\partial_k \hat{A}_0 \cdot (\partial_i \hat{A}_0 \times \partial_j \hat{A}_0) \equiv -\rho_{\mathrm{mag}}\,. \tag{21}$$

Hence ρ_{mag} is the nonvanishing magnetic charge density. However, all the $\partial_i \hat{A}_0$ $(i=1,2,3)$ cannot be independent because they are all orthogonal to $\hat{A}_0$. Therefore ρ_{mag} vanishes everywhere except for isolated points where A_0 is zero, and ρ_{mag} is the density of a set of point magnetic monopoles. It can be shown[7] that they have positive or negative integral charge and are topological in origin. The total magnetic charge is the Brouwer degree of the system, and the charges of the individual monopoles are the Poincaré-Hoff indices of the zeros of the Higgs field $\vec{A}_0$ located at those points.

Finally, in order to interpret the second part of our ansatz, we take the inner product of Eq. (3b) with $\hat{A}_0$ to find

$$\partial^j F_{ij} = \vec{j}_i \cdot \hat{A}_0 \equiv j_i\,. \tag{22}$$

These are the determining equations for the remaining degrees of freedom. Taking the cross product of $\hat{A}_0$ with Eq. (3b), however, gives

$$\vec{j}_i \times \hat{A}_0 = 0\,. \tag{23}$$

Thus

$$\vec{j}_i = j_i \hat{A}_0\,.$$

Also, by Eq. (9), we have

$$\partial^i j_i = 0\,. \tag{24}$$

These conditions on the static current density may be satisfied in a variety of ways: for instance, by taking $\vec{j}_i$ to vanish everywhere. Therefore, we will still have a nontrivial magnetic field even though there are no spatial-current sources. The magnetic monopoles, necessary for the consistency of the nonlinear equations, are the source. The a_i are now determined from Eq. (22) and the locations and strengths of the topological charges.

IV. EXAMPLES

We show some details of the previous considerations in specific examples.

Let the static potential be given by[8]

$$\begin{aligned} (\vec{A}_0)^a &= kx^a e^{-r/\sigma} \quad (a=1,2,3)\,,\\ r &\equiv (x^a x^a)^{1/2}\,. \end{aligned} \tag{25}$$

The constants k and σ are parameters whose specification determines a particular $\vec{A}_0$. This vector function of position has a simple zero at the origin, where we expect to find a magnetic monopole. The magnitude of $\vec{A}_0$ is radially symmetric and equal to

$$A_0 = |k|\,r e^{-r/\sigma}\,. \tag{26}$$

We operate on A_0 with $\partial^i\partial_i$ and use Eq. (16) to obtain

$$\mathrm{sgn}(\alpha)\rho = -|k|\frac{e^{-r/\sigma}}{r}\left[\left(\frac{r}{\sigma}\right)^2 - 4\left(\frac{r}{\sigma}\right) + 2\right]. \tag{27}$$

This function is continuous and differentiable except at the origin. Graphs of A_0 and $\mathrm{sgn}(\alpha)\rho$ are given in Fig. 1. It is seen that the quantity $\mathrm{sgn}(\alpha)\rho$ vanishes on the surfaces of two concentric spheres. Their radii are determined by σ. Also, $\mathrm{sgn}(\alpha)$ changes when the non-negative ρ goes through zero. Following a radial line from the origin, $\mathrm{sgn}(\alpha)$ is first negative then positive between the two spheres of zeros and negative again to infinity. Hence, $\mathrm{sgn}(\alpha)\rho$ appears smooth.

The direction of $\vec{\rho}$ is given by Eq. (17b), and since $\hat{A}_0$ is radial we have

$$(\hat{\rho})^a = \mathrm{sgn}(\alpha)\frac{x^a}{r}. \tag{28}$$

Thus $\hat{\rho}$ reverses when $\mathrm{sgn}(\alpha)$ changes. Other quantities are calculated in a straightforward manner; we list some of them:

$$(\partial_i \hat{A}_0)^a = \frac{1}{r}\left(\delta_i^a - \frac{x^a x_i}{r^2}\right), \tag{29}$$

$$(\vec{A}_i)^a = -\epsilon^{abc}\delta_i^c \frac{x^b}{r^2} + a_i\frac{x^a}{r}, \tag{30}$$

$$F_{ij} = f_{ij} - \epsilon^{abc}\delta_i^b\delta_j^c\frac{x^a}{r^2}, \tag{31}$$

$$B^k = b^k - \tfrac{1}{2}\epsilon^{kij}\epsilon^{abc}\delta_i^b\delta_j^c\frac{x^a}{r^2}, \tag{32}$$

$$b^k \equiv \tfrac{1}{2}\epsilon^{kij}f_{ij}, \tag{33}$$

$$\partial_k B^k = -\frac{1}{4\pi}\delta^3(x), \tag{34}$$

and

$$\partial^j f_{ij} = j_i, \tag{35}$$

which, given j_i, determines a_i up to a gradient.

We conclude with one more example. It has been seen that the simple potential of Eq. (25) is generated by a rather complicated charge distribution. We propose another monopole which exists in conjunction with a simpler charge distribution:

$$\mathrm{sgn}(\alpha)\rho = -\frac{q}{32\pi\sigma^3}\left(\frac{2\sigma^2}{r} - r\right)e^{-r/\sigma},$$
$$(\hat{\rho})^a = \mathrm{sgn}(\alpha)\frac{x_a}{r}. \tag{36}$$

As a function of r, the distribution has only one finite zero. Therefore $\mathrm{sgn}(\alpha)\rho$ has one sphere of zeros in space. The quantity $\mathrm{sgn}(\alpha)$ changes on the sphere, which divides space into two parts.

Substituting Eq. (36) into Eq. (17a), we calculate

$$A_0 = \tfrac{1}{2}q\frac{\sigma}{r}\left[1 - \tfrac{1}{4}e^{-r/\sigma}\left(2 + \frac{r}{\sigma}\right)^2\right]. \tag{37}$$

This potential has a simple zero at the origin, which marks the location of a unit magnetic charge. Unlike the first example, which is not a dyon, this monopole is a dyon, with electric charge q. However, since they are radial in direction, both examples have vanishing total isotopic spin vector

$$\vec{T} \equiv \int d^3x\vec{\rho}(x) = 0. \tag{38}$$

Graphs of Eqs. (36) and (37) are given in Fig. 2.

V. SUMMARY AND CONCLUSIONS

In the preceding sections we have treated the problem of Yang-Mills theory with source current density in a static limit. The conditions which we have used remind us of electrostatics.

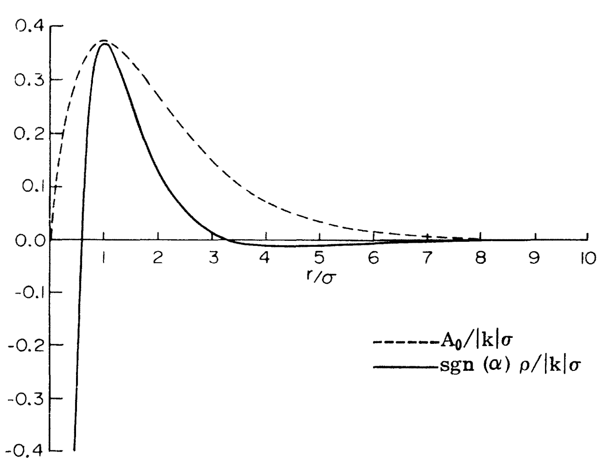

FIG. 1. Wu–Yang–'t Hooft monopole potential and charge distribution.

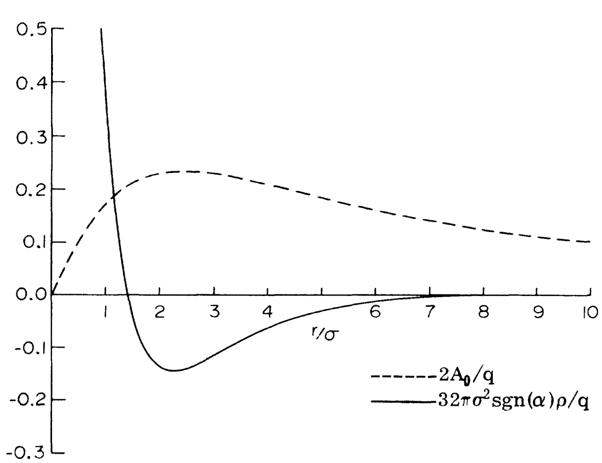

FIG. 2. Dyon potential and charge distribution.

After applying our compound ansatz we recover a non-Abelian Poisson equation for $\vec{A}_0$ as well as a subsidiary condition which causes $\vec{A}_0$ and the non-Abelian electric field to be parallel in group space. The electric field is in turn given by the negative, spatial covariant derivative of $\vec{A}_0$.

It is possible to solve these equations. First, we see that $\vec{A}_0$ acts like a Higgs field in our limit and transforms linearly under the set of time-independent elements of the group. The transformed $\vec{A}_i$, however, still displays the inhomogeneous piece of a gauge transformation of the second kind. Now, consistency demands that the quantity $\vec{A}_i$ contains a term which is orthogonal to $\vec{A}_0$. Other objects in the theory, such as $D_i\vec{A}_0$, $\vec{F}_{ij}$, $\vec{\rho}$, and $\vec{j}_i$ are parallel to $\vec{A}_0$. Therefore the magnitude of these latter quantities along with a relative sense (±) is all that is required to specify them, once $\vec{A}_0$ is obtained. If however, there are points in space where A_0 vanishes, the nonparallel part of $\vec{A}_i$ is generated by conserved topological charges: net integral magnetic monopoles. We obtain, in the end, a theory with magnetic charges, no Dirac strings, and single-valued potential (up to an Abelian gauge transformation).

After establishing the above interpretation, we proceed to study examples which contain "Poisson" distributions. First we consider a single monopole of the Wu–Yang–'t Hooft type. This object is seen to be generated by a source-charge distribution with two concentric spherical shells of zeros—a rather complicated function. We expect that a simpler geometry can be generated by a $\vec{\rho}$ which contains only one spherical shell of zeros. The corresponding potential for this charge distribution is constructed, and a dyon is reported to be situated at the origin.

In both examples the charge density becomes large in the neighborhood of the monopole. However, both static charge distributions are normalizable and lead to nonzero magnetic field without spatial current density. Also, for both, the total isotopic spin vector vanishes.

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[1] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1965).

[2] The more general configurations involving time-dependent operator source charges have been treated semiclassically: I. B. Khriplovich, Zh. Eksp. Teor. Fiz. 74, No. 1 (1978) S. L. Adler, Phys. Rev. D 18, 411 (1978); R. Giles and L. McLerran, MIT Report No. MIT-CTP 711, 1978 (unpublished).

[3] Our metric is $\eta_{00} = -\eta_{ii} = 1$.

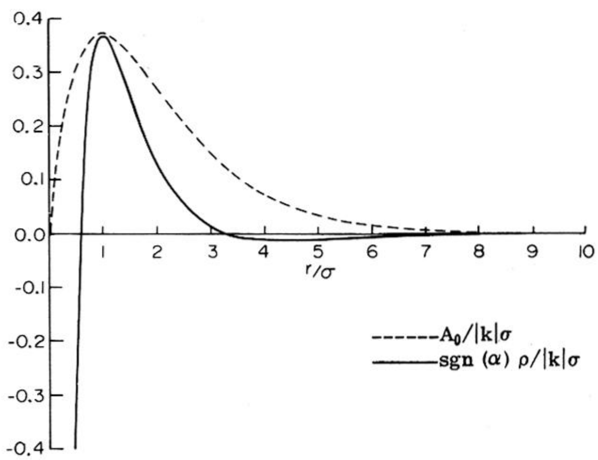
[4] Y. Nambu, invited talk given at the international symposium, "Five Decades of Weak Interactions", The City College of the City University of New York, 1977 (unpublished).

[5] The equations which follow can be obtained by taking the inner product of $\hat{A}_0$ with Eq. (5a) and noting that Eq. (5b) implies that $D_i\hat{A}_0 = 0$.

[6] It is possible for this term to be pure gauge. However, as we shall see, this is not always the case.

[7] J. Arafune, P. G. O. Freund and C. J. Goebel, J. Math. Phys. 16, 433 (1975).

[8] T. T. Wu and C. N. Yang, in *Properties of Matter under Unusual Conditions*, edited by S. Fernbach and H. Mark (Interscience, New York, 1969), p. 349; G. 't Hooft, Nucl. Phys. B79, 276 (1974).



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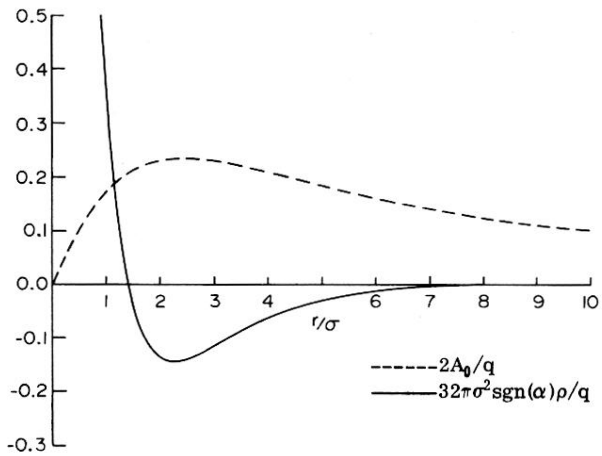

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