Source-free Yang-Mills theories

Ezra T. Newman
Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
(Received 10 May 1978)

Using techniques developed in recent years in general relativity we study the general GL(R,n) and GL(C,n) gauge theories with no external sources. In particular, after casting the field equations into the spin-coefficient formalism and adapting natural (to the formalism) gauge conditions we show how most of the equations can be integrated in the case of real asymptotically "flat" solutions. In the self-dual (or anti-self-dual) case the field equations are reduced to a single nonlinear wave equation for a Hertz-type potential. As a final point we show the relationship between this potential and our version of the Atiyah-Ward method of finding self-dual fields.

I. INTRODUCTION

It is the purpose of this paper to study the general[1] GL(R,n) and GL(C,n) gauge theories (with no external interactions) and to show in particular how many of the ideas and techniques developed in recent years in general relativity (GR) can be usefully applied to them. More specifically, we will show how to cast the field equations of the gauge theory into a null coordinate and tetrad form which in turn define a natural or canonical choice of gauge frame. The resulting equations, which now resemble the spin-coefficient versions of the Einstein or Maxwell equations, can then be integrated rather completely (by using techniques familiar from GR) in the two important cases (1) self-dual (or anti-self-dual) fields and (2) real asymptotically vanishing fields.

In Sec. II we will review[2] some relevant material from vector-bundle theory which will be needed to fix the gauge.

In Sec. III null coordinates associated with a timelike world line and the related null tetrad are used to produce the spin-coefficient version of the field equations, while in Secs. IV and V, respectively, we show how the cases of the self-dual and the asymptotically vanishing fields can be integrated. Finally, Sec. VI is devoted to discussing the relation of the work presented here to Sparling's version of the Atiyah-Ward twistor approach to gauge theories.

II. MATHEMATICAL PRELIMINARIES (SEE REF. 2)

We will consider the trivial vector bundle[3] B (each fiber being n-dimensional real or complex for the time being) over Minkowski space M, i.e., $B = M\times R^n$ or $M\times C^n$, or more intuitively we consider that at each point of M there is an n-dimensional vector space and that there exists n linearly independent vector fields, global over M. These vector fields, $e_A (A=1,\ldots,n)$, form a basis set, as does

$$e'_A = G_A{}^B e_B\,, \tag{2.1}$$

for $G_A{}^B$ in GL(R,n) or GL(C,n). A connection or parallel transfer of vectors in B is introduced by defining ∇_a by

$$\nabla_a e_A = \gamma_{Aa}{}^B e_B \tag{2.2a}$$

or in form notation

$$\nabla e_A = \gamma_A{}^B e_B\,,\quad \gamma_A{}^B = \gamma_{Aa}{}^B dx^a \tag{2.2b}$$

with $\gamma_A{}^B$ being an arbitrary matrix-valued one-form. Equation (2.2) allows one to take the covariant derivative of an arbitrary vector $V = V^A e_A$ by

$$\nabla_a V = (V^A{}_{,a} + V^B \gamma_{Ba}{}^A) e_A \tag{2.3a}$$

or

$$\nabla V = (dV^A + V^B \gamma_B{}^A) e_A\,. \tag{2.3b}$$

Under a change in basis (2.1) one can easily show from (2.2) and (2.3) that

$$\gamma'_A{}^B = G_A{}^C \gamma_C{}^D G^{-1}{}_D{}^B + dG_A{}^C G^{-1}{}_C{}^B \tag{2.4a}$$

or with matrix notation

$$\gamma' = G\gamma G^{-1} + dG G^{-1}\,. \tag{2.4b}$$

The "curvature" tensor of this connection is defined by

$$F = d\gamma - \gamma\wedge\gamma \tag{2.5a}$$

or

$$F_{ab} = \gamma_{b,a} - \gamma_{a,b} - [\gamma_a, \gamma_b] \tag{2.5b}$$

or

$$F_A{}^B{}_{ab} = \gamma_A{}^B{}_{b,a} - \gamma_A{}^B{}_{a,b} - (\gamma_{Aa}{}^C \gamma_{Cb}{}^B - \gamma_{Ab}{}^C \gamma_{Ca}{}^B)\,. \tag{2.5c}$$

Under (2.1) one can easily show from (2.4) and (2.5) that

$$F' = GFG^{-1}. \quad (2.6)$$

The (generalized) Yang-Mills field equations can be written either as

$$dF^* + F^* \wedge \gamma - \gamma \wedge F^* = 0 \quad (2.7a)$$

with $F^*_{ab} = \frac{1}{2}\eta_{abcd}F^{cd}$, $\eta_{0123} = -\sqrt{-g}$, or more conventionally as

$$F^{ab}{}_{,b} + F^{ab}\gamma_b - \gamma_b F^{ab} = 0. \quad (2.7b)$$

In the special cases of self-dual or anti-self-dual fields, i.e., $F_{ab} = \pm iF^*_{ab}$ the field equations (2.7) are automatically satisfied because of the generalized Bianchi identities

$$dF + F \wedge \gamma - \gamma \wedge F \equiv 0 \quad (2.8)$$

which follow directly from (2.5).

We now investigate how to select a basis set or gauge. The basic idea is to choose a vector V at a point on a specific curve and then parallel transfer it along the curve, via

$$t^a\nabla_a V \equiv (V^A{}_{,a}t^a + V^B\gamma_{Ba}{}^A t^a)e_A = 0 \quad (2.9)$$

which comes from (2.3b), with t^a being the tangent vector to the curve.[4] If this is now generalized to n linearly independent vectors chosen at one point of each curve of a space filling congruence and then parallel propagated along the curve we obtain a basis set, satisfying

$$t^a\nabla_a e_A = 0. \quad (2.10)$$

which then implies from (2.2) that

$$\gamma_{Aa}{}^B t^a = 0. \quad (2.11)$$

There, of course, remains the GL(n) freedom in the choice of the e_A at the starting point of each curve. This remaining freedom will be used to eliminate some constants of integration when the Yang-Mills equations are solved.

III. A NULL-TETRAD FORMULATION OF YANG-MILLS THEORY

Though the material of this section can be presented in a much more general null-tetrad system (see Appendix A) we will confine ourselves here to the simplest case—the null tetrad based on the null cones emanating from a timelike geodesic L in Minkowski space.

Beginning with Minkowski coordinates x^a, we introduce null polar coordinates $(u, r, \zeta, \bar\zeta)$ by

$$x^a = uv^a + rl^a(\zeta, \bar\zeta) \quad (3.1)$$

with

$$l^a = \frac{1}{2\sqrt{2}P_0}(1+\zeta\bar\zeta, \zeta+\bar\zeta, i(\bar\zeta-\zeta), -1+\zeta\bar\zeta),$$
$$P_0 = \tfrac{1}{2}(1+\zeta\bar\zeta) \quad (3.2)$$

(such that $l_a l^a = 0$) and $v_a v^a = 2$. ζ and $\bar\zeta$ are complex stereographic coordinates, $\zeta = e^{i\phi}\cot\frac{1}{2}\theta$, while u is the proper time along the world line L ($x^a = uv^a$), and r is the affine length measured along the null rays leaving L. In this coordinate system, the Minkowski metric becomes

$$ds^2 = 2\,du^2 + 2\,du\,dr - \frac{r^2}{2P_0{}^2}d\zeta d\bar\zeta. \quad (3.3)$$

In addition to the vector field l^a we consider three further null fields n, m, and $\bar m$ such that

$$l^a n_a = -m^a\bar m_a = 1$$

and all other scalar products vanishing, with n^a being the inward-pointing radial null vector field and m^a and $\bar m^a$ being complex spacelike null fields. In the null coordinate system the four fields l, n, m, and $\bar m$ take the form

$$l = \frac{\partial}{\partial r}, \quad n = \frac{\partial}{\partial u} - \frac{\partial}{\partial r},$$
$$m = -\frac{P_0}{r}\frac{\partial}{\partial \zeta}, \quad \bar m = -\frac{P_0}{r}\frac{\partial}{\partial \bar\zeta}. \quad (3.4)$$

We are now in position to translate the Yang-Mills field equations into a form associated with the above null-tetrad and coordinate system. Defining the three curvature tensor matrices

$$||\chi_{0A}{}^B|| \equiv \chi_0 = F_{ab}l^a m^b,$$
$$||\chi_{1A}{}^B|| \equiv \chi_1 = \tfrac{1}{2}F_{ab}(l^a n^b + \bar m^a m^b), \quad (3.5)$$
$$||\chi_{2A}{}^B|| \equiv \chi_2 = F_{ab}\bar m^a n^b$$

and their complex conjugates $\bar\chi_0, \bar\chi_1, \bar\chi_2$ as well as the connection matrices by

$$||\gamma_{00'A}{}^B|| \equiv \gamma_{00'} = \gamma_a l^a,$$
$$||\gamma_{11'A}{}^B|| \equiv \gamma_{11'} = \gamma_a n^a,$$
$$||\gamma_{10'A}{}^B|| \equiv \gamma_{10'} = \gamma_a \bar m^a,$$
$$||\gamma_{01'A}{}^B|| \equiv \gamma_{01'} = \gamma_a m^a, \quad (3.6)$$

we can write (2.7) as

$$\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r} - \frac{1}{r}\right)\chi_0 + \frac{1}{r}\eth\chi_1 = [\chi_0, \gamma_{11'}] - [\chi_1, \gamma_{01'}], \quad (3.7a)$$

$$\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r} - \frac{2}{r}\right)\chi_1 + \frac{1}{r}\eth\chi_2 = [\chi_1, \gamma_{11'}] - [\chi_2, \gamma_{01'}], \quad (3.7b)$$

$$\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\chi_1 + \frac{1}{r}\bar\eth\chi_0 = [\chi_1, \gamma_{00'}] - [\chi_0, \gamma_{10'}], \quad (3.7c)$$

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\chi_2 + \frac{1}{r}\bar\eth\chi_1 = [\chi_2, \gamma_{00'}] - [\chi_1, \gamma_{10'}]. \quad (3.7d)$$

[,] represents the commutator. $\eth$ and $\bar{\eth}$ are angular differential operators which act on spin-weight s functions η by

$$\eth\eta = 2P_0^{1-s}\frac{\partial}{\partial\zeta}P_0^{s}\eta\,,$$

$$\bar{\eth}\eta = 2P_0^{1+s}\frac{\partial}{\partial\bar{\zeta}}P_0^{-s}\eta\,.$$

The spin weights of $\chi_0, \chi_1, \chi_2, \gamma_{00'}, \gamma_{01'}, \gamma_{10'}, \gamma_{11'}$ are, respectively, $s=1,0,-1,0,1,-1,0$.

To complete the null-tetrad version of the Yang-Mills equations we need the translation of (2.5), namely,

$$\chi_0 = \frac{1}{r}\eth\gamma_{00'} + \left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\gamma_{01'} + [\gamma_{01'}, \gamma_{00'}]\,, \tag{3.8a}$$

$$2\chi_1 = \left(\frac{\partial}{\partial r}-\frac{\partial}{\partial u}\right)\gamma_{00'} - \frac{1}{r}(\bar{\eth}\gamma_{01'} - \eth\gamma_{10'}) + \frac{\partial}{\partial r}\gamma_{11'} + [\gamma_{11'}, \gamma_{00'}] - [\gamma_{10'}, \gamma_{01'}]\,, \tag{3.8b}$$

$$\chi_2 = \left(\frac{\partial}{\partial r}-\frac{\partial}{\partial u}+\frac{1}{r}\right)\gamma_{10'} - \frac{1}{r}\bar{\eth}\gamma_{11'} + [\gamma_{11'}, \gamma_{10'}]\,. \tag{3.8c}$$

Equations (3.7) amd (3.8) are completely equivalent to the ordinary GL(n) Yang-Mills theory.[5]

We now use the techniques of Sec. II to partially fix the gauge. For the space-filling family of curves we choose the null geodesics leaving L [i.e., l^a being the tangent vectors t^a of (2.10)] and hence from (2.11)

$$\gamma_{Aa}{}^{B} l^a = \gamma_{A00'}{}^{B} = 0 \tag{3.9a}$$

or

$$\gamma_{00'} = 0\,. \tag{3.9b}$$

The remaining freedom of gauge transformations can easily be seen from (2.4) to be a $G_A{}^B$ subject to

$$G_A{}^B{}_{,a}\, l^a \equiv \frac{\partial}{\partial r} G_A{}^B = 0\,. \tag{3.10}$$

This freedom plays an important role in Secs. IV and V.

Using (3.9), Eqs. (3.7) and (3.8) simplify to

$$\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\chi_1 + \frac{1}{r}\bar{\eth}\chi_0 = [\gamma_{10'}, \chi_0]\,, \tag{3.11a}$$

$$\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\chi_2 + \frac{1}{r}\bar{\eth}\chi_1 = [\gamma_{10'}, \chi_1]\,, \tag{3.11b}$$

$$\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}-\frac{1}{r}\right)\chi_0 + \frac{1}{r}\eth\chi_1 = [\gamma_{01'}, \chi_1] - [\gamma_{11'}, \chi_0]\,, \tag{3.11c}$$

$$\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}-\frac{2}{r}\right)\chi_1 + \frac{1}{r}\eth\chi_2 = [\gamma_{01'}, \chi_2] - [\gamma_{11'}, \chi_1]\,, \tag{3.11d}$$

$$\chi_0 = \left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\gamma_{01'}\,, \tag{3.12a}$$

$$2\chi_1 = -\frac{1}{r}(\bar{\eth}\gamma_{01'} - \eth\gamma_{10'}) + \frac{\partial}{\partial r}\gamma_{11'} - [\gamma_{10'}, \gamma_{01'}]\,, \tag{3.12b}$$

$$\chi_2 = \left(\frac{\partial}{\partial r}-\frac{\partial}{\partial u}+\frac{1}{r}\right)\gamma_{10'} - \frac{1}{r}\bar{\eth}\gamma_{11'} + [\gamma_{11'}, \gamma_{10'}] \tag{3.12c}$$

as well as the conjugate equations.

In the next two sections we show how (3.11) and (3.12 can be largely integrated for the two important case of self-dual (or anti-self-dual) fields and real asymptotically flat fields.

IV. ASYMPTOTICALLY VANISHING FIELDS

In this section we will study the asymptotic behavior of real Yang-Mills fields in (real) Minkowski space.

The basic idea is to first concentrate on a single null cone $u=u_0$ and integrate those equations from (3.11) and (3.12) that do not involve $\partial/\partial u$, namely, (3.11a), (3.11b), (3.12a), (3.12b), in terms of some arbitrarily chosen data. The remaining equations then are used to determine the time evolution of the system.

If at $u=u_0$ we choose χ_0 $(u=u_0, r, \zeta, \bar{\zeta})$ as an arbitrary function of r, ζ, and $\bar{\zeta}$ [later we will restrict the r behavior to $O(r^{-3})$], regular in ζ and $\bar{\zeta}$, then (3.12a) easily yields

$$\gamma_{01'} = \frac{\gamma^0}{r} + \frac{1}{r}\int_\infty^r r\chi_0\, dr\,, \tag{4.1}$$

where $\gamma^0(u_0, \zeta, \bar{\zeta})$ is an arbitrary function of integration. If now (4.1) is used in (3.11a) we have immediately (remembering that $\gamma_{10'} = \bar{\gamma}_{01'}$)

$$\chi_1 = \frac{\chi_1^0}{r^2} - \frac{1}{r^2}\int_\infty^r r^2 dr\left(\frac{1}{r}\bar{\eth}\chi_0 + [\chi_0, \gamma_{10'}]\right) \tag{4.2}$$

which when used in (3.11b) yields

$$\chi_2 = \frac{\chi_2^0}{r} - \frac{1}{r}\int_\infty^r r\, dr\left(\frac{1}{r}\bar{\eth}\chi_1 + [\chi_1, \gamma_{10'}]\right) \tag{4.3}$$

with χ_1^0 and χ_2^0 as arbitrary functions of ζ and $\bar{\zeta}$. If we now write the sum and difference of (3.12b) with its complex-conjugate equation (remembering $\gamma_{11'}$ is real and $\bar{\gamma}_{10'} = \gamma_{01'}$) we obtain

$$\chi_1 - \bar{\chi}_1 = -\frac{1}{r}(\bar{\bar{\eth}}\gamma_{01'} - \eth\gamma_{10'}) - [\gamma_{10'}, \gamma_{01'}] \tag{4.4}$$

which we will see imposes a restriction on the functions of integration and

$$\chi_1 + \bar{\chi}_1 = +\frac{\partial}{\partial r}\gamma_{11'} \tag{4.5}$$

and hence

$$\gamma_{11}' = \gamma_{11'}^0 + \int_\infty^r (\chi_1 + \bar{\chi}_1)\, dr \tag{4.6}$$

$\gamma^0_{11'}(\zeta,\bar\zeta)$ being arbitrary but real. $\gamma^0_{11'}$, however, can be made to vanish by using some of the remaining gauge freedom (3.10), i.e., by solving

$$\frac{\partial G}{\partial u}+G\gamma^0_{11'}=0 \qquad (4.7)$$

the remaining freedom thus restricted by

$$\frac{\partial}{\partial u}G=\frac{\partial}{\partial r}G=0\,.$$

Equation (4.6) becomes

$$\gamma_{11}=+\int_\infty^r(\chi_1+\bar\chi_1)dr\,. \qquad (4.8)$$

We see that at this point given $\chi_0(r,\zeta,\bar\zeta)$ the radial behavior of all the functions is determined via (4.1), (4.2), (4.3), and (4.8) with γ^0, χ_1^0, and χ_2^0 being the arbitrary functions of integration. We now find the restrictions on these functions and their evolution.

From (4.4), using (4.1) and (4.2), we obtain

$$\chi_1^0-\bar\chi_1^0=-(\bar\eth\gamma^0-\eth\bar\gamma^0)-[\bar\gamma^0,\gamma^0] \qquad (4.9)$$

which determines the imaginary part of χ_1^0 in terms of γ^0.

From (3.12c), using (4.1), (4.2), (4.3), and (4.8) we obtain

$$\chi_2^0=-\frac{\partial}{\partial u}\gamma^0 \qquad (4.10)$$

the definition of χ_2^0 in terms of γ^0 and from (3.11d) we obtain

$$\frac{\partial}{\partial u}\chi_1^0=-\eth\chi_2^0-[\chi_2^0,\gamma^0] \qquad (4.11)$$

the evolution equation of χ_1^0 in terms of γ^0. Note that (4.11) is compatible with (4.9).

The final equation to be studied (and also the most difficult to solve) is (3.11c). This equation determines the time evolution of χ_0, i.e., from (3.11c) one can find $(\partial/\partial u)\chi_0$ in terms of χ_0, χ_1, and γ^0 at $u=u_0$. If, for example, one has

$$\chi_0=\frac{\chi_0^0}{r^3}+O(r^{-4}) \qquad (4.12)$$

then from (3.11c) we obtain

$$\frac{\partial}{\partial u}\chi_0^0=-\eth\chi_1^0-[\chi_1^0,\gamma^0]\,, \qquad (4.13)$$

and in general if

$$\chi_0=\sum_n\frac{\chi_0^n}{r^{3+n}}$$

we obtain ordinary evolution equations for the χ_0^n.

If $\chi_0=O(r^{-3})$ then it is easily checked from (4.1), (4.2), and (4.3) that

$$\chi_1=\frac{\chi_1^0}{r^2}+O(r^{-3})\,,\qquad \chi_2=\frac{\chi_2^0}{r}+O(r^{-2}) \qquad (4.14)$$

and that as in Maxwell theory there are three zones, the far or radiation zone where one considers only the r^{-1} term in the field, the intermediate zone where terms of r^{-3} and higher are excluded, and the near zone.

If we consider the eigenvalue problem of the form

$$F_A{}^B{}_{ab}k^b=\lambda_A{}^Bk_a\,, \qquad (4.15)$$

then in general for each matrix element, i.e., (A,B), there are two independent eigenvectors.

However, in the radiation zone they coalesce to one, simply being $k_a=l_a$ and obviously independent of (A,B). Furthermore, $\lambda_A{}^B=0$.

In the intermediate zone *one* of the *two* eigenvectors is l_a [independent of (A,B)], the other depends on the matrix element in question. In the near zone the eigenvectors are different from l_a and depend on the matrix element.

This behavior of the eigenvectors, both being l_a in the radiation zone,[6] one being l_a in the intermediate zone and none in the near zone is the Yang-Mills version of the well-known peeling theorem of general relativity and Maxwell theory.

[As an aside we note that if a Yang-Mills field is null or degenerate in the sense that there is a degenerate null eigenvector of F_{ab} independent of the matrix element, then it follows immediately from the equations in Appendix A that the degenerate eigenvector must be the tangent vector to a shear-free null geodesic congruence. The argument is simply that degeneracy implies that $\chi_0=\chi_1=0$ (assuming that l_a is the degenerate eigenvector) then from (A1) one has $\kappa=\sigma=0$, the conditions for l_a to be tangent to a shear-free null geodesic congruence.[7]]

V. SELF-DUAL YANG-MILLS FIELDS

The condition for a self-dual field, e.g., $F^*_{ab}=iF_{ab}$ takes the following form in the null-tetrad notation

$$\chi_0=\chi_1=\chi_2=0 \qquad (5.1)$$

with the $\tilde\chi_0$, $\tilde\chi_1$, $\tilde\chi_2$ in general nonvanishing. (For anti-self-dual fields $\tilde\chi_0$, $\tilde\chi_1$, $\tilde\chi_2=0$.)

Since in this case the F_{ab} are complex and hence the χ's and $\gamma_{01'}$ are independent of the $\tilde\chi$'s and $\gamma_{10'}$, we must use (3.11) and (3.12) as well as their conjugates — actually since (3.11) is identically satisfied by (5.1) all we need are the conjugate equations to (3.11) and (3.12) and its conjugate,

$$\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\tilde{\chi}_1+\frac{1}{r}\eth\tilde{\chi}_0=[\gamma_{01'},\tilde{\chi}_0], \tag{5.2a}$$

$$\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\tilde{\chi}_2+\frac{1}{r}\eth\tilde{\chi}_1=[\gamma_{01'},\tilde{\chi}_1], \tag{5.2b}$$

$$\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}-\frac{1}{r}\right)\tilde{\chi}_0+\frac{1}{r}\bar{\eth}\tilde{\chi}_1=[\gamma_{10'},\tilde{\chi}_1]-[\gamma_{11'},\tilde{\chi}_0], \tag{5.2c}$$

$$\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}-\frac{2}{r}\right)\tilde{\chi}_1+\frac{1}{r}\bar{\eth}\tilde{\chi}_2=[\gamma_{10'},\tilde{\chi}_2]-[\gamma_{11'},\tilde{\chi}_1], \tag{5.2d}$$

$$0=\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\gamma_{01'}, \tag{5.3a}$$

$$0=-\frac{1}{r}(\bar{\eth}\gamma_{01'}-\eth\gamma_{10'})+\frac{\partial}{\partial r}\gamma_{11'}-[\gamma_{10'},\gamma_{01'}], \tag{5.3b}$$

$$0=\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}-\frac{1}{r}\right)\gamma_{10'}+\frac{1}{r}\bar{\eth}\gamma_{11'}-[\gamma_{11'},\gamma_{10'}], \tag{5.3c}$$

$$\tilde{\chi}_0=+\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\gamma_{10'}, \tag{5.4a}$$

$$2\tilde{\chi}_1=-\frac{1}{r}(\eth\gamma_{10'}-\bar{\eth}\gamma_{01'})+\frac{\partial}{\partial r}\gamma_{11'}-[\gamma_{01'},\gamma_{10'}], \tag{5.4b}$$

$$\tilde{\chi}_2=\left(\frac{\partial}{\partial r}-\frac{\partial}{\partial u}+\frac{1}{r}\right)\gamma_{01'}-\frac{1}{r}\eth\gamma_{11'}+[\gamma_{11'},\gamma_{01'}]. \tag{5.4c}$$

Since (as was pointed out in Sec. II) the Yang-Mills equations are identically satisfied by virtue of the Bianchi identities for self-dual or anti-self dual fields, we could dispense with (5.2) (which are now identities) and simply integrate (5.3) for the γ's and use (5.4) to obtain the fields. We will in fact now show that (5.3) can be reduced to a single nonlinear wave equation for a "Hertz-type" potential.[8]

Equation (5.3a) integrates to

$$\gamma_{01'}=\frac{\gamma^0_{01'}(u,\zeta,\bar{\zeta})}{r} \tag{5.5}$$

with $\gamma^0_{01'}$, the function of integration, being an arbitrary function of u, ζ, $\bar{\zeta}$. By now using the remaining gauge freedom (from Sec. III) and satisfying

$$\eth G_A{}^B+G_A{}^C\gamma^{0D}_{C\,01'}=0 \tag{5.6}$$

we can make

$$\gamma^0_{01'}=0. \tag{5.7}$$

[We point out here that Eq. (5.6) is the starting point of the Sparling[9] version of the Atiyah-Ward[10] twistor approach to self-dual Yang-Mills fields.]

From (5.7), (5.3b) becomes

$$\frac{\partial}{\partial r}\gamma_{11'}=-\frac{1}{r}\eth\gamma_{10'}. \tag{5.8}$$

By defining the potential $\mathfrak{F}$ by

$$\gamma_{10'}=-rD\mathfrak{F}\equiv-r\frac{\partial}{\partial r}\mathfrak{F} \tag{5.9}$$

we have

$$\gamma_{11'}=\eth\mathfrak{F}. \tag{5.10}$$

Finally, using (5.9) and (5.8), (5.3c) becomes

$$Dr^2D\mathfrak{F}-r^2D\frac{\partial}{\partial u}\mathfrak{F}+\bar{\eth}\,\eth\mathfrak{F}=r^2[D\mathfrak{F},\eth\mathfrak{F}] \tag{5.11}$$

our promised nonlinear wave equation. Obviously knowing $\mathfrak{F}$ allows, from (5.9) and (5.10), the calculation of the γ's and hence from (5.4) the calculation of the $\tilde{\chi}$. [One could easily check that (5.2) are identities.]

Explicitly we have for the fields

$$\tilde{\chi}_0=-\frac{1}{r}Dr^2D\mathfrak{F}, \tag{5.12a}$$

$$\tilde{\chi}_1=\eth D\mathfrak{F}, \tag{5.12b}$$

$$\tilde{\chi}_2=-\frac{1}{r}\eth^2\mathfrak{F} \tag{5.12c}$$

or alternately

$$\tilde{\chi}_0=+\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\gamma_{10'},\qquad \gamma_{10'}=-\frac{1}{r}\int_0^r r\tilde{\chi}_0\,dr, \tag{5.13a}$$

$$\tilde{\chi}_1=+\frac{1}{r^2}\eth\int_0^r r\tilde{\chi}_0\,dr, \tag{5.13b}$$

$$\tilde{\chi}_2=-\frac{1}{r}\eth^2\left(\int_0^r\tilde{\chi}_0\,dr-\frac{1}{r}\int_0^r r\tilde{\chi}_0\,dr\right). \tag{5.13c}$$

Though this appears to be quite pretty and simple, we unfortunately do not yet see a way to integrate Eq. (5.11).

VI. ATIYAH-WARD PROCEDURE

In this section we will describe our version (of the Sparling version) of the procedure due to Atiyah and Ward for producing self-dual solutions of the $GL(n,C)$ Yang-Mills equations. Unfortunately, we must use the full technology associated with the operators $\eth$ and $\bar{\eth}$ and spin-weight spherical harmonics.

We begin with an arbitrary (spin-weight one) matrix-valued function of the coordinates $u,\zeta,\bar{\zeta}$ that we will call $A(u,\zeta,\bar{\zeta})$. Though it is not necessary to do so, one could think of this function as the coefficient of r^{-1} in $\gamma_{01'}$, i.e., γ°, of Sec. IV. We will show that from $A(u,\zeta,\bar{\zeta})$ one can generate a unique self-dual solution of the Yang-Mills equations. In other words, we will have shown that from the radiation field of a real asymptotically "flat" Yang-Mills field there is associated uniquely a self-dual field.[11]

In Sec. III, Eq. (3.2), we introduce the parametrized null-cone tangent vectors $l_a(\zeta,\bar\zeta)$, from which we now define

$$u(x^a,\zeta,\bar\zeta)=l_a x^a\,, \tag{6.1}$$

where x^a is an arbitrary point of Minkowski space. We now ask for a matrix function $G(x^a,\zeta,\bar\zeta)$ which satisfies the linear matrix equation

$$\eth G+GA=0 \tag{6.2}$$

with $A=A(l_a x^a,\zeta,\bar\zeta)$, i.e., with (6.1) put into the u-slot of A. We will require that G be a regular function (in the sense of having a spherical harmonic expansion) of ζ and $\bar\zeta$. Though it is difficult to solve (6.2) with the regularity conditions [and we will avoid a discussion of methods of solving (6.2)], we will nevertheless assume that regular functions $G(x^a,\zeta,\bar\zeta)$, solutions of (6.2) are known or can be found. We will show that by purely algebraic and differential operations on G, a self-dual Yang-Mills solution can be produced. (Why this procedure works is not at all clear though it seems as if it is closely related to coding all the information about the field into the gauge transformation [see (5.6)] required to make $\gamma_a l^a$ and $\gamma_a m^a$ vanish, for any null vector fields l^a and m^a, with l^a, m^a being two legs of a null tetrad system l^a, n^a, m^a, $\overline{m}^a$, with $l^a n^a=-m^a\overline{m}^a=1$, other products vanishing.)

We first introduce some preliminary technology. In addition to $l_a(\zeta,\bar\zeta)$ of (3.2) we introduce

$$\begin{aligned} m_a&=\eth l_a\,,\\ \overline{m}_a&=\bar\eth l_a\,.\\ n_a&=l_a+\eth\bar\eth l_a \end{aligned} \tag{6.3}$$

which formally defines for each value of ζ and $\bar\zeta$ four independent vectors at any point of Minkowski space. (l_a is treated as if it is a spin-weight-zero function.) Note that, for each ζ and $\bar\zeta$ they satisfy $l^a n_a=-m^a\overline{m}_a=1$, other products vanishing, and that any vector $v^a(x^a)$ can be written

$$v^a=(v\cdot n)l^a-(v\cdot\overline{m})m^a-(v\cdot m)\overline{m}^a+(v\cdot l)n^a\,.$$

Further, it can easily be shown from the properties of $\eth$ and the form of l_a that

$$\eth^2 v=0\rightarrow v=v_a(x^a)l^a \tag{6.4}$$

if v is a regular spin-weight-zero function.

Our first claim is that $G_{,a}(x^a,\zeta,\bar\zeta)$ can be expressed by

$$G_{,a}G^{-1}=\gamma'_a(x^a)+jl_a+km_a\,, \tag{6.5}$$

with

$$\begin{aligned} j&=-m^a\bar\eth(G_{,a}G^{-1})\,,\\ k&=l^a\bar\eth(G_{,a}G^{-1})\,. \end{aligned} \tag{6.6}$$

To prove this we first note that from (6.2) we have

$$G^{-1}\eth G_{,a}=G^{-1}G_{,a}A-\dot{A}l_a\,, \tag{6.7}$$

where we have used

$$A_{,a}=\mathring{A}l_a,\quad \mathring{A}=\frac{\partial}{\partial u}A, \tag{6.8}$$

$$G^{-1}{}_{,a}=-G^{-1}G_{,a}G^{-1}\,. \tag{6.9}$$

From (6.7) it easily follows (using $\eth G^{-1}=-G^{-1}\eth GG^{-1}$) that

$$\eth(G_{,a}G^{-1})=-G\mathring{A}G^{-1}l_a\,. \tag{6.10}$$

Defining $V=G_{,a}G^{-1}l^a$, we see [using (6.10)] that

$$\begin{aligned} \eth V&=G_{,a}G^{-1}m^a+\eth(G_{,a}G^{-1})l^a\\ &=G_{,a}G^{-1}m^a \end{aligned} \tag{6.11}$$

and

$$\eth^2 V=0 \tag{6.12}$$

from which it follows from the regularity of G that

$$\begin{aligned} G_{,a}G^{-1}l^a&=\gamma'_a(x^a)l^a\,,\\ G_{,a}G^{-1}m^a&=\gamma'_a m^a\,. \end{aligned} \tag{6.13}$$

Equation (6.5) follows immediately from (6.13) using the orthognality properties of l, n, m, and $\overline{m}$. By applying $\bar\eth$ to (6.5),

$$\bar\eth(G_{,a}G^{-1})=\bar\eth jl_a+j\overline{m}_a+\bar\eth km_a+k(n_a-l_a)$$

and contracting with l^a and m^a, we obtain (6.6), thus proving our contention.

Our second claim is that $\gamma'_a(x^a)$, now written as

$$\gamma'_a(x^a)=G_{,a}G^{-1}+\eth hl_a-hm_a \tag{6.14}$$

with

$$h=l^a\bar\eth(G_{,a}G^{-1}) \tag{6.15}$$

is automatically the (matrix valued) vector potential for a self-dual Yang-Mills field. To prove this we must simply construct the field and show that it is self-dual. This, though relatively straightforward, is unfortunately rather tedious. We will sketch the proof.

Directly from (2.5) with (6.14) we have, after a lengthy calculation,

$$\begin{aligned} \tfrac{1}{2}F'_{ab}=\;&l_{[b}m_{a]}\{2a-\eth\bar b-c+[h,\eth h]+[\bar\beta,\eth h]-[\alpha,h]\}\\ &+l_{[a}n_{b]}\{b-\eth c+[\gamma,\eth h]\}+m_{[a}\overline{m}_{b]}\{-b+[\beta,h]\}\\ &+l_{[a}\overline{m}_{b]}\{\eth b-[\beta,\eth h]\}+n_{[a}m_{b]}\{-c+[\gamma,h]\} \end{aligned} \tag{6.16}$$

with

$$\begin{aligned} h_{,a}&=al_a-b\overline{m}_a-\bar bm_a+cn_a\,,\\ G_{,a}G^{-1}&=\alpha l_a-\beta\overline{m}_a-\bar\beta m_a+\gamma n_a\,. \end{aligned} \tag{6.17}$$

We now show that F_{ab} is self-dual and hence satisfies the Yang-Mills equations. It is not difficult to show that the bivectors

$$l_{[a}m_{b]};\ (l_{[a}n_{b]}+\overline{m}_{[a}m_{b]});\ n_{[a}\overline{m}_{b]} \tag{6.18}$$

are anti-self-dual, while

$$l_{[a}\overline{m}_{b]};\ (l_{[a}n_{b]}+m_{[a}\overline{m}_{b]});\ n_{[a}m_{b]} \tag{6.19}$$

are self-dual and hence the conditions for (6.16) to be self-dual are

$$2b-\eth c+[\gamma,\eth h]-[\beta,h]=0\,, \tag{6.20}$$

$$2a-\eth\overline{b}-c+[h,\eth h]+[\overline{\beta},\eth h]-[\alpha,h]=0\,. \tag{6.21}$$

By simply using the defining equations (6.17), one can directly show that (6.20) is an identity. To prove (6.21) involves a further step. Call the left side of (6.21) L; by direct calculation one can show that $\eth L=0$. From the fact that h and hence L is a regular spin-weight -1 function it then follows that $L=0$. Here also we have used the regularity condition on G.

We have thus shown that

$$\begin{aligned}\tfrac{1}{2}F'_{ab}=\,&l_{[a}\overline{m}_{b]}\{\eth b-[\beta,\eth h]\}\\&+(l_{[a}n_{b]}+m_{[a}\overline{m}_{b]})\{-b+[\beta,h]\}\\&+m_{[a}n_{b]}\{c-[\gamma,h]\}\end{aligned} \tag{6.22}$$

which is clearly self-dual. Introducing

$$\begin{aligned}\tilde{\chi}_0'&=c-[\gamma,h]\,,\\\tilde{\chi}_1'&=-b+[\beta,h]\,,\\\tilde{\chi}_2'&=\eth b-[\beta,\eth h]\end{aligned} \tag{6.23}$$

it is straightforward to show that

$$\begin{aligned}\tilde{\chi}_0'&=l^a l^b[\,\overline{\eth}(G_{,ab}G^{-1})-2G_{,a}G^{-1}\,\overline{\eth}(G_{,b}G^{-1})]\,,\\\tilde{\chi}_1'&=-\tfrac{1}{2}\eth\tilde{\chi}_0'\,,\\\tilde{\chi}_2'&=-\eth\tilde{\chi}_1'\,.\end{aligned} \tag{6.24}$$

As a final point we show the connection between the present section and Sec. V, i.e., we will relate G to $\mathfrak{F}$.

We first note that the freedom in the solution of Eq. (6.2) is

$$G(x^a,\zeta,\overline{\zeta})\rightarrow G'(x^a,\zeta,\zeta)=g(x^a)G(x^a,\zeta,\zeta)\,, \tag{6.25}$$

where $g(x^a)$ is an arbitrary nonsingular matrix function of x^a. This freedom in the solution for G corresponds to the gauge freedom.

We next introduce for x^a the null polar coordinates of Sec. III, namely $(u,r,\zeta,\overline{\zeta})$. We will, however, call them $(u,r,\eta,\overline{\eta})$ to avoid confusion with the parameters ζ and $\overline{\zeta}$ of this section. Actually we will be interested in evaluating functions of x^a and ζ at $\eta=\zeta$ and $\overline{\eta}=\overline{\zeta}$. Great care must be taken to perform appropriate differentiations, i.e., $\partial/\partial x^a$ and $\overline{\eth}$, before restricting η and $\overline{\eta}$ to ζ and $\overline{\zeta}$.

If now the gauge is chosen as in Sec. V, so that $\gamma'_a=\gamma_a$, i.e., $\gamma'_a l^a=\gamma'_a m^a=0$ with $\eta=\zeta$, $\overline{\eta}=\overline{\zeta}$, we have, from Eqs. (6.17) and (6.14),

$$\begin{aligned}&\gamma=0\,,\\&c=h_{,a}l^a=Dh=\frac{\partial}{\partial r}h\,.\end{aligned} \tag{6.26}$$

Comparing Eq. (6.23) with Eq. (5.12a) we have

$$\hat{h}\equiv h|_{\eta=\zeta,\overline{\eta}=\overline{\zeta}}=-D(r\mathfrak{F}) \tag{6.27}$$

or

$$\mathfrak{F}=\frac{-1}{r}\int^r\hat{h}\,dr\,. \tag{6.28}$$

By comparing the $\gamma_{10'}$ of Secs. V and VI, i.e., Eq. (5.9) with $\gamma'_a\overline{m}^a$ computed from (6.14), and using Eq. (6.27) we can obtain the alternate expression for $\mathfrak{F}$,

$$\mathfrak{F}=\frac{1}{r}(\overline{\eth}_{\overline{\eta}}G)\cdot G^{-1}|_{\eta=\zeta,\overline{\eta}=\overline{\zeta}}\,, \tag{6.29}$$

where $\overline{\eth}_{\overline{\eta}}$ refers to differentiation with respect to $\overline{\eta}$.

This work was supported by a grant from the National Science Foundation.

APPENDIX A

We present here for completeness the general GL(n,C) Yang-Mills equations in spin-coefficient[12] form with arbitrary choice of the tetrad vectors l_a, n_a, m_a, and $\overline{m}_a$. They are

$$\begin{aligned}D\chi_1-\overline{\delta}\chi_0=&(\pi-2\alpha)\chi_0+2\rho\chi_1-\kappa\chi_2\\&+[\chi_1,\gamma_{00'}]-[\chi_0,\gamma_{10'}]\,,\end{aligned} \tag{A1a}$$

$$\begin{aligned}D\chi_2-\overline{\delta}\chi_1=&-\lambda\chi_0+2\pi\chi_1+(\rho-2\epsilon)\chi_2\\&+[\chi_2,\gamma_{00'}]-[\chi_1,\gamma_{10'}]\,,\end{aligned} \tag{A1b}$$

$$\begin{aligned}\delta\chi_1-\Delta\chi_0=&(\mu-2\gamma)\chi_0+2\tau\chi_1-\sigma\chi_2\\&-[\chi_0,\gamma_{11'}]+[\chi_1,\gamma_{01'}]\,,\end{aligned} \tag{A1c}$$

$$\begin{aligned}\delta\chi_2-\Delta\chi_1=&-\nu\chi_0+2\mu\chi_1+(\tau-2\beta)\chi_2\\&-[\chi_1,\gamma_{11'}]+[\chi_2,\gamma_{01'}]\end{aligned} \tag{A1d}$$

with

$$\begin{aligned}\chi_0=&D\gamma_{01'}-\delta\gamma_{00'}+\gamma_{11'}\kappa-\gamma_{01'}(\epsilon-\overline{\epsilon}+\overline{\rho})\\&+\gamma_{00'}(\overline{\alpha}+\beta-\pi)-\gamma_{10'}\sigma\\&+[\gamma_{01'},\gamma_{00'}]\,,\end{aligned} \tag{A2a}$$

$$2\chi_1 = D\gamma_{11'} - \Delta\gamma_{00'} + \bar{\delta}\gamma_{01'} - \delta\gamma_{10'} + \gamma_{11'}(\epsilon + \bar{\epsilon} + \rho - \bar{\rho}) - \gamma_{01'}(\pi + \bar{\tau} + \alpha - \bar{\beta}) + \gamma_{10'}(\bar{\pi} + \tau + \bar{\alpha} - \beta) + \gamma_{00'}(\gamma + \bar{\gamma} + \mu - \bar{\mu}) + [\gamma_{11'}, \gamma_{00'}] - [\gamma_{10'}, \gamma_{01'}], \tag{A2b}$$

$$\chi_2 = \bar{\delta}\gamma_{11'} - \Delta\gamma_{10'} + \gamma_{11'}(\alpha + \bar{\beta} - \bar{\tau}) - \gamma_{01'}\lambda - \gamma_{10'}(\bar{\mu} + \gamma - \bar{\gamma}) + \gamma_{00'}\nu + [\gamma_{11'}, \gamma_{10'}]. \tag{A2c}$$

---

[1]No essential use is made of GL(n); we could restrict ourselves at any time to any of its subgroups.

[2]S. S. Chern, *Complex Manifolds without Potential Theory* (Van Nostrand, Princeton, N.J., 1967), Chap. 5.

[3]Actually, all bundles over Minkowski space, M, are trivial; it can be nontrivial only when one deletes portions of M or compactifies it.

[4]In the case of a timelike $t^a = (1,0,0,0)$ or a spacelike $t^a = (0,1,0,0)$, the respective gauges are frequently referred to as temporal and axial.

[5]Equations (3.7) and (3.8) have independently been derived by M. Carmeli, Phys. Lett. 68B, 463 (1977).

[6]J. Anandan and R. Roskies, Phys. Rev. D 18, 1152 (1978).

[7]This result was stimulated by an inquiry from R. Roskies and J. Plebanski.

[8]See P. Tod [Twistor Newsletter No. 6, Dec. 1977] for an alternate but equivalent approach to the potential.

[9]G. Sparling, report (unpublished).

[10]M. F. Atiyah and R. S. Ward, Commun. Math. Phys. 55, 117 (1977).

[11]Under special circumstances, i.e., for special $A(u, \zeta, \bar{\zeta})$ the self-dual field will have the same radiation field as the real asymptotically "flat" starting field. We will explore this question in a future paper.

[12]E. T. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962). The definitions of the $D, \Delta, \delta, \bar{\delta}$ etc. are found here.