Separable quasipotential formulation of the relativistic dynamics for three point particles

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In this paper I develop a relativistic Hamiltonian and Schrödinger equation for three particles using a generalization of the assumptions that give rise to Todorov's relativistic quasipotential equation for two particles. This formalism is treated by a method that leads to a generalization of the separable Lagrange equilateral-triangle solution from gravitational nonrelativistic to relativistic nongravitational bound-state problems in quantum as well as classical mechanics. As an example I derive an analytic solution for the spectrum of a three-body system bound by scalar Coulomb forces.

I. INTRODUCTION

Todorov's homogeneous quasipotential or relativistic Schrödinger equation for two particles was originally proposed, in part, as a formulation from which the relativistic eikonal expansion could be derived in a very efficient manner.[1] The Schrödinger-type equation comes from a homogeneous form of the inhomogeneous quasipotential equation

$$T+V+VGT=0 \tag{1}$$

by making certain assumptions about the Green's function. The connection between the potential V and a field theory is set by requiring that T be the two-body on-shell scattering amplitude. Its applications to quantum electrodynamics have been studied in recent work by Rizov, Todorov, and Aneva.[2]

With an eye toward phenomenological applications, in which the potential is not necessarily derived from a field theory, an alternative derivation of Todorov's relativistic Schrödinger equation has been given based on a Hamiltonian formulation.[3] Basically, we derived a two-body "Hamiltonian" as a sum of two one-body "Hamiltonians". The formalism developed was particularly suited to strongly coupled systems involving world scalar forces such as the scalar Coulomb force for large α. In a later paper, I applied this formalism to a phenomenological scalar linear potential.[4] A fit to the two lowest-lying charmonium states, $\psi/J(3095)$ and $\psi'(3684)$, allowed higher excited states predicted to be tested. A third excited state ψ''' was calculated to be at 4080 MeV. This agreed very well with the previously measured charmonium state at 4140 MeV.

The aim of this paper is twofold. The first is to develop a three-body "Hamiltonian" and Schrödinger equation along the lines of Todorov's quasipotential approach. Todorov has recently presented a derivation of his two-body relativistic Schrödinger equation by a method of generalized mass-shell constraints.[5] His approach is general enough to be applied to the three-body problem. In Sec. II, I present a slightly modified review of this approach as it applies to the two-body problem. I limit myself in this paper to scalar particles and scalar interactions. In doing so I also relate the work in this section to earlier work on the generalized form of Todorov's quasipotential equation for scalar interactions.[6] In Sec. III, the three-body "Hamiltonian" is given as a natural generalization of the two-body "Hamiltonian" is given as a natural generalization of the two-body case. It is required that the kinematics and dynamics reduce to that of two bodies when the mass and coupling of the third goes to zero. This eliminates some of the arbitrariness in the specification of the potential functions as well as in the relativistic kinematics. As with the nonrelativistic three-body problem, the accompanying equations of motion as well as the corresponding Schrödinger equation are not separable in relative coordinates. This brings me to the second aim of this paper.

The existence of simple, exactly soluble configurations in the gravitational three-body system is well known. These are the Lagrange equilateral-triangle and the Euler collinear configurations. Such equilibrium points, in fact, are observed in the solar system even though they require special initial conditions. The sun, Jupiter, and a cluster of asteroids sharing Jupiter's orbit known as the Trojan asteroids form a rotating equilateral triangle system.[7] The possibility that such rotating equilibrium points may exist for forces other than gravitational was demonstrated in a recent paper.[8] In particular, the general conditions under which the equations of motion separate into three two-body equations were given. Implications of the existence of quantum analogs to these separable solutions were also examined there. In Sec. IV those results are reviewed in the context of the relativistic "Hamiltonian" of Sec. III, and I compare those equations with the corresponding nonrelativistic equations. The consequences of separability of the equations of motion in classical mechanics translates into a separable Schrödinger equation in quantum mechanics. From this, in Sec. V I derive as an example an analytic solution for the spectrum of a three-body system bound by scalar Coulomb forces.

II. TWO-PARTICLE DYNAMICS

For two particles, the relativistic dynamics is established by imposing constraints of the form of generalized mass-shell conditions:[5,9]

$$2\phi_i = p_i^2 + m_i^2 + \Phi_i = 0, \quad i = 1, 2. \tag{2}$$

The functions Φ_i are Poincaré-invariant functions of the particles' coordinates and momenta. With no external forces this function vanishes when the interparticle separation goes to infinity. Certain combinations of the functions ϕ_i are used to define an effective relativistic "Hamiltonian".

The square of the total momentum,

$$P = p_1 + p_2, \tag{3}$$

is defined as $-w^2$ where w is the center-of-momentum (c.m.) value of the total energy of the two particles. Energies of the separate particles in the c.m. frame are

$$\begin{aligned} E_1 &= -\frac{1}{w} p_1 P = \frac{1}{2w}(w^2 - p_1^2 + p_2^2), \\ E_2 &= -\frac{1}{w} p_2 P = \frac{1}{2w}(w^2 - p_2^2 + p_1^2). \end{aligned} \tag{4}$$

The fundamental Poisson brackets are

$$\begin{aligned} &\{x_i^\mu, x_j^\nu\} = 0 = \{p_i^\mu, p_j^\nu\}, \\ &\{x_i^\mu, p_j^\nu\} = g^{\mu\nu}\delta_{ij}, \quad i, j = 1, 2. \end{aligned} \tag{5}$$

The relative position variable is

$$x = x_1 - x_2 \tag{6}$$

and has a vanishing Poisson bracket with the total momentum P. The relative momentum variable is defined by

$$p = \frac{E_2 p_1 - E_1 p_2}{2w}. \tag{7}$$

From (4) follows the orthogonality condition

$$P \cdot p = 0. \tag{8}$$

The two-body "Hamiltonian" is defined in terms of the functions ϕ_i as

$$\mathcal{H} = \frac{\phi_1}{E_1} + \frac{\phi_2}{E_2} = 0. \tag{9}$$

This definition is used because it is easily generalized to three bodies. This differs by a factor of $m_1 m_2/E_1 E_2$ from the "Hamiltonian" described in Ref. 3 and by a factor of $E_1 E_2/w$ from that given by Todorov in Ref. 5. This has no effect other than to scale the proper time (called τ_w here and in Ref. 3) associated with the equation of motion

$$\frac{dF}{d\tau_w} = \dot{F} = \{F, \mathcal{H}\} \tag{10}$$

for any dynamical variable F.

In the absence of external forces that may distinguish between the two particles one can choose

$$\Phi_1 = \Phi_2 = \Phi. \tag{11}$$

The set of invariants on which Φ may depend is, for scalar particles,

$$\begin{aligned} &I_1 = x^2 + (x \cdot P)^2/w^2 \equiv \tilde{x}^2, \\ &I_2 = x \cdot P, \quad I_3 = x \cdot p, \\ &I_4 = p^2, \quad P^2 = -w^2 = \text{const} \end{aligned} \tag{12}$$

with the derivatives with respect to these invariants defined as

$$\Phi^1, \Phi^2, \Phi^3, \Phi^4, \tag{13}$$

respectively. The function Φ^2 can be eliminated if the c.m. energies are required to be constants of the motion. Since

$$\dot{P} = \{P, \mathcal{H}\} = 0, \tag{14}$$

one has

$$\begin{aligned} \dot{E}_i &= -\frac{P}{w} \cdot \{p_i, \mathcal{H}\} \\ &= \frac{P}{2E_1 E_2} \cdot \{[2x + 2(x \cdot P)P]\Phi^1 \\ &\qquad + P\Phi^2 + p\Phi^3\}. \end{aligned} \tag{15}$$

Hence E_1 and E_2 are constants of the motion if $\Phi^2 = 0$ or

$$\Phi = \Phi(\tilde{x}^2, x \cdot p, p^2; w^2). \tag{16}$$

The orthogonality condition (8) is also a constant of motion as

$$\begin{aligned} \{P \cdot p, \mathcal{H}\} &= -\frac{w}{E_1 E_2} P \cdot \left\{ \left[2x + 2\left(x \cdot \frac{P}{w^2}\right) P \right] \Phi^1 + p\Phi^3 \right\} \\ &= 0. \end{aligned} \tag{17}$$

The constraint functions ϕ_1 and ϕ_2 have zero Poisson bracket with each other and hence with the Hamiltonian. This is also a consequence of (16).

The energies E_1 and E_2 are constants of the motion and functions of p_1 and p_2. Any such function must depend on the constant of motion $P^2 = -w^2$. Hence E_1 and E_2 must be functions of w, m_1, and m_2. In the case of two bodies, the constraints provide these constants. In particular, the combination

$$\phi_1 - \phi_2 = 0 = p_1{}^2 - p_2{}^2 + m_1{}^2 - m_2{}^2 \tag{18}$$

leads to

$$E_1 = \frac{1}{2w}(w^2 + m_1{}^2 - m_2{}^2)\,, \quad E_2 = \frac{1}{2w}(w^2 + m_2{}^2 - m_1{}^2)\,. \tag{19}$$

Other useful functions of the invariant w^2 are

$$E_1{}^2 - m_1{}^2 = E_2{}^2 - m_2{}^2 = b^2 = E^2 - m_w{}^2\,, \tag{20}$$

where

$$b^2 = \frac{1}{4w^2}(w^4 - 2(m_1{}^2 + m_2{}^2)w^2 + (m_1{}^2 - m_2{}^2)^2) \tag{21}$$

and

$$m_w = \frac{m_1 m_2}{w}\,,\ E^2 = \frac{1}{4w^2}(w^2 - m_1{}^2 - m_2{}^2)^2\,. \tag{22}$$

The functions b^2, m_w, and E are called respectively the on-shell c.m. value of the relative momentum squared, the relativistic reduced mass, and the energy of the fictitious particle of relative motion.[1]

Using (19), (21), and

$$p_1 = \frac{E_1}{w} P + p\,,\quad p_2 = \frac{E_2}{w} P - p \tag{23}$$

and substituting into the "Hamiltonian" (9) yields

$$\mathcal{H} = \frac{p^2 + \Phi - b^2}{2E_w} = 0\,, \tag{24}$$

where

$$E_w = E_1 E_2 / w\,. \tag{25}$$

The variable $x \cdot P$ is a constant of the motion if

$$\{x \cdot P, \mathcal{H}\} = \frac{P}{E_w} \cdot (2p + x\,\Phi^3 + 2p\Phi^4) = \frac{P \cdot x}{E_w}\,\Phi^3 = 0\,. \tag{26}$$

Hence if it vanishes initially, it vanishes for all time. By imposing this initial condition one has two orthogonality conditions:[5,10]

$$x \cdot P = 0 = p \cdot P\,. \tag{27}$$

These two conditions are conjugate in the sense

$$\{x \cdot P,\ p \cdot P\} = P^2 = -w^2\,. \tag{28}$$

Given the "Hamiltonian" (9) or (24), these two conditions restrict the accessible regions the system may occupy in phase space.

The physical interpretation of τ_w as the proper time of the c.m. system can be verified most readily if the p dependence of Φ is ignored. Consider

$$\frac{d}{d\tau_w}(x_1 - x_2) = \dot{x} = p/E_w = \frac{p_1}{E_1} - \frac{p_2}{E_2}\,. \tag{29}$$

Since

$$p_i = m_i \frac{dx_i}{d\tau_i}\,,\quad i = 1, 2 \tag{30}$$

the relationship between τ_w and the proper times of the two particles is

$$d\tau_i = \frac{m_i}{E_i}\,d\tau_w\,,\quad i = 1, 2\,. \tag{31}$$

For scalar potentials, the form of Φ can be inferred by rewriting (24) for $\Phi = 0$ as

$$\frac{p^2 - b^2}{2E_w} = \frac{\tilde{p}^2 + m_w{}^2}{2E_w}\,, \tag{32}$$

where

$$\tilde{p} = p + \frac{E}{w}\,P\,. \tag{33}$$

The scalar potential is introduced by the modification of the reduced mass[6]

$$m_w \to m_w + V_w\,. \tag{34}$$

Then Eq. (24) becomes

$$\frac{1}{2E_w}(\tilde{p}^2 + m_w{}^2 + 2m_w V_w + V_w{}^2) = \frac{p^2 + \Phi - b^2}{2E_w}\,. \tag{35}$$

Hence

$$\Phi = 2m_w V_w + V_w{}^2\,. \tag{36}$$

The motion takes place in the hyperplane defined by (27). Thus in the c.m. frame the relative time and energy variables x^0 and p^0 are zero and the fundamental dynamical variables are $\vec{x}$ and $\vec{p}$. With this in mind the transition to quantum mechanics becomes straightforward. In the Schrödinger representation,

$$\langle\psi|\psi\rangle = \int (dx)\psi^*(x)\psi(x)\delta(P \cdot x) = \int (d\vec{x})\psi^*(\vec{x})\psi(\vec{x}) \tag{37}$$

in the c.m. frame, and the relativistic Schrödinger equation

$$\mathcal{H}\psi = 0 \tag{38}$$

has the three-dimensional form

$$(-\vec{\nabla}^2 + \Phi)\psi = b^2\psi\,. \tag{39}$$

For the scalar Coulomb forces[11]

$$\Phi = -2m_w\frac{\alpha}{r} + \frac{\alpha^2}{r^2}\,, \tag{40}$$

and the resulting spectrum is[6]

$$w^2 = m_1{}^2 + m_2{}^2 + 2m_1m_2\left(1 - \frac{\alpha^2}{n'^2}\right)^{1/2}, \tag{41}$$

where

$$n' = n - l - \tfrac{1}{2} + [(l+\tfrac{1}{2})^2 + \alpha^2]^{1/2}, \quad n = 1, 2, \ldots. \tag{42}$$

The purpose of this section has been a review of recent recastings and expositions by Todorov and to a lesser extent myself on the question of two-body relativistic dynamics. It will provide a "two-body limit", so to speak, which I will demand the three-body formulation to satisfy.

III. THREE-PARTICLE DYNAMICS

As with the two body problem, the square of the total momentum

$$P = p_1 + p_2 + p_3 \tag{43}$$

is defined as $-w^2$ where w is the c.m. value of the total energy of the three particles. Relative position and momentum are defined by

$$x_{ij} = x_i - x_j, \tag{44}$$

$$p_{ij} = (E_j p_i - E_i p_j)/w. \tag{45}$$

The definition (the particle subscripts i, j, k are in cyclic order throughout this paper)

$$\begin{aligned} 2\phi_i &= p_i{}^2 + m_i{}^2 + \Phi_{ij}(\tilde{x}_{ij}{}^2, x_{ij}p_{ij}, p_{ij}{}^2) \\ &\quad + \Phi_{ik}(\tilde{x}_{ik}{}^2, x_{ik}p_{ik}, p_{ik}{}^2) \\ &\equiv p_i{}^2 + m_i{}^2 + \Phi_i = 0 \end{aligned} \tag{46}$$

of the constraint variables with $\Phi_{ij} = \Phi_{ji}$ gives the ordinary two-body dynamics for particles i and j if $\Phi_{ik} = 0 = \Phi_{jk}$. For scalar interactions I assume[12]

$$\Phi_{ij} = \frac{2m_i m_j}{E_i + E_j} V_{ij} + V_{ij}{}^2. \tag{47}$$

The c.m. energies E_1, E_2, and E_3 are

$$\begin{aligned} E_i &= -\frac{1}{w} P \cdot p_i \\ &= \frac{1}{3w}(w^2 - 2p_i{}^2 + p_j{}^2 + p_k{}^2 \\ &\quad + 2p_j p_k - p_k p_i - p_i p_j). \end{aligned} \tag{48}$$

From the cyclic nature of this equation it is easy to see that

$$E_1 + E_2 + E_3 = w. \tag{49}$$

Also notice that one has orthogonality condition

$$P \cdot p_{ij} = 0. \tag{50}$$

The fundamental Poisson brackets are as in (5) except that there are now three variables. The three-body "Hamiltonian" is defined as

$$\mathcal{H} = \frac{\phi_1}{E_1} + \frac{\phi_2}{E_2} + \frac{\phi_3}{E_3} = 0. \tag{51}$$

The equations of motion are

$$\begin{aligned} \frac{dx_i}{d\tau_w} &= \dot{x}_i = \{x_i, \mathcal{H}\} \\ &= \frac{p_i}{E_i} + \left(\Phi_{ij}{}^3 \frac{x_{ij}E_j}{2} + \Phi_{ij}{}^4 p_{ij} E_j\right) \Big/ E_{ij} + (j \to k), \end{aligned} \tag{52}$$

$$\begin{aligned} \frac{dp_i}{d\tau_w} &= \dot{p}_i = \{p_i, \mathcal{H}\} \\ &= -(\tilde{x}_{ij}\Phi_{ij}{}^1 + \tfrac{1}{2}p_{ij}\Phi_{ij}{}^3)/E_{ij} - (j \to k), \end{aligned}$$

where

$$E_{ij} = E_i E_j/(E_i + E_j).$$

The superscripts on the Φ's designate derivatives with respect to the appropriate invariants [see (13)]. Using the latter equation, it is clear that the c.m. energies E_i are constants of the motion just as in the case of two bodies, that is,

$$\dot{E}_i = -\frac{P}{w} \cdot \{p_i, \mathcal{H}\} = 0. \tag{53}$$

Hence E_1, E_2, and E_3 must be functions of w, m_1, m_2, and m_3. Unlike the case of two bodies, the constraints do not provide explicit forms for these constants. In any event, when the mass and coupling of particle k vanishes, the c.m. energies of particle i and j should be given by the two-body forms

$$\begin{aligned} E_i &= \frac{1}{2w}(w^2 + m_i{}^2 - m_j{}^2), \\ E_j &= \frac{1}{2w}(w^2 + m_j{}^2 - m_i{}^2). \end{aligned} \tag{54}$$

As with the two-body problem there are other constants of the motion. The Poisson bracket of $P \cdot p_{ij}$ with $\mathcal{H}$ vanishes:

$$\begin{aligned} \{P \cdot p_{ij}, \mathcal{H}\} &= \frac{P}{w} \cdot (E_j\{p_i, \mathcal{H}\} - E_i\{p_j, \mathcal{H}\}) \\ &= 0. \end{aligned} \tag{55}$$

Thus, this orthogonality condition holds for all time. Is the invariant $P \cdot x_{ij}$ also a constant of the motion? Its Poisson bracket with $\mathcal{H}$ can be found by using the equation of motion for x_i and x_j. Observing that

$$\frac{p_i}{E_i} - \frac{p_j}{E_j} = \frac{w}{E_i E_j} p_{ij} \text{ and } P \cdot p_{ij} = 0, \tag{56}$$

it is clear from (52) that it is a constant of the motion if initially $P \cdot x_{ij} = 0 = P \cdot x_{ik}$:

$$\{P\cdot x_{ij},\mathcal{H}\}=P\cdot\left[x_{ij}\frac{E_j^2-E_i^2}{2E_iE_j}\Phi_{ij}{}^3+x_{ik}\frac{E_k^2-E_i^2}{2E_kE_i}\Phi_{ik}{}^3\right]=0. \quad (57)$$

By requiring it to vanish one has the two sets of orthogonality conditions

$$x_{ij}\cdot P=0=p_{ij}\cdot P,\quad i,j=i,2,3,\quad i\neq j. \quad (58)$$

I will restrict consideration in the rest of the paper to potentials Φ that do not depend on the relative momenta p_{ij}. Using this and the orthogonality condition $x_{ij}\cdot P=0$ implies

$$\mathcal{H}=\frac{p_1{}^2+m_1{}^2}{2E_1}+\frac{p_2{}^2+m_2{}^2}{2E_2}+\frac{p_3{}^2+m_3{}^2}{2E_3}+\mathcal{V}_{12}(x_{12})+\mathcal{V}_{23}(x_{23})+\mathcal{V}_{31}(x_{31})=0, \quad (59)$$

where

$$\mathcal{V}_{ij}=\frac{\Phi_{ij}}{2E_{ij}}. \quad (60)$$

The equations of motion (52) simplify to

$$\dot{x}_i=p_i/E_i,\qquad \dot{p}_i=-\partial_i\mathcal{V}_{ij}-\partial_i\mathcal{V}_{ik}. \quad (61)$$

If I define

$$q_k=x_i-x_j, \quad (62)$$

then combining the equations of motion from (59) and (60) leads to

$$\ddot{q}_i+\frac{\partial}{\partial q_i}\frac{\mathcal{V}_{jk}(q_i)}{E_jE_k}w=E_iZ,\quad i,j,k \text{ in cyclic order} \quad (63)$$

where

$$Z=\frac{\partial}{\partial q_1}\frac{\mathcal{V}_{23}(q_1)}{E_2E_3}+\frac{\partial}{\partial q_2}\frac{\mathcal{V}_{31}(q_2)}{E_3E_1}+\frac{\partial}{\partial q_3}\frac{\mathcal{V}_{12}(q_{13})}{E_1E_2}=Z(q_1,q_2,q_3). \quad (64)$$

Equations (63) and (64) are the relativistic generalization of

$$\ddot{\vec{q}}_i+\vec{\nabla}_{q_i}\frac{\mathcal{V}_{jk}(\vec{q}_i)}{m_jm_k}M=m_i\vec{Z}, \quad (65)$$

where

$$\vec{Z}=\frac{\vec{\nabla}_{q_1}\mathcal{V}_{23}(\vec{q}_1)}{m_2m_3}+\frac{\vec{\nabla}_{q_2}\mathcal{V}_{31}(\vec{q}_2)}{m_3m_1}+\frac{\vec{\nabla}_{q_3}\mathcal{V}_{12}(\vec{q}_3)}{m_1m_2}. \quad (66)$$

These equations can be derived from the nonrelativistic-three body Hamiltonian

$$H=\frac{\vec{p}_1{}^2}{2m_1}+\frac{\vec{p}_2{}^2}{2m_2}+\frac{\vec{p}_3{}^2}{2m_3}+\mathcal{V}_{12}(\vec{q}_3)+\mathcal{V}_{23}(\vec{q}_1)+\mathcal{V}_{31}(\vec{q}_1) \quad (67)$$

by manipulating Hamilton's equation as in (61)–(64). In fact, since in the c.m. frame $q_1^0=\dot{q}_1^0=0$, Eqs. (63) and (64) are of the same form as (65) and (66) except that the c.m. energies E_i replace the masses m_i and the total c.m. energy w replaces M.

IV. SEPARABLE SOLUTIONS FOR THE PROBLEM OF THREE BODIES

In a recent paper on the nonrelativistic three-body problem, I showed that the special types of configurations of Euler and Lagrange that allowed exact and separable solutions of the three-body problem are not unique to gravitational forces.[8] The equations of motion for these special configurations could be regarded as derived formally by applying Hamilton's equations to a separable Hamiltonian. Because of the analogies between (63) and (64) and the nonrelativistic equations (65) and (66) [Eqs. (10) and (11) in Ref. 8], the arguments given there are easily adaptable to the relativistic case.

The "Hamiltonian" (59) and the resultant equations (63) and (64) are not separable. This is, of course, the primary difficulty with the three-body problem. In the two-body problem a separation is achieved by introducing c.m. and relative coordinates. The analogous substitutions here are

$$p_i=\frac{E_iP}{w}-\mathfrak{p}_j+\mathfrak{p}_k,\quad i,j,k \text{ cyclic} \quad (68)$$

where

$$\mathfrak{p}_j=p_{ki},\quad \mathfrak{p}_k=p_{ij}. \quad (69)$$

Substituting into (59) gives

$$\mathcal{H}=\frac{1}{2E_1}(\mathfrak{p}_2-\mathfrak{p}_3)^2+\frac{1}{2E_2}(\mathfrak{p}_3-\mathfrak{p}_1)^2+\frac{1}{2E_3}(\mathfrak{p}_1-\mathfrak{p}_2)^2+\mathcal{V}_{12}(q_3)+\mathcal{V}_{23}(q_1)\mathcal{V}_{31}(q_2)-B=0 \quad (70)$$

with

$$B=\frac{1}{2}\left(w-\frac{m_1{}^2}{E_1}-\frac{m_2{}^2}{E_2}-\frac{m_3{}^2}{E_3}\right). \quad (71)$$

The nonseparability in (70) resides in the momentum terms rather than the coordinate terms as with (59). The variables $\mathfrak{p}$ and q are not independent. In particular,

$$q_1+q_2+q_3=0 \quad (72)$$

or from (61)

$$E_1 \mathfrak{p}_1 + E_2 \mathfrak{p}_2 + E_3 \mathfrak{p}_3 = 0 . \tag{73}$$

Adding $1/2E_1E_2E_3$ times the square of this to (70) results in

$$\mathcal{H} = \frac{p_1^2}{2\mu_1} + \frac{p_2^2}{2\mu_2} + \frac{p_3^2}{2\mu_3} + \mathcal{V}_{12}(\mathfrak{q}_3) + \mathcal{V}_{23}(\mathfrak{q}_1) + \mathcal{V}_{31}(\mathfrak{q}_2) - B , \tag{74}$$

where

$$\mu_i = \frac{E_j E_k}{w} . \tag{75}$$

The fact that this appears separable is just an optical illusion. The variables $\mathfrak{q}_i$ and $\mathfrak{p}_i$ cannot be treated as independent canonically conjugate variables because in arriving at (74) I used the triangle constraint equations (72) and (73). They can be treated as independent if a Lagrange multiplier of the form $\lambda \circ (\mathfrak{q}_1 + \mathfrak{q}_2 + \mathfrak{q}_3)$ is added to this Hamiltonian. The resulting equations of motion are then

$$\mu_i \ddot{\mathfrak{q}}_i + \frac{\partial}{\partial \mathfrak{q}_i} \mathcal{V}_{jk}(\mathfrak{q}_i) = -\lambda . \tag{76}$$

Comparing this with (63) implies

$$\lambda = -\frac{E_1 E_2 E_3}{w} Z(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3) . \tag{77}$$

These same equations of motion can also be derived from (70) by using Hamilton's equations and treating $\mathfrak{q}_i, \mathfrak{p}_i$ as canonically conjugate. The reason this is true is that the triangle constraint is not used in arriving at (70). In fact, that constraint follows as a consequence of the first of Hamilton's equations. Consider

$$\frac{\partial}{\partial \mathfrak{p}_i} \mathcal{H} = \dot{\mathfrak{q}}_i = \frac{1}{E_k}(\mathfrak{p}_i - \mathfrak{p}_j) - \frac{1}{E_j}(\mathfrak{p}_k - \mathfrak{p}_i) . \tag{78}$$

This implies

$$\dot{\mathfrak{q}}_1 + \dot{\mathfrak{q}}_2 + \dot{\mathfrak{q}}_3 = 0 . \tag{79}$$

Hence the constraint $\mathfrak{q}_1 + \mathfrak{q}_2 + \mathfrak{q}_3 = c_1$ is a first integral of the equation of motion. The initial conditions of the three-body problem imply that $\vec{c}_1 = 0$ in c.m. coordinates and hence $c_1 = 0$ in all frames. The second set of Hamilton's equations

$$\frac{\partial}{\partial \mathfrak{p}_i} H = \dot{\mathfrak{p}}_i \tag{80}$$

when combined with (78) leads to (63) and (64).

Including the Lagrange multiplier, the three-body Hamiltonian (74) is

$$\mathcal{H} = \frac{\mathfrak{p}_1^2}{2\mu_1} + \frac{\mathfrak{p}_2^2}{2\mu_2} + \frac{\mathfrak{p}_3^2}{2\mu_3} + \mathcal{V}_{12}(\mathfrak{q}_3) + \mathcal{V}_{23}(\mathfrak{q}_2) + \mathcal{V}_{31}(\mathfrak{q}_3) - \frac{E_1 E_2 E_3}{w} Z \circ (\mathfrak{q}_1 + \mathfrak{q}_2 + \mathfrak{q}_3) - B = 0 . \tag{81}$$

This Hamiltonian is separable if $Z = 0$. It is just such a condition that, in the nonrelativistic equations (65) and (66), leads to the special configuration of a rotating equilateral triangle in the case of gravitational forces. That is called the Lagrange solution. With

$$\mathcal{V}_{ij}(\vec{\mathfrak{q}}_k) = -\frac{\kappa m_i m_j}{|\vec{\mathfrak{q}}_k|} , \tag{82}$$

one finds that

$$\vec{Z} = \kappa \left(\frac{\vec{\mathfrak{q}}_1}{|\vec{\mathfrak{q}}_1|^3} + \frac{\vec{\mathfrak{q}}_2}{|\vec{\mathfrak{q}}_2|^3} + \frac{\vec{\mathfrak{q}}_3}{|\vec{\mathfrak{q}}_3|^3} \right) = \vec{0} \tag{83}$$

for $|\vec{\mathfrak{q}}_1| = |\vec{\mathfrak{q}}_2| = |\vec{\mathfrak{q}}_3|$ since $\vec{\mathfrak{q}}_1 + \vec{\mathfrak{q}}_2 + \vec{\mathfrak{q}}_3 = \vec{0}$.

In Ref. 8 I gave several examples of nongravitational forces which could lead to a separable nonrelativistic Hamiltonian. The $\vec{Z} = 0$ restriction may or may not imply an equilateral-triangle configuration depending on the mass ratios. For example, if I assume a scalar Coulomb potential of the form

$$\mathcal{V}_{ij}(\vec{\mathfrak{q}}_k) = \alpha_{ij} \mathcal{V}(\vec{\mathfrak{q}}_k) = \frac{-\alpha_{ij}}{|\vec{\mathfrak{q}}_k|} , \tag{84}$$

then from (66)

$$\vec{Z} = \frac{\vec{\mathfrak{q}}_1}{|\vec{\mathfrak{q}}_1|^3} \frac{\alpha_{23}}{m_2 m_3} + \frac{\vec{\mathfrak{q}}_2}{|\vec{\mathfrak{q}}_2|^3} \frac{\alpha_{31}}{m_3 m_1} + \frac{\vec{\mathfrak{q}}_3}{|\vec{\mathfrak{q}}_3|^3} \frac{\alpha_{12}}{m_1 m_2} . \tag{85}$$

In the case of a general triangular solution

$$\vec{\mathfrak{q}}_1 + \vec{\mathfrak{q}}_2 + \vec{\mathfrak{q}}_3 = 0, \quad |\vec{\mathfrak{q}}_1| = \lambda |\vec{\mathfrak{q}}_3| , \quad |\vec{\mathfrak{q}}_2| = \rho |\vec{\mathfrak{q}}_3 ,$$

and

$$\vec{Z} = \frac{\alpha_{12}}{m_1 m_2} \frac{1}{|\vec{\mathfrak{q}}_3|^3} \left(\frac{m_1}{m_3} \frac{\alpha_{23}}{\alpha_{12}} \frac{\vec{\mathfrak{q}}_1}{\lambda^3} + \frac{m_2}{m_3} \frac{\alpha_{31}}{\alpha_{12}} \cdot \frac{\vec{\mathfrak{q}}_2}{\rho_3} + \vec{\mathfrak{q}}_3 \right) . \tag{86}$$

The Hamiltonian is separable $(\vec{Z} = \vec{0})$ if

$$\lambda = \left(\frac{m_1}{m_3} \frac{\alpha_{23}}{\alpha_{12}} \right)^{1/3} , \quad \rho = \left(\frac{m_2}{m_3} \frac{\alpha_{31}}{\alpha_{12}} \right)^{1/3} . \tag{87}$$

The triangle inequality implies

$$\lambda + \rho \geq 1 . \tag{88}$$

It places restrictions on the coupling constants and masses. For example, if the masses are equal, then

$$\left(\frac{\alpha_{23}}{\alpha_{12}} \right)^{1/3} + \left(\frac{\alpha_{31}}{\alpha_{12}} \right)^{1/3} \geq 1 , \tag{89}$$

which is clearly satisfied if the couplings are equal.

The most attractive feature concerning applicability to the quantum system, whether nonrelativistic or relativistic, is the separability of the equations of motion. This would mean that the corresponding Schrödinger equation would be separable. In Ref. 8 I demonstrated that in the case of helium, a spectrum derived from such a procedure is moderately accurate, in fact almost as good as perturbation theory. Questions that I left unanswered there concerned the nature of this approximation, in particular, how the constraint $\vec{q}_1+\vec{q}_2+\vec{q}_3=\vec{0}$ is to be imposed on the wave functions and eigenvalues. That will not be dealt with in this paper either. The purpose here is to obtain a relativistic three-body quantum spectrum by applying this separable approximation to the three-body relativistic Schrödinger equation whose classical "Hamiltonian" form is given in (81).

The quantum analog of (81) is

$$\mathcal{H}\psi=0\ . \tag{90}$$

It is a three-body homogeneous quasipotential or relativistic Schrödinger equation. If I write this equation in the c.m. system where $q_i^0=p_i^0=0$, then imposing the separability assumption $Z=0$ leads to

$$\left(\frac{-\vec{\nabla}_1^{\,2}}{2\mu_1}\frac{-\vec{\nabla}_2^{\,2}}{2\mu_2}\frac{-\vec{\nabla}_3^{\,2}}{2\mu_3}+\mathcal{V}_{12}(\vec{q}_3)+\mathcal{V}_{23}(\vec{q}_1)+\mathcal{V}_{31}(\vec{q}_2)\right)\times\psi(\vec{q}_1,\vec{q}_2,\vec{q}_3)=B\psi(\vec{q}_1,\vec{q}_2,\vec{q}_3)\ . \tag{91}$$

In the nonrelativistic limit

$$\mu_i=\frac{m_jm_k}{M} \tag{92}$$

and the variable B becomes the binding energy. To see this let

$$w=M-\Delta E\ ,\qquad E=m_i-\Delta E_i\ , \tag{93}$$

where

$$\Delta E=\Delta E_1+\Delta E_2+\Delta E_3\ . \tag{94}$$

Substituting into (71) and assuming $\Delta E\ll M$ leads to $B=-\Delta E$. For the scalar Coulomb potential (84)

$$B=B_1+B_2+B_3\ , \tag{95}$$

where

$$B_i=\frac{-\mu_i\alpha_{jk}^{\,2}}{2n_i^{\,2}}\ . \tag{96}$$

(I suspect that the principal quantum numbers n_1, n_2, n_3, are not arbitrarily related. This is one of those problem areas mentioned above.) Hence

$$w=M\left(1-\frac{m_1m_2\alpha_{12}^{\,2}}{2M^2n_3^{\,2}}-\frac{m_2m_3\alpha_{23}^{\,2}}{2M^2n_1^{\,2}}-\frac{m_3m_1\alpha_{31}^{\,2}}{2M^2n_2^{\,2}}\right)\ . \tag{97}$$

Notice that if $m_3, \alpha_{13}, \alpha_{23}\to 0$ then

$$w=m_1+m_2-\frac{m_1m_2}{2(m_1+m_2)}\frac{\alpha_{13}^{\,2}}{n_3^{\,2}}\ . \tag{98}$$

That is, the three-body spectrum (97) has the expected two-body "limit" (98).

V. A RELATIVISTIC SPECTRUM FOR THE SEPARABLE THREE-BODY PROBLEM

This is not a trivial kinematical generalization of the nonrelativistic spectrum. The reason is that the E_i's are unknown functions of the c.m. total energy w and the m_i's. To determine what these functions might be I shall require that the formalism have the correct two-body limit. This was mentioned earlier below Eq. (53). Notice further that the three-body "Hamiltonian" (70) reduces to the two-body form

$$\mathcal{H}=\frac{p_{12}^{\,2}+\Phi_{12}-b^2}{2E_w} \tag{99}$$

if the mass and coupling of particle 3 goes to zero in such a way that E_1 and E_2 reduce to the forms given in (19).[13] In particular,

$$B\to\frac{1}{2}\left(w-\frac{m_1^{\,2}}{E_1}-\frac{m_2^{\,2}}{E_2}\right)=\frac{b^2}{2E_w}\ . \tag{100}$$

The primary problem, insofar as the kinematics is concerned, is the determination of the functional dependence of E_1, E_2, E_3 or w, m_1, m_2, m_3. As mentioned above, the constraints do not provide an answer as in the two-body case. It is, however, important to obtain these functions so that the eigenvalue B is expressed in terms of the total c.m. energy. The variable B is related to binding energy. In particular, if the particles are unbound and static then

$$E_i=\frac{1}{3w}\left(w^2+2m_i^{\,2}-m_j^{\,2}-m_k^{\,2}-2m_jm_k+m_im_j+m_im_k\right)\Big|_{w=M}=m_i\ . \tag{101}$$

Obviously B vanishes in this case. In the general case, $w\neq M=m_1+m_2+m_3$, so the choice

$$E_i=\frac{1}{3w}\left(w^2+2m_i^{\,2}-m_j^{\,2}-m_k^{\,2}-2m_jm_k+m_im_j+m_im_k+f_i(w^2-m^2)\right) \tag{102}$$

is an obvious extension of (101). The functions f_i are not entirely arbitrary. If the kth particle decouples and has a zero mass and energy then

$$E_k=\frac{1}{3w}\left[w^2-m_i^{\,2}-m_j^{\,2}-2m_im_j+f_k(w^2-(m_i+m_j)^2)\right]=0 \tag{103}$$

or

$$f_k(w,m_i,m_j,m_k)\big|_{m_k=0}=-1 \tag{104}$$

and

$$E_i=\frac{1}{3w}\left[w^2+2m_i^2-m_j^2+m_im_j+f_i(w^2-(m_i+m_j)^2)\right]$$
$$=\frac{1}{2w}(w^2+m_i^2-m_j^2), \tag{105}$$

or

$$f_i(w,m_i,m_j,m_k)\big|_{m_k=0}=\tfrac{1}{2}\,. \tag{106}$$

Furthermore, (49) implies

$$f_1+f_2+f_3=0\,. \tag{107}$$

In the application of this to a bound-state system, the individual E_i's are not observable quantities. Thus, as long as their sum is w it should, in principle, not matter what functions are chosen as long as those choices are consistent with the definition (48) and the limiting forms in Eq. (19). In particular, for equal masses the choice $E_1=E_2=E_3=w/3$ is consistent. This implies that

$$f_i(w,m,m,m,)=0. \tag{108}$$

An example of a class of functions that satisfy (104), (106), (107), and (108) is

$$f_i(w,m_1,m_2,m_3)=\frac{1}{2}\,\frac{(m_im_j)^\eta+(m_im_k)^\eta-2(m_jm_k)^\eta}{(m_1m_2)^\eta+(m_2m_3)^\eta+(m_3m_1)^\eta}\,,\quad \eta>0. \tag{109}$$

The separable relativistic three-body Schrödinger equation (91) is equivalent to three two-body equations of the form

$$\left(\frac{\vec{\nabla}_i^2}{2\mu_i}+\mathcal{V}_{jk}(\vec{q}_i)\right)\psi(\vec{q}_i)=B_i(\vec{q}_i) \tag{110}$$

with

$$B=B_1+B_2+B_3. \tag{111}$$

Now the relativistic scalar potential in the c.m. frame is, from (47),

$$\mathcal{V}_{jk}(q_i)=\frac{\Phi_{jk}}{2E_jE_k}(E_j+E_k)$$
$$=\frac{m_jm_k}{E_jE_k}V_{jk}+V_{jk}^2\,\frac{E_j+E_k}{2E_jE_k} \tag{112}$$

with

$$V_{jk}=-\frac{\alpha_{jk}}{|\vec{q}_i|}\,. \tag{113}$$

The radial form of the relativistic Schrödinger equation (110) is ($q_i=|\vec{q}_i|$here)

$$\left[-\frac{w}{2E_jE_k}\frac{d^2}{dq_i^2}+\frac{w}{2E_jE_k}\frac{l_i(l_i+1)}{q_i^2}-\frac{m_jm_k}{E_jE_k}\frac{\alpha_{jk}}{q_i}+\frac{E_j+E_k}{2E_jE_k}\frac{\alpha_{jk}^2}{q_i^2}\right]u(q_i)=B_iu(q_i). \tag{114}$$

To determine the spectrum compare it with the nonrelativistic Schrödinger equation of the form

$$\left[-\frac{1}{2m}\frac{d^2}{dq^2}+\frac{l(l+1)}{2mq^2}-\frac{\alpha}{q}+\frac{\beta^2}{2mq^2}\right]u=\epsilon u. \tag{115}$$

It has the eigenvalue solution

$$\epsilon=-\frac{m\alpha^2}{2n'^2}\,, \tag{116}$$

where by analytic continuation of the angular momentum

$$n'=n-l-\tfrac{1}{2}+[(l+\tfrac{1}{2})^2+\beta^2]^{1/2}. \tag{117}$$

Applying this to (114) implies

$$B_i=-\frac{E_jE_k}{2w}\left(\frac{m_jm_k}{E_jE_k}\right)^2\alpha_{jk}^2\,\frac{1}{n_i'^2}$$
$$=\frac{-m_j^2m_k^2}{2wE_jE_k}\,\frac{\alpha_{jk}^2}{n_i'^2}\,, \tag{118}$$

where

$$n_i'=n_i-l_i-\tfrac{1}{2}+[(l_i+\tfrac{1}{2})^2+\beta_i^2]^{1/2} \tag{119}$$

and

$$\beta_i^2=\alpha_{jk}^2\,\frac{E_j+E_k}{w}\,. \tag{120}$$

Hence the eigenvalues for the total c.m. energy w are given by the solution to

$$B=B_1+B_2+B_3 \tag{121}$$

or to

$$w-\frac{m_1^2}{E_1}-\frac{m_2^2}{E_2}-\frac{m_3^2}{E_3}=-\frac{m_1^2m_2^2\alpha_{12}^2}{wE_1E_2n_3'^2}-\frac{m_2^2m_3^2\alpha_{23}^2}{wE_2E_3n_1'^2}-\frac{m_3^2m_1^2\alpha_{31}^2}{wE_3E_1n_2'^2}\,. \tag{122}$$

Now if E_3, m_3, α_{23}, $\alpha_3\to 0$ and E_1 and E_2 take on their limiting form (19) then solving this equation for w yields (41).[13] In the general case it is necessary to have explicit forms for E_i.

In general

$$E_i=\frac{1}{3w}\left(w^2+2m_i^2-m_j^2-m_k^2-2m_jm_k+m_km_i+m_im_j+f_i(w^2-M^2)\right). \tag{123}$$

For small coupling, the nonrelativistic result (97) should come from (122). The binding energy is

$$\Delta E=M-w\ll M \tag{124}$$

and using the approximation in (123) gives

$$E_i \to m_i - \frac{\Delta E}{M} h_i m_i , \tag{125}$$

where

$$h_i = \frac{2}{3}\frac{M}{m_i}(1+f_i) - 1. \tag{126}$$

Using this in (122) and retaining only terms of order $\Delta E/M$ leads to

$$\Delta E = \frac{1}{2}\frac{m_1 m_2 \alpha_{12}^2}{M n_3^2} + \frac{1}{2}\frac{m_2 m_3 \alpha_{23}^2}{M n_1^2} + \frac{1}{2}\frac{m_3 m_1 \alpha_{31}^2}{M n_2^2} \tag{127}$$

and w is the same as given in (97). Notice that the arbitrary functions f_i do not appear. The reason is that the linearity of their appearance allows (107) to be imposed.

In the relativistic case I define a variable Δ^2 by

$$w^2 = M^2 - 3\Delta^2. \tag{128}$$

Then

$$E_i = \frac{1}{w}\left(m_i M - \Delta^2(1+f_i)\right). \tag{129}$$

Equation (122) can be rewritten as

$$wE_1E_2E_3 - m_1^2E_2E_3 - m_2^2E_3E_1 - m_3^2E_1E_2 = -\frac{m_1^2m_2^2E_3\alpha_{12}^2}{wn_3'^2} - \frac{m_2^2m_3^2E_1\alpha_{23}^2}{wn_1'^2} - \frac{m_3^2m_1^2E_2\alpha_{31}^2}{wn_2'^2}. \tag{130}$$

Substituting (128) leads to

$$\Delta^6 - \Delta^4(m_1'(m_2+m_3) + m_2'(m_3+m_1) + m_3'(m_1+m_2)) + \Delta^2\left(3m_1'm_2'm_3'M + \frac{m_1'm_2'm_1m_2\alpha_{12}^2}{n_3'^2} + \frac{m_2'm_3'm_2m_3m_3\alpha_{23}^2}{n_1'^2} + \frac{m_3'm_1'm_3m_1\alpha_{31}^2}{n_2'^2}\right) - m_1'm_2'm_3M\left(\frac{m_1m_2\alpha_{12}^2}{n_3'^2} + \frac{m_2m_3\alpha_{23}^{12}}{n_1'^2} + \frac{m_3m_1\alpha_{31}^2}{n_2'^2}\right) = 0, \tag{131}$$

where

$$m_i' = m_i/(1+f_i). \tag{132}$$

The spectrum that this equation implies is not uniquely determined unless the set of functions f_i can be uniquely specified. In the case of equal masses $(m_1 = m_2 = m_3 \equiv m)$ $f_1 = f_2 = f_3 = 0$. I further assume for simplicity that $\alpha_{12} = \alpha_{23} = \alpha_{31} = \alpha$ and that $n_1 = n_2 = n_3 = n'$ and $l_1 = l_2 = l_3 = l$ where

$$n' = n - l - \tfrac{1}{2} + [(l+\tfrac{1}{2})^2 + \tfrac{2}{3}\alpha^2]^{1/2}. \tag{133}$$

The factor of $\frac{2}{3}$ follows from (120). Then (131) implies

$$\Delta^6 - 6m^2\Delta^4 + m^4(9 + 3\alpha^2/n'^2)\Delta^2 - 9(\alpha^2/n'^2)m^6 = 0. \tag{134}$$

This equation has three roots. The only root that vanishes as $\alpha^2 \to 0$ (so that $w^2 \to M^2$) is

$$\Delta^2 = 2m^2\left[1 - \left(1 - \frac{\alpha^2}{n'^2}\right)^{1/2} \cos\tfrac{1}{3}\tan^{-1}\left(\frac{(\alpha^2/n'^2)(1 - \frac{3}{4}\alpha^2/n'^2)^{1/2}}{1 - \frac{3}{2}\alpha^2/n'^2}\right)\right] \tag{135}$$

or

$$w^2 = 3m^2 + 6m^2\left(1 - \frac{\alpha^2}{n'^2}\right)^{1/2} \cos\tfrac{1}{3}\tan^{-1}\left(\frac{\alpha^2}{n'^2}\frac{(1 - \frac{3}{4}\alpha^2/n'^2)^{1/2}}{1 - \frac{3}{2}\alpha^2/n'^2}\right). \tag{136}$$

This is to be compared with the two-body spectrum

$$w^2 = 2m^2 + 2m^2(1 - \alpha^2/n'^2)^{1/2}. \tag{137}$$

VI. CONCLUSION

The primary results of this paper are summarized in Eqs. (59), (63), (70), (71), (81), and (90) which give the various exact forms of the relativistic three-body "Hamiltonian" and Schrödinger equation. In general, even for simple potentials, there are no exact solutions. Numerical methods are needed. Because the classical equations of motions or the corresponding Schrödinger equation are not separable, even this type of analysis can become quite involved especially for simple model calculations. The forms (63), (81), and (90) become separable and more tractable when

$Z=0$. In particular, the separable relativistic Schrödinger equation (91) becomes as easy to handle as an ordinary reduced one-particle Schrödinger equation. The separability assumption cannot be classified as an approximation in any perturbation sense. Rather, when applied to any particular problem it amounts to a rather strong physical assumption or restriction on the nature of the orbits. In the case of gravitational forces in nonrelativistic mechanics, the orbits in the equilateral-triangle configuration are known to be stable, at least with respect to linear perturbations.[14] The gulf between this classical system and a relativistic submicroscopic quantum system is enormous but the formalisms are similar. One might speculate that such special three-body solutions are of fundamental importance in the quark model of baryons.[15] The forces are not restricted by the forms to be gravitational.

Besides the unanswered questions alluded to in the paragraph below Eq. (89), there are nontrivial problems that remain with the relativistic three-body formalism developed here. The determination of the functional dependence of the c.m. energies E_i on w, m_1, m_2, m_3 is not clearly resolved as it is with the case of two bodies. In that case, the functional dependence was fixed by imposing the constraints in a form other than in that of a "Hamiltonian". The problem with three bodies is that there are no combinations of the constraints that involve the cross terms p_1p_2, p_2p_3, p_1p_3 which appear in (48). If that problem is resolved, the uniqueness problem for unequal masses inherent in the definitions (102) and (109) will, it is hoped, be eliminated. These are subjects for future papers.

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