PHYSICAL REVIEW D VOLUME 18, NUMBER 1 1 JULY 1978

Inverse scattering problem for quarkonium systems. II. Applications to ψ and Υ families

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(Received 16 January 1978)

The inverse scattering formalism for reflectionless potentials is extended to the reconstruction of central potentials in three space dimensions. An approximate interquark potential is derived from the $\psi(3.095)$ and $\psi'(3.684)$ levels and leptonic decay widths. Remaining ambiguities are discussed in detail. Consequences for the Υ family and prospects for refining the interquark potential are explored.

I. INTRODUCTION

An extensive literature[1] now supports the notion[2] that mesons which are composed of massive quarks may be described in terms of the nonrelativistic Schrödinger equation. An impressive phenomenology of the psion family has been constructed following the analogy between (e^+e^-) positronium and $(c\bar{c})$ charmonium. Several approaches to the problem have been fruitful. The most thoroughly explored of these has been the explicit-potential technique in which a specific form is assumed for the interquark potential. In most applications, this potential (which is thought to result from the exchange of massless gluons) has been assumed to be a superposition of a Coulomb term and a linear confining potential.[3] However, no compelling derivation of this form from the underlying field theory has been given, and alternative suggestions[4] have met with some degree of phenomenological success. Consequently, it has been of interest to obtain general results which permit the properties of the potential to be inferred from experiment. For example, the scaling of observables with quark mass has been investigated by a number of authors,[5] and several important theorems on the order of levels and on leptonic widths have been proved.[6] In addition, general results (which do not depend upon details of the potential) on the number of levels below new-flavor threshold[7] and on other connections between observables[8] have been exhibited. We present here, as a complement to these approaches, the first application of the inverse scattering method to the determination of the interquark potential.

In the preceding paper[9] (hereafter denoted as I) we developed a technique for reconstructing a symmetric, confining, one-dimensional potential $V(x)$ from the energy spectrum of its bound states. The energies E_n of the N lowest-lying bound states determine an approximate potential $V_N(x)$ which is a rational function of exponentials. The approximation $V_N(x)$ is a symmetric, reflectionless potential which supports N bound states at the first N bound-state energies of $V(x)$. It is not confining, but approximates $V(x)$ over a range which is roughly delimited by the classical turning point $|x_N|$ of the highest level included, where

$$V(\pm|x_N|)=E_N. \tag{1.1}$$

(For simplicity we consider a potential which is monotonically increasing for $x>0$.) Beyond the classical turning point, $V_N(x)$ approaches a value E_0 which lies between the highest level included in the approximation and the lowest level omitted,

$$E_N<E_0<E_{N+1} \tag{1.2}$$

In particular examples it was seen that the choice

$$E_0=\tfrac{1}{2}(E_N+E_{N+1}) \tag{1.3}$$

resulted in excellent approximations. For values of x within the expected range of validity, the sequence of approximations $V_N(x)$ was shown to approach rapidly the exact result, yielding faithful representations of $V(x)$ for N as small as 3 or 4. We now apply this method to the problem for which it was conceived.

Within the framework of the nonrelativistic Schrödinger equation with a central potential,

$$\left(\frac{-\nabla^2}{2\mu}+V(r)\right)\Psi(\vec{r})=E\Psi(\vec{r}), \tag{1.4}$$

the procedure we shall describe for calculating $V(r)$ is explicit and essentially unambiguous. We restrict our attention to spin-triplet quarkonium states. The possibility of going beyond (1.4) to incorporate spin-spin, spin-orbit, and other relativistic effects will not be discussed. In Sec. II we collect some important formulas derived for the one-dimensional problem in I and make the necessary extensions to the S-wave radial equation in three dimensions. The approximate potential $V_4(r)$ deduced from the masses and leptonic decay widths of ψ and ψ' is the subject of Sec. III. Assuming the

interquark potential to be independent of quark flavor, as it would be in quantum chromodynamics, we apply the reconstructed $c\bar{c}$ potential to predict properties of the new quarkonium system suggested by the discovery[10] of the Υ family. We also solve the P-wave Schrödinger equation in the reconstructed potential to determine the positions of ${}^3P_J$ levels and radiative decay rates. The manner in which such derived predictions may be used to resolve ambiguities in the reconstructed potential is explained. Section IV contains a summary and conclusions.

II. INVERSE PROBLEM FOR THE RADIAL SCHRÖDINGER EQUATION

We first summarize the procedure derived in I whereby a symmetric, confining, one-dimensional potential $V(x)$ is locally reconstructed from the energies E_n of its first N bound states. The reconstructed potential is specified by the N bound-state parameters

$$\kappa_n^2 = 2\mu(E_0 - E_n), \tag{2.1}$$

where μ is the reduced mass and E_0 has been chosen according to (1.3). We define an $N \times N$ matrix A with elements

$$A_{mn} = \delta_{mn} + \frac{\lambda_m \lambda_n}{\kappa_m + \kappa_n}, \tag{2.2}$$

where

$$\lambda_n(x) = c_n \exp(-\kappa_n x), \tag{2.3}$$

and the constants c_n are given in terms of the κ's by

$$\frac{c_n^2}{2\kappa_n} = \prod_{m \neq n} \left| \frac{\kappa_m + \kappa_n}{\kappa_m - \kappa_n} \right| . \tag{2.4}$$

We showed using the Gel'fand-Levitan inverse scattering formalism that a symmetric, reflectionless potential which supports bound states at $E_1, E_2, \ldots, E_N$ is given by

$$V_N(x) = E_0 - 2\frac{d^2}{dx^2} \ln(\mathrm{Det}\, A). \tag{2.5}$$

The corresponding normalized bound-state wave functions, which obey the condition

$$\int_{-\infty}^{\infty} dx [\psi_n(x)]^2 = 1, \tag{2.6}$$

are obtained from the formula

$$\psi_n(x) = -\frac{1}{\lambda_n} \frac{\mathrm{Det}\, A^{(n)}}{\mathrm{Det}\, A} . \tag{2.7}$$

The matrix $A^{(n)}$ is simply given by A, with the elements of the nth column replaced by their derivatives with respect to x. Equations (2.5) and (2.7) provide a sequence of approximations to $V(x)$ and its bound-state wave functions.

The reduced radial equation which follows from (1.4) upon substitution of

$$\Psi(\vec{r}) = R(r) Y_{lm}(\theta, \phi) \tag{2.8}$$

and

$$u(r) = rR(r) \tag{2.9}$$

is

$$-\frac{1}{2\mu} u''(r) + \left[\frac{l(l+1)}{2\mu r^2} + V(r) - E \right] u(r) = 0 . \tag{2.10}$$

For S waves, Eq. (2.10) is identical to the one-dimensional Schrödinger equation. As a result, formulas (2.5) and (2.7) may be applied to the study of quarkonium systems. However, because of the boundary condition

$$u(0) = 0 \tag{2.11}$$

imposed by the finiteness of the radial wave function at the origin, only the even numbered parameters $\kappa_2, \kappa_4, \ldots$ correspond to energy levels of physical states. The remaining parameters κ_1, $\kappa_3, \ldots$ describe states of a one-dimensional system which have even parity and hence do not satisfy (2.11). Consequently, in order to apply the one-dimensional formalism to the S-wave charmonium system, we must regard ψ and ψ' as the second and fourth levels of a symmetric one-dimensional potential $V(r) = V(-r)$. The even-parity levels which occur in the one-dimensional problem are interleaved with the physical psions, one below the ψ, one between ψ and ψ', and so on.

The values of the parameters $\kappa_2, \kappa_4, \ldots$ that correspond to physical states are determined directly from particle masses (with a given choice of E_0 and charmed-quark mass). The others $(\kappa_1, \kappa_3, \ldots)$ do not have immediate physical significance. However, the wave functions of the physical states depend through (2.7) upon the "unphysical" parameters $\kappa_1, \kappa_3, \ldots$. The square of a ${}^3S_1$ wave function at the origin is measured by the leptonic decay rate as[11]

$$|\Psi(0)|^2 = \frac{M_V^2}{16\pi\alpha^2 e_Q^2} \Gamma(V \to e^+ e^-), \tag{2.12}$$

where M_V is the vector-meson mass and e_Q is the charge of the constituent quark. This piece of information permits the determination of the odd numbered κ's from experimental data.

To illustrate these points let us construct the $N = 2$ approximation to the charmonium potential from the mass and leptonic width of $\psi(3.095)$. We must first choose a charmed quark mass m_c and a parameter E_0. According to the rule(1.3), the lat-

ter should lie about halfway between $E_2 = M_\psi$ and the unphysical level at E_3. The parameter κ_2 is then given by

$$\kappa_2 = [m_c(E_0 - M_\psi)]^{1/2} . \tag{2.13}$$

To compute κ_1 we employ Eq. (2.7) to construct the approximate wave function. Imposing the condition

$$\int d^3r [\Psi(\vec{r})]^2 = 1 , \tag{2.14}$$

which allows the identification

$$u_n(r) = \sqrt{2}\, \psi_n(r) , \tag{2.15}$$

we obtain

$$[\Psi(0)]^2 = \frac{\kappa_2(\kappa_1^2 - \kappa_2^2)}{4\pi} , \tag{2.16}$$

from which

$$\kappa_1^2 = \kappa_2^2 + \frac{4\pi[\Psi(0)]^2}{\kappa_2} . \tag{2.17}$$

The formula (2.5) yields a reconstructed potential $V_2(r)$ in terms of κ_1 and κ_2.

In the next section we shall construct the approximate interquark potential $V_4(r)$. For $N > 2$ we have found no simple analog of (2.17) for the odd-numbered κ's. We shall determine them implicitly through (2.7) and (2.12).

III. HEAVY-QUARK SPECTROSCOPY

A. Calculation of central potentials

The $N = 4$ approximation to a central potential is calculated from the masses and wave functions at the origin of the two lowest-lying S-wave bound states. For the $^3S_1$ states of the charmonium system, the parameters κ_2 and κ_4 are obtained from the ψ and ψ' masses,

$$\kappa_2 = [m_c(E_0 - M_\psi)]^{1/2} , \tag{3.1}$$

$$\kappa_4 = [m_c(E_0 - M_{\psi'})]^{1/2} . \tag{3.2}$$

The remaining parameters κ_1 and κ_3 are determined implicitly by the inverse scattering formulas which express the wave functions at $r = 0$ in terms of the κ's. Introducing the notation

$$\gamma_{ij} = \frac{\kappa_i + \kappa_j}{\kappa_i - \kappa_j} , \tag{3.3}$$

we may write the wave functions at the origin using (2.7) as

$$\Psi_\psi(0) = \frac{2(\kappa_2\gamma_{12}\gamma_{23}\gamma_{24})^{1/2}}{D(0)\sqrt{\pi}} \times [\gamma_{13}\gamma_{14}(\kappa_1 - \kappa_3 - \kappa_4) + \gamma_{13}\gamma_{34}(\kappa_1 + \kappa_4 - \kappa_3) + \gamma_{14}\gamma_{34}(\kappa_1 + \kappa_3 - \kappa_4) + (\kappa_1 + \kappa_3 + \kappa_4)] , \tag{3.4}$$

$$\Psi_{\psi'}(0) = \frac{2(\kappa_4\gamma_{14}\gamma_{24}\gamma_{34})^{1/2}}{D(0)\sqrt{\pi}} \times [\gamma_{12}\gamma_{13}(\kappa_1 - \kappa_2 - \kappa_3) + \gamma_{12}\gamma_{23}(\kappa_2 - \kappa_1 - \kappa_3) + \gamma_{13}\gamma_{23}(\kappa_3 - \kappa_1 - \kappa_2) + (\kappa_1 + \kappa_2 + \kappa_3)] . \tag{3.5}$$

Here we have abbreviated

$$D(x) = \mathrm{Det}A(x) , \tag{3.6}$$

where A is defined in (2.2). In particular, we have

$$D(0) = 2[1 + \gamma_{12}\gamma_{13}\gamma_{14} + \gamma_{12}\gamma_{23}\gamma_{24} + \gamma_{13}\gamma_{23}\gamma_{34} + \gamma_{14}\gamma_{24}\gamma_{34} + \gamma_{12}\gamma_{13}\gamma_{24}\gamma_{34} + \gamma_{12}\gamma_{23}\gamma_{14}\gamma_{34} + \gamma_{13}\gamma_{23}\gamma_{14}\gamma_{24}] . \tag{3.7}$$

The parameters κ_1 and κ_3 are fixed by solving (3.4) and (3.5) numerically with the experimentally measured wave functions at the origin.

Before displaying the $N = 4$ approximation to the charmonium potential, we shall briefly discuss two three-dimensional examples which illustrate the technique. They also indicate the response of the inverse scattering equations to potentials singular at $r = 0$. This is an issue of some practical importance because general arguments based upon quantum chromodynamics suggest that the interquark potential will have such a singularity. For these examples we set $2\mu = 1$.

We first consider the Coulomb potential

$$V(r) = -1/r , \tag{3.8}$$

which is neither confining nor reflectionless. Our concern, however, is not the convergence of the method to the exact potential, but to learn how the singularity is imitated. In this case there is a natural choice for the parameter E_0, namely,

$$E_0 = 0 . \tag{3.9}$$

The κ's are determined[12] by the bound-state energies

$$E_1 = -\tfrac{1}{4} , \tag{3.10a}$$

$$E_2 = -\tfrac{1}{16} , \tag{3.10b}$$

and wave functions at the origin

$$|\Psi_1(0)|^2 = 1/8\pi , \tag{3.11a}$$

$$|\Psi_2(0)|^2 = 1/64\pi . \tag{3.11b}$$

The resulting approximation $V_3(r)$ is compared with the true potential (3.8) in Fig. 1. The manner in which the pole at the origin is simulated is noteworthy.

As a second example we consider the logarithmic potential

$$V(r) = \ln(r) . \tag{3.12}$$

An interquark potential of this form is suggested[13]

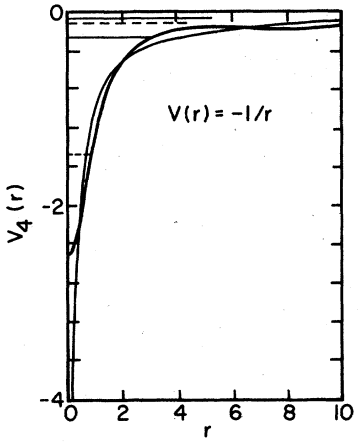

FIG. 1. Two-bound-state approximate reconstruction $V_4(r)$ of the Coulomb potential compared with the exact potential (3.8). The physical and unphysical levels are indicated by solid and dashed lines, respectively.

by the equality of the ψ-ψ' and Υ-Υ' level spacings. Numerical evaluation of the energy levels and wave functions leads to the appropriate parameters[14] for the inverse scattering equations. The approximation $V_4(r)$ is compared with the true potential (3.12) in Fig. 2.

B. The charmonium system

The observables from which we shall reconstruct the charmonium potential are the masses

$$M(\psi) = 3.095 \text{ GeV}/c^2, \tag{3.13a}$$

$$M(\psi') = 3.684 \text{ GeV}/c^2, \tag{3.13b}$$

and leptonic decay widths[15]

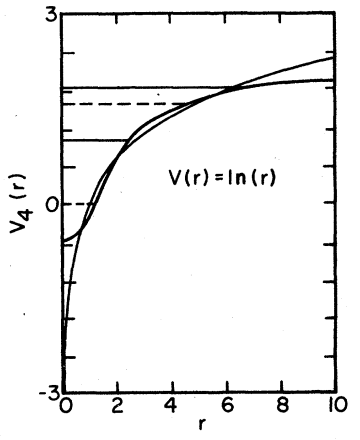

FIG. 2. Two-bound-state approximate reconstruction $V_4(r)$ of the logarithmic potential compared with the exact potential (3.12). The physical and unphysical levels are indicated by solid and dashed lines, respectively.

$$\Gamma(\psi \to e^+e^-) = 4.8 \pm 0.6 \text{ keV}, \tag{3.14a}$$

$$\Gamma(\psi' \to e^+e^-) = 2.1 \pm 0.3 \text{ keV}. \tag{3.14b}$$

For a given choice of E_0 and m_c, the parameters κ_2 and κ_4 are given by (3.1) and (3.2). The value of E_0 certainly must lie between $M(\psi')$ and 4.03 GeV/c^2, the position of the $3\,^3S_1$ level. In practice we find it sufficient to restrict our attention to the slightly smaller range

$$3.75 \text{ GeV} \leq E_0 \leq 3.9 \text{ GeV}. \tag{3.15}$$

It remains to choose the charmed-quark mass.

In Fig. 3 we show 20 distinct charmonium potentials corresponding to the choices $E_0 = 3.75$, 3.80, 3.85, 3.90 GeV and $m_c = 1.1$, 1.2, 1.3, 1.4, 1.5 GeV/c^2. All of these reproduce—by construction—the observables (3.13) and (3.14). It is striking that smooth potentials of such diverse character ranging from Coulombic ($m_c = 1.1$, $E_0 = 3.75$) to linear ($m_c = 1.5$, $E_0 = 3.8$) and beyond ($m_c = 1.4$, $E_0 = 3.85$), are achieved.

We shall also explore the implications of the reconstructed charmonium potentials for the Υ system. The $\Upsilon(9.4)$ and $\Upsilon'(10.0)$ are regarded as the $1\,^3S_1$ and $2\,^3S_1$ levels of a $Q\bar{Q}$ system. The appropriate value of the heavy-quark mass m_Q for each of the 20 potentials displayed in Fig. 3 is chosen by requiring $M(\Upsilon) = 9.4$ GeV/c^2. The ordinates for the ψ and Υ systems are then related by

$$E_0(\Upsilon) = E_0(\psi) + 2(m_Q - m_c). \tag{3.16}$$

We find m_Q/m_c essentially independent of $E_0(\psi)$, and varying between 4.1 (for $m_c = 1.1$ GeV/c^2) and 3.2 (for $m_c = 1.5$ GeV/c^2).

The presence of four Υ levels in all the potentials of Fig. 3 is a consequence of the choice of E_0, m_c, and of the stipulation that $m(\Upsilon) = 9.4$ GeV/c^2.[7] The value of E_0 is not necessarily correlated with flavor threshold. However, it is possible to estimate the number of *narrow* Υ levels (those below flavor threshold) directly from Fig. 3 if the two flavor thresholds differ by $2m_Q - 2m_c = E_0(\Upsilon) - E_0(\psi)$.[16] The flavor threshold for the charmonium system is a line lying 45 MeV above the ψ' on the left-hand side of each picture in Fig. 3. The corresponding flavor threshold for the Υ family is the extension of this line to the right. Thus, one would expect four narrow Υ levels for small m_c and E_0 (lower left-hand corner of Fig. 3), and three for large m_c and E_0 (upper right-hand corner of Fig. 3). This is in accord with the expectation of three or four narrow Υ levels obtained in Ref. 7 in a semiclassical approximation.

For a given value of E_0, Fig. 3 shows that smaller values of m_c are correlated with deeper potentials. Since the levels

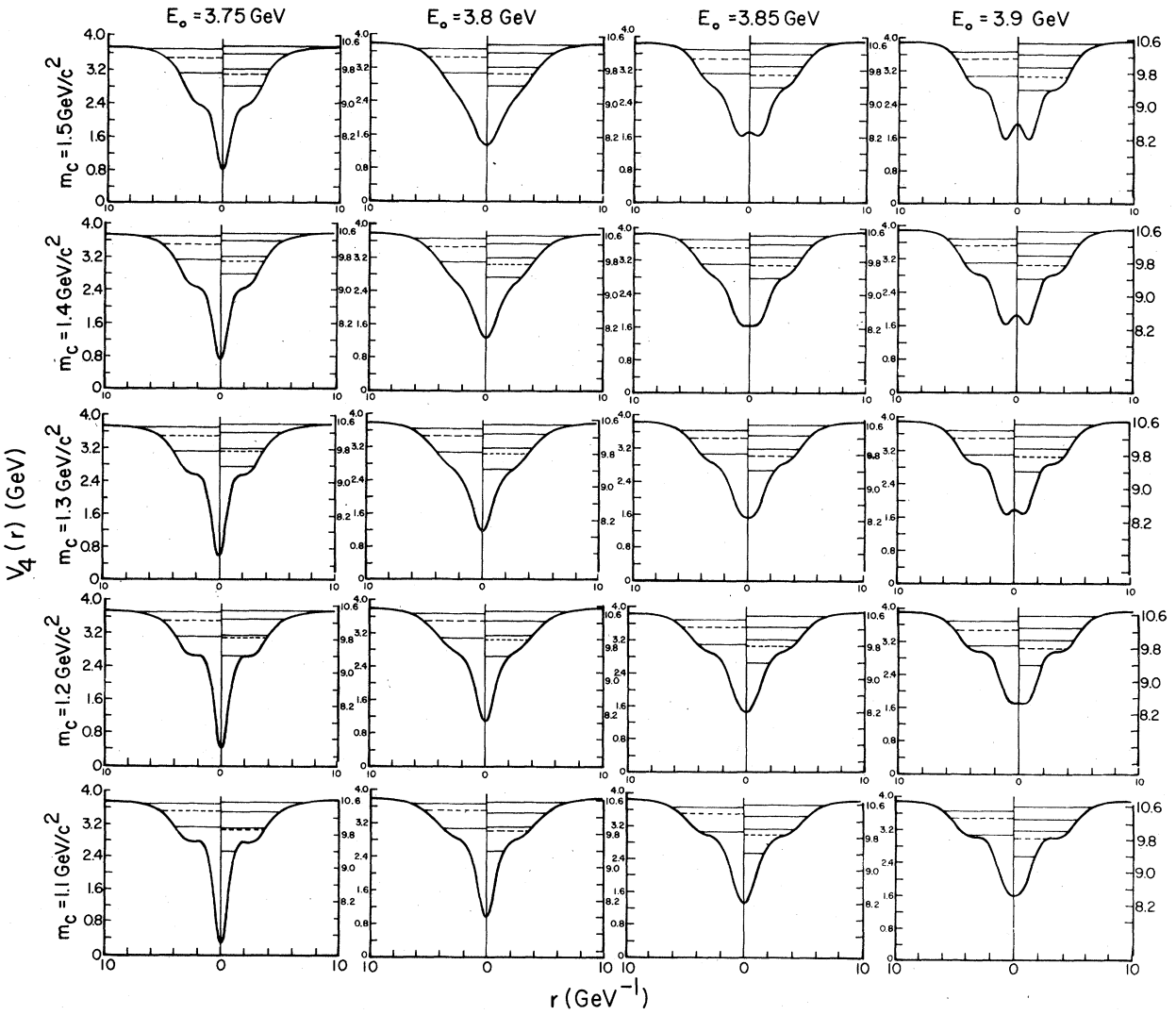

FIG. 3. Interquark potentials reconstructed from the masses and leptonic widths of $\psi(3.095)$ and $\psi'(3.684)$. The levels of charmonium are indicated on the left-hand side of each graph. Those of the Υ family are shown on the right-hand side of each graph. The solid lines denote $^3S_1$ levels; dashed lines indicate the $2\,^3P_J$ levels. The 20 potentials depicted correspond to the choices $E_0 = 3.75$, 3.8, 3.85, 3.9 GeV and $m_c = 1.1$, 1.2, 1.3, 1.4, 1.5 GeV/c^2.

$$E_n = \langle T + V \rangle_n = \left\langle \frac{r}{2} \frac{dV}{dr} \right\rangle + \langle V \rangle_n \tag{3.17}$$

are fixed, decreasing m_c is correlated here with more negative $\langle V \rangle$ and larger $\langle (r/2)(dV/dr) \rangle$ (steeper potential, greater kinetic energy). The potentials also become deeper for fixed m_c and decreasing E_0, corresponding to a decreasing ratio of κ_4/κ_2.

In order to sharpen the estimates of m_c and E_0 we now focus on two spectral quantities which are, to some degree, known from available data: the P-wave charmonium levels and the Υ-Υ' mass difference.

The predicted mass of the $2P$ charmonium state χ_c is shown as a function of m_c and E_0 in Fig. 4. To compare it with experiment, we note that spin-orbit and tensor force contributions vanish for the combination

$$\langle M(2\,^3P) \rangle \equiv \tfrac{1}{9}[M(^3P_0) + 3M(^3P_1) + 5M(^3P_2)] \simeq 3.52 \text{ GeV}/c^2 . \tag{3.18}$$

The numerical value in (3.18) comes from masses quoted in Ref. 17. Values of m_c and E_0 in the lower right-hand corner of Fig. 4 are preferred. Many of the models noted in Ref. 3 predict too low a value of $M(\chi_c)$; this may be connected with the higher charmed-quark masses occurring in such models.

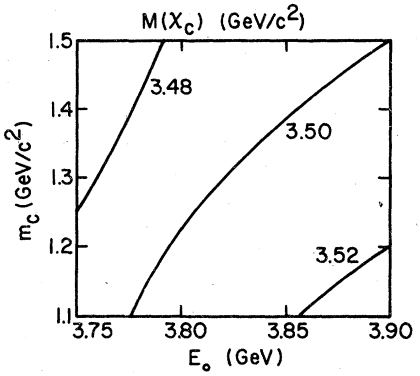

FIG. 4. Contours of the predicted mass of the $2\,^3P_J$ (χ_c) level of the charmonium system as functions of the parameters E_0 and m_c.

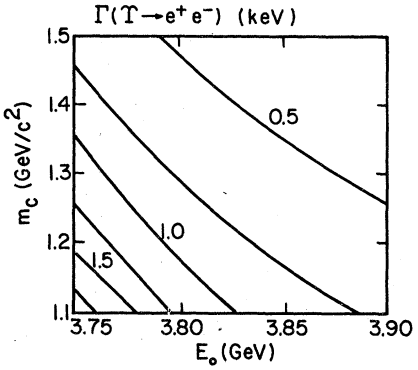

FIG. 6. Contours of the predicted leptonic width of Υ as functions of the parameters E_0 and m_c.

A contour plot of the predicted Υ-Υ′ mass difference is shown in Fig. 5. The shapes of the contours are similar to those in Fig. 4. The experimental values[18]

$$M(\Upsilon')-M(\Upsilon)=\begin{cases}0.61\pm0.04\ \text{GeV}/c^2\\ \text{(three-peak hypothesis)},\\ 0.65\pm0.03\ \text{GeV}/c^2\\ \text{(two-peak hypothesis)}\end{cases}\quad(3.19)$$

again favor values of E_0 and m_c in the lower right-hand corner of the figure. A specific potential which reproduced the result (3.18) when constructed to give the observed Υ′-Υ spacing already has been noted in Ref. 13.

The small values of $m_c \simeq 1.1$–1.3 GeV/c^2 implied by comparison of Figs. 4 and 5 with (3.18) and (3.19) tend to weaken somewhat the case for a nonrelativistic approach to charmonium spectroscopy. However, these small values have been encountered previously: They are obtained from sum rules for e^+e^- annihilation,[8] and are required if the specific model of Ref. 13 is constrained to fit $\Gamma(\psi\to e^+e^-)$.[19]

The similarity of contours in Figs. 4 and 5 prevents an unambiguous choice of E_0 and m_c. A very different dependence is exhibited by the leptonic width of Υ, shown in Fig. 6. This quantity is particularly sensitive to short-distance behavior of the potential not probed by existing data. A clear correlation may be noted between large values of $\Gamma(\Upsilon\to e^+e^-)$ and highly singular potentials. (See the lower left-hand corners of Figs. 6 and 3, respectively.[20])

The higher-lying Υ′ samples values of the potential that include those related to charmonium spectroscopy. Indeed, the predicted values of $\Gamma(\Upsilon'\to e^+e^-)$, shown in Fig. 7, vary less strikingly than those of $\Gamma(\Upsilon\to e^+e^-)$. This relative insensitivity to parameters may be useful for a test of the heavy-quark charge e_Q.[21]

Additional information on the preferred values of m_c and E_0, very different from that provided by present data, will come from a measurement of the $2S$-$2P$ splittings in the Υ system. The predicted values are shown in Fig. 8. For singular (Coulomb-type) potentials, such as occur in the lower left-hand corner of Fig. 3, the $2S$ and $2P$ levels are nearly degenerate. They move apart

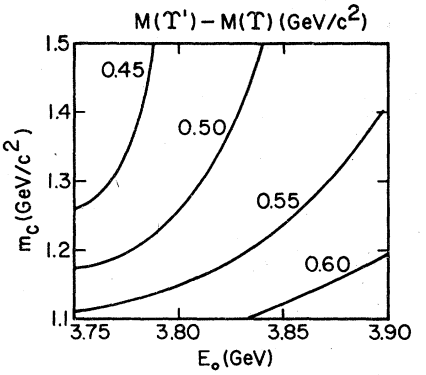

FIG. 5. Contours of the predicted Υ-Υ′ level splitting as functions of the parameters E_0 and m_c.

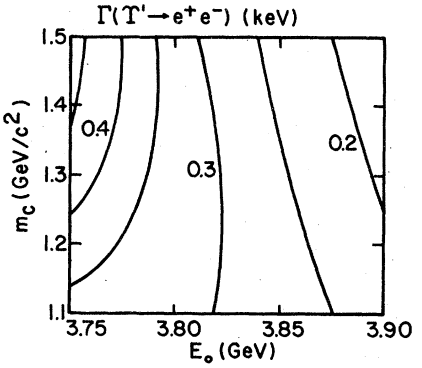

FIG. 7. Contours of the predicted leptonic width of Υ′ as functions of the parameters E_0 and m_c.

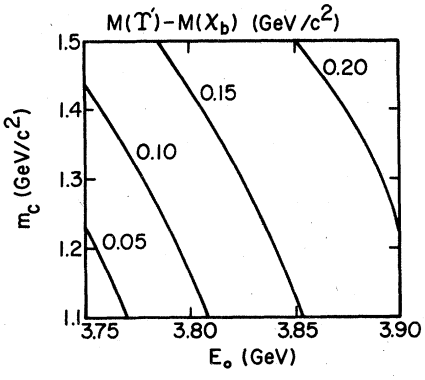

FIG. 8. Contours of the predicted 2S-2P splittings of the Υ family as functions of the parameters E_0 and m_c.

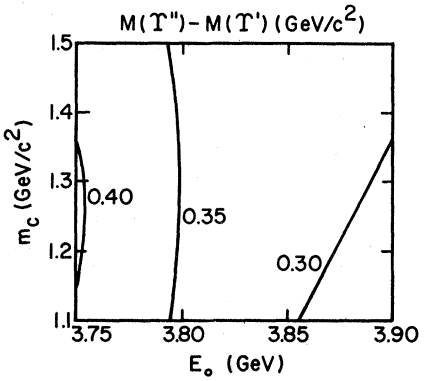

FIG. 9. Contours of the predicted Υ' - Υ'' level splitting as functions of the parameters E_0 and m_c.

as the potentials become shallower.

The Υ''-Υ' splitting is displayed in Fig. 9. It is slowly varying over the range of interest, a feature compatible with previous expectations based on specific models.[22] The three-peak hypothesis of Ref. 18 gives

$$M(\Upsilon'') - M(\Upsilon') = 0.39 \pm 0.13 \text{ GeV}/c^2. \qquad (3.20)$$

For the purpose of further discussion we have chosen two specific values of m_c and E_0, lying along approximate contours of Figs. 4 and 5: $(m_c, E_0) = (1.1 \text{ GeV}/c^2, 3.8 \text{ GeV})$ and $(1.2 \text{ GeV}/c^2, 3.85 \text{ GeV})$. These choices ensure (i) approximate agreement with the constraints (3.18), (3.19), (ii) reasonably smooth behavior with r, and (iii) a pair of potentials between which new experimental data can provide a reasonable distinction. Some properties of levels in these two potentials are shown in Table I.

The uncertainty in the value of $\Gamma(\Upsilon \to e^+e^-)$ apparent from Table I already has been noted in connection with Fig. 6. The other leptonic widths are more stable. They fail to decrease monotonically; compare Υ' and Υ''. This effect is an artifact of the oscillating convexity of the reconstructed potential.[23]

The predicted radiative decay widths of the ψ' states into $\chi_c\gamma$ are considerably too large. A similar discrepancy arises in specific potential models,[3,13] and may indicate a general shortcom-

TABLE I. Predicted properties of levels in two quarkonium potentials.

	$m_c = 1.1$ GeV/c^2, $E_0 = 3.8$ GeV	$m_c = 1.2$ GeV/c^2, $E_0 = 3.85$ GeV	Experiment[a]
Υ M (GeV/c^2)	9.40 (input)	9.40 (input)	
Υ Γ_{ee}[b] (keV)	1.19	0.69	
Υ' M (GeV/c^2)	9.98	9.96	
Υ' Γ_{ee}[b] (keV)	0.32	0.27	
Υ'' M (GeV/c^2)	10.32	10.27	
Υ'' Γ_{ee}[b] (keV)	0.33	0.30	
Υ''' M (GeV/c^2)	10.58	10.54	
Υ''' Γ_{ee}[b] (keV)	0.18	0.18	
$\chi_b(2P)$ $\langle M \rangle$ (GeV/c^2)	9.89	9.81	
$\Gamma(\psi' \to \chi_c \gamma)$[c] (keV) $\chi_c = {}^3P_0$	76	71	15±5
$\Gamma(\psi' \to \chi_c \gamma)$[c] (keV) $\chi_c = {}^3P_1$	64	60	15±5
$\Gamma(\psi' \to \chi_c \gamma)$[c] (keV) $\chi_c = {}^3P_2$	50	47	15±4

[a] Ref. 17.

[b] $e_Q = -\frac{1}{3}$ is assumed.

[c] In the expression $\Gamma(\psi' \to \chi_c ({}^3P_J)\gamma) = 4\alpha e_Q^2 (2J+1) k_\gamma^3 |\langle \psi' | r | \chi \rangle|^2/27$, the experimental values of the photon energy k_γ derived from the particle masses of Ref. 17 are used.

ing in the nonrelativistic Schrödinger bound-state picture of charmonium.[24] However, radiative decays are particularly demanding tests of structure,[25] probably requiring more pieces of information than the four (less two free parameters) at our disposal.

IV. SUMMARY AND CONCLUSIONS

With the wealth of charmonium data now available and the prospects for measurement of a still richer Υ spectrum, it seems likely that future efforts toward a theory of heavy-quark bound states will fall along two main lines of investigation, one theoretical, the other phenomenological. Attempts to relate the interquark potential function $V(r)$ to fundamental theory[26] will be complemented by phenomenological determination of this function from the measured bound-state parameters. The work described here and in I is directed toward the phenomenological investigation. Within the framework of the nonrelativistic Schrödinger equation with a central interquark potential $V(r)$, we have developed a systematic method for reconstructing $V(r)$ from the masses and leptonic decay widths of S-wave bound states. With information about N bound-state levels, the method provides an explicit formula for a reconstructed potential $V_{2N}(r)$. A mathematical proof that $V_{2N}(r)$ converges to the exact potential $V(r)$ is still lacking. However, the examples studied in Sec. IV of I leave little doubt that this is the case for any reasonably smooth function $V(r)$. More importantly, these examples clearly show that the number of bound states needed for the practical application of this method is very small, with $V_4(r)$ (two bound states) already providing a rather accurate approximation to $V(r)$ over some range of r.

The charmonium potential $V_4(r)$ constructed from ψ and ψ' data was discussed extensively in Sec. III. The ambiguities in $V_4(r)$ associated with the choice of charmed-quark mass and E_0 parameter may be viewed as a commentary on the limits of our present knowledge of $V(r)$ and the manner in which this knowledge will be refined and extended by future measurements of the Υ system. Already the combined evidence of the Υ-Υ' splitting and the P-wave charmonium levels suggests a rather small value for the charmed-quark mass, $m_c \approx 1.1$–1.2 GeV/c^2. The sensitivity of quantities such as $\Gamma(\Upsilon \to e^+e^-)$ and the Υ'-χ_b ($2S$-$2P$) splitting to the remaining ambiguities in $V_4(r)$ serves to emphasize that these quantities probe values of r which are not explored in the charmonium system.

We conclude that inverse scattering techniques provide a valuable tool for analyzing and correlating currently available quarkonium data and for using these data to estimate the spectral parameters of the Υ system. This approach can complement the more familiar explicit-potential techniques that allow the incorporation of theoretical prejudices regarding the form of the potential at short and long distances. But in our view, the most encouraging aspect of the present work is the prospect of reconstructing the quark potential from forthcoming data on the Υ system. According to general arguments,[7] which are borne out by the specific potentials studied in Sec. III, this system is expected to have at least three and possibly four narrow $^3S_1$ bound-state levels. From the examples described in Sec. IV of I, we expect $V_6(r)$ or $V_8(r)$ to provide a very accurate representation of the true potential. Moreover, the assumptions and approximations which go into a nonrelativistic potential model should be much more reliable for the heavier quarks which form the Υ states. Thus, when the Υ levels are accessible to e^+e^- machines, they will provide an extremely detailed and accurate measurement of the potential which binds quarkonium.

ACKNOWLEDGMENT

One of us (J. L. R.) wishes to thank Ivan Muzinich for an informative discussion and to acknowledge the kind hospitality of the Fermilab Theoretical Physics Department. The work of J. L. R. was supported in part by the Energy Research and Development Administration under Contract No. $E(11$-$1)$-1964. The work of C. Q. was supported in part by the Alfred P. Sloan Foundation.

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†Operated by Universities Research Association Inc. under contract with the Energy Research and Development Administration.

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[20]In this context the logarithmic potential of Ref. 13 is seen to be very similar to the choice $(m_c, E_0) = (1.1\ \mathrm{GeV}/c^2, 3.8\ \mathrm{GeV})$ in the present approach. An alternative modified Coulomb potential discussed in Ref. 13 predicts $\Gamma(\psi \to e^+e^-) = 8.8$ keV if Eq. (2.12) is used, and hence does not satisfy the input criteria used here to determine $\kappa_1, \ldots, \kappa_4$.

[21]Thus, when combined with the previous discussion, Fig. 7 would imply $\Gamma(\Upsilon' \to e^+e^-) = 0.2$–$0.5$ keV for $e_Q = -\frac{1}{3}$, and 0.7–1.9 keV for $e_Q = \frac{2}{3}$. Lower bounds on leptonic widths of the upsilons are derived by J. L. Rosner, C. Quigg, and H. B. Thacker, Phys. Lett. 74B, 350 (1978).

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