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Presymmetry of classical relativistic particles

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The invariant methods of modern differential geometry are used to study both a noninertial single particle and Fermi-Walker transport in special relativity. The presymmetry, especially the acceleration group, is formulated in the tangent manifold of the observer. Our geometrical construction of the presymmetry of Ekstein also gives Newton's second law as a theorem. In addition, we obtain the exact, closed-form expression for a noninertial particle in special relativity and an exact, closed-form expression for Fermi-Walker transport in special relativity.

I. INTRODUCTION

The purpose of this study is to relate ideas on modern differential geometry to the ideas of *presymmetry* of Ekstein and collaborators,[1-3] for the case of special-relativistic particle mechanics.[5-7] The fundamental basis for presymmetry is the clarification of the association between the laboratory procedures and the self-adjoint observables. Physically, this means that instead of blindly and uncritically hanging physical names upon mathematical objects, one searches for a framework in which the correspondence between physical and mathematical entities is reasonable and explicit. Presymmetry is probably the only existing attempt to establish such a framework. Ekstein[1] has shown how to idealize and extrapolate from empirical facts to turn the laboratory procedure collection and the observation procedure collection into "hardware spaces" where $\mathcal{O}$ is an algebra of operation procedures and $\mathcal{S}$ is a convex set of state preparation procedures. If $\mathfrak{A}$ is the $*$-algebra of observables then the many-one map Φ defined such that

$$\Phi:\ (\mathcal{O},\mathcal{S})\to\mathfrak{A}$$

is the *presymmetry* of the theory.

In special-relativistic mechanics,[1-7] the equations of motion have the same form in all inertial frames. A mathematical solution to an equation of motion is a family of world lines. An element of the Poincaré group has a natural action in the transformation of this family of world lines. The equation of motion is said to be relativistically covariant if and only if the set of all solutions is closed under the the transformations induced by Poincaré transformations. As Avishai and Ekstein[3] have pointed out, this fails in the presence of external forces because in this case "there is a privileged frame." The content of presymmetry is a precise and nontrivial formulation of the residual covariance of a system subject to external forces. This structure is also applicable to subsystems of free systems which are subject to internal forces from other subsystems. In this context, Newton's second law follows as a theorem.

The concept of local independence was greatly clarified by presymmetry. Several authors[1-4,8] following Ekstein's lead on local field theories discussed the following:

The Ekstein proposition. Causal independence is neither necessary nor sufficient for local commutativity.

Perhaps the most striking consequence of presymmetry is due to Avishai and Ekstein.[3] They proved that presymmetry, including a postulate of local acceleration covariance, imply the equivalence principle as a theorem.

Another recent interesting collection of the application of geometrical ideas are due to Estabrook and collaborators,[9-13] Kiehn,[14,15] Post,[16,17] and Yang *et al.*[18,19] These authors have forcefully advocated the use of modern invariant differential forms for a variety of fundamental physical problems.

In this paper we will study presymmetry using modern differential geometry. In addition to being asthetically pleasing to make the noninertial-frame nature manifest, we easily find a generalization of the usual textbook formulas[20-22] for Fermi-Walker transport and generalize the recent paper by Ni and Zimmerman.[23] Given the global nature of invariant differential geometry this might seem surprising were it not for the rich variety of local information on partial differential equation systems which Estabrook *et al.* have obtained. This also has practical importance since relativistic particles near the earth's surface are subject to exactly these noninertial forces. See also the recent paper by Ni and Zimmerman.[23] Finally, we will not introduce differential forms to the reader because of the availability of Ref. 7, where an excellent introduction appears.

The organization of this paper is the following: In Sec. II we will present a geometrical presymmetry structure. In Sec. III the invariant methods are used to give a geometrical realization of the acceleration group and in Sec. IV these methods are applied to the Fermi-Walker transport problem. In Sec. V our conclusions are given.

II. GENERAL PRESYMMETRY

Let us paraphrase Avishai and Ekstein[3] to motivate the discussion of presymmetry. We are considering experiments which are not necessarily shielded from external influences. Nevertheless, we assume that the external influences can be controlled, or understood, to the extent that sequences of experiments are reproducible. The causality of a theory gives the conditions for control.[3] A particular experiment will consist of a state preparation procedure and an observation procedure. The complete state preparation procedure includes a set of blueprints b for the specific experiment and a set of points $\{p_1, p_2, \ldots, p_n\}$ in Minkowski space at which the blueprint instructions are to be carried out. The act of observation in the experiment extracts a real number s from the prepared state. An equivalence class s' of a state-preparing procedure $(b,\{p_n\})$ are those procedures with the same mean values $\{\bar{s}_n\}$ of the infinite (random) sequence of measured values. An equivalence class α' of observation procedures consists of those observation procedures which create the same expectation values $\{\bar{s}_n\}$ for each state-preparation procedure. This gives the state-preparation collection $\mathcal{S}'$ as the equivalence class of state preparations with elements s' and the collection of observation procedure $\mathcal{O}'$ as the equivalence class of observation with elements α'. Then the mean values $\mathcal{E}'$ are a map of the Cartesian product of $\mathcal{S}'$ and $\mathcal{O}'$ into the real numbers R^1, i.e., such that

$$\mathcal{E}': \mathcal{S}' \times \mathcal{O}' \to R^1 .$$

The time translation $\mathcal{T}_\tau$: $(\vec{x}, t) \to (\vec{x}, t+\tau)$ induces a change on each procedure via

$$(b, \{p_n\}) \to (b, \{\mathcal{T}_\tau p_n\}) , \qquad (2.1a)$$

where

(a) the altered instruction is an observation procedure, and

(b) equivalence classes of procedures remain equivalence classes under this transformation. Note that the points p_n are the representatives of the Minkowski space points in the tangent space. A similar transformation can be considered to act upon the convex set $\mathcal{S}'$. The concept of *symmetry* is realized by pairs of (symmetry) transformation (S_a^*, S_{a*}) which act upon the pair $(\mathcal{O}', \mathcal{S}')$ such that the expectation $\mathcal{E}'$ is left invariant, i.e.,

$$\mathcal{E}'(S_a^*\alpha', S_{a*}s') = \mathcal{E}'(\alpha', s') , \qquad (2.1b)$$

for all $\alpha' \in \mathcal{O}'$ and for all $s' \in \mathcal{S}'$. The concept of presymmetry is realized by pairs of transformations which act upon canonical subsets $\mathcal{O}_c' \subset \mathcal{O}'$ and $\mathcal{S}_c' \subset \mathcal{S}'$ which leave $\mathcal{E}'$ invariant and which satisfy conditions (a) and (b) above, i.e.,

$$\mathcal{E}'(S_a^*\alpha'', S_{a*}s'') = \mathcal{E}'(\alpha'', s'') \qquad (2.1c)$$

for each $\alpha'' \in \mathcal{O}_c'$ and for each $s'' \in \mathcal{S}_c'$. Predictability is provided by these canonical subsets by the fact that a transformation of $\mathcal{O}_c'$ induces some corresponding transformation of the orbit sets in $\mathcal{S}_c'$. To quote Avishai and Ekstein from Ref. 3 about Eq. (2.1a), because it cannot be said better,

"The existence of such transformations is the basic fact that makes physics possible. Indeed, physics compares the results of observations at different instants by the *same* instrument, and the test for, 'sameness,' is the agreement between different instruments of the same equivalence class. Experimenters spend most of their time, 'checking their instruments,' which is a shorthand expression for verifying the preservation of equivalence classes. Note that this property is far more general than time-translation invariance of the equations of motion. It is the prime example of presymmetry."

Now let us use the invariant methods of modern differential geometry in addition to the methods of Ekstein[1] and Avishai and Ekstein.[2-4] It is necessary to lift the Lorentz manifold into its tangent (or cotangent) bundle, i.e., we must also distinquish the "space-time" from "observation of the space-time."

Let us consider a general space-time (M, η) where M is the manifold of space-time points and η is the metric. Suppose $\varphi|_z$ is a diffeomorphism of M restricted to a timelike curve z of an observer. We allow all diffeomorphisms φ which satisfy the presymmetry conditions set forth in this section. To determine the allowed class of

φ's it is necessary to construct representations of the objects $\mathcal{O}'$ and $\mathcal{S}'$.

In Minkowski space-time (L,g), it is easy to construct the representations of $\mathcal{O}'$ and $\mathcal{S}'$ if one uses the fact that L is algebraically a vector space. One simply describes an *event* as a unique vector $X \in L$ and the dual basis vector dx^i as the observation of the coordinate X^i so that

$$X^i = dx^i(X)$$

as the *measurement of the event* X. For a general space-time (M,g), this algebraic structure is only available in $T(M)$, its tangent space. This suggests that it might prove instructive to analyze Minkowski space-time in $T(L)$. Thus it is necessary to lift L into its tangent space $T(L)$ which, in effect, distinquishes between the "space-time," L and the "observation of the space-time," $T(L)$. This lifting is accomplished by identifying each event $q \in L$ with its tangent vector $X_p \in T_p(L)$ by

$$X_p = (\exp_p)^{-1} q .$$

This means that p represents an observer event and q represents a particle being observed. We will call X_p the "laboratory representation" of a particle at q. In flat Minkowski space-time, the exp map reduces to the identity and $X_p = q - p$ is a relative vector. Although the only function of the exp map here is to remind us that X "lives in" $T(L)$, it will prove absolutely essential in a future work which generalizes the present paper to curved space-time.[25] Using this construct we now introduce the presymmetry group A_G.

Definition. The presymmetry group A_G is the relativistic rigid-body motions of $\mathcal{T}_{z(\tau)}(L)$, where $z(\tau)$ is the nongeodesic curve obtained from the geodesic (τ') by the composition map

$$z = \varphi \circ \gamma \circ \sigma ,$$

where τ and τ' are arc-length parametrizations, σ is the reparametrization $\tau' = \sigma(\tau)$ and φ: $L \to L$ is the presymmetry diffeomorphism.

A vector field is called *complete* if and only if it is the infinitesimal generator of a one-parameter group of transformations on the underlying manifold. In Minkowski space-time, a natural choice of complete vector fields in the tangent space are given by

$$\vec{t} = \frac{\partial}{\partial t} ,$$

and

$$\vec{e}_i = \frac{\partial}{\partial x_i} \quad (i = 1, 2, 3) . \tag{2.2}$$

Under the diffeomorphism φ, the tangent vector $\vec{t}$ becomes $\varphi_* \vec{t}$ and the diagram

$$\begin{array}{ccc} L & \xleftarrow{\exp} & T(L) \\ \varphi \downarrow & & \downarrow \varphi_* \\ L & \xleftarrow[\exp]{} & T^*(L) \end{array}$$

commutes, i.e.,

$$\exp \circ \varphi_* = \varphi \circ \exp . \tag{2.3}$$

Therefore, the effect of φ on the laboratory representation is given by

$$\varphi_* X = [\exp_{\varphi(p)}]^{-1} [\varphi \exp(X_p)] . \tag{2.4}$$

The tangent vectors $\{\vec{t}_i |\ i = 1, 2, 3\}$ span a family of initial-data surfaces for the geodesic observer γ. Since the basis in Eq. (2.2) is complete, it follows that the set of tangent vectors $\{\varphi_* \vec{t}_i | i = 1, 2, 3\}$ are complete and provide families initial-data surfaces for an observer z. This demonstrates the existence of a causality condition for $\varphi \in A_G$.

If $\varphi \in A_G$ and $\mathcal{P}$ is a Poincaré transformation, we can write

$$\varphi = \mathcal{P} \circ \varphi_a ,$$

where

$$\varphi_a(p) = p ,$$

$$\varphi_{a*} \vec{t}_p = \vec{t}_p ,$$

for some $p \in L$. The map φ_a carries any hypersurface Σ orthogonal to $\vec{t}_p$ into itself, and, therefore, represents the act of imparting an acceleration to a particle in its instantaneous restframe. Under such a transformation, the restriction $\varphi' = \varphi_a |_\Sigma$ satisfies the presymmetry condition

$$\varphi'_* w(\varphi'_* X_p) = w_p(X) ,$$

where $w \in T^*_p(L)$ is an observation procedure.

Let $\mathcal{G}$, $\mathcal{G} \subset \mathcal{L}$, be the set of geodesics in L and let $z \in \mathcal{G}$. Under φ each $z \in \mathcal{G}$ is mapped into a nongeodesic curve $z\gamma = \varphi \circ \gamma$, so that, clearly, φ does not preserve the inertial connection on L. However, a connection ∇' exists such that z_γ is geodesic with respect to ∇' because

$$\varphi_*(\nabla_{\dot\gamma} \dot\gamma) = \nabla'_{\varphi * \dot\gamma} \varphi^* \dot\gamma = 0 . \tag{2.5}$$

Thus, a new "free" Hamiltonian exists whose flows are geodesic of ∇'. This is a geometrical statement of the equivalence principle; the presymmetry proof of the equivalence principle was given by Avishai and Ekstein in Ref. 3.

To show that our version of A_G is parametrized by a function, so that we can use the published proof by Avishai and Ekstein[2] that $\dim(A_G)$ is infinite, it is necessary to show that the diffeomorphism φ preserves the symplectic structure of L. Recall that a *symplectic structure*[7,10] on any manifold M, here L, is a global nondegenerate closed two-form ω on $T^*(M)$. In local coordinates (X^a, p_a) on $T^*(M)$, ω is given by

$$\omega = dx^a \wedge dp_a\,,$$

where the p_a's are the components of the natural one-form

$$\vartheta = p_a dx^a\,.$$

The observation of an *event* in $T^*(S_{z(\tau)})$ is a tangent vector

$$X = X^a \frac{\partial}{\partial x^a} + P \frac{\partial}{a \partial P_a}\,, \tag{2.6}$$

where (X^a, P_a) are the relative position and momentum of the particle, and where $S_{z(\tau)}$ is the local spatial rest frame of the particle with world line $z(\tau)$. Thus

$$\vartheta(X) = P_a X^a\,,$$

is the observation of "relative phase" and

$$\omega(X_1, X_2) = \tfrac{1}{2}(X_1^a P_a{}^2) \tag{2.7}$$

is an observation of neighboring geodesics. Therefore, the preservation of ω ensures, and explains, the proviso concerning preservation of rigid-body structure in presymmetry.

This shows that a "natural" geometrical formulation of presymmetry exists and, therefore, that all of the usual local structure is irrelevant for its formulation. Next, let us turn to developing this structure for the acceleration group A_G. Then we will be able to calculate easier by having eliminated unneeded details.

III. INVARIANT METHODS AND THE ACCELERATION GROUP

In this section modern invariant methods are used to derive the exact noninertial-frame equations of motion. Toward this end a geometrical construction and physical discussion of noninertial reference frames is given. Although these ideas are widely discussed in relativity tests,[20,21] lecture notes,[22] and a recent publication,[23] our general expression is not available, rather only special cases based upon drastic assumptions.

Let z: $R \to L$ be the trajectory of an accelerated observer in Minkowski space-time. At each instant τ an observer $z(\tau)$ has a local laboratory frame consisting of three orthonormal spatial vectors and a unit timelike vector which is recorded by a standard clock at $z(\tau)$. These four unit vectors span the tangent space $T_{z(\tau)}(L)$ as a vector space. Therefore, $T_{z(\tau)}(L)$ is called the "laboratory frame" of the accelerating observer A_G as in the preceding section. Both mathematically and physically the laboratory frame is distinct from the space-time L in which the accelerated observer moves. In general this distinction cannot be ignored, although in certain cases for special relativity one can ignore it. One important consequence of the postulates of special relativity is that at each instant τ and isomorphism exists between $T_{z(\tau)}(L)$ and L.

We now describe the geometry of the noninertial world line $z(\tau)$. If the connection on L is denoted by ∇, then the timelike velocity vector, $w \in T_{z(\tau)}(L)$, tangent to z is given by

$$z_* \frac{d}{d\tau} = u = \frac{\nabla z}{\partial \tau}\,, \tag{3.1}$$

where $\langle u, u\rangle = 1$. The accelerated observer also carries a triad of spatial vectors $\vec{e}_i \in T_{z(\tau)}(L)$, $i = 1, 2, 3$ which satisfy

$$\langle \vec{e}_i(\tau),\ \vec{e}_j(\tau)\rangle = -\delta_{ij} \tag{3.2}$$

and

$$\langle u(\tau),\ \vec{e}_i(\tau)\rangle = 0\,.$$

At each instant τ the $\vec{e}_i$ span the three-dimensional vector space $S_{z(\tau)}$ orthogonal to u as shown in Fig. 1. Physically, $S_{z(\tau)}$ is the spatial rest frame of the observer at $z(\tau)$. Further, the $\vec{e}_i$ are unique modulo a constant spatial SO(3) rotation which reflects the Galilean principle of relativity. Vectors lying entirely in $S_{z(\tau)}$ are indicated by overarrows.

The orthogonal tetrad $\{u, \vec{e}_i\}$ is propagated along z via the equations

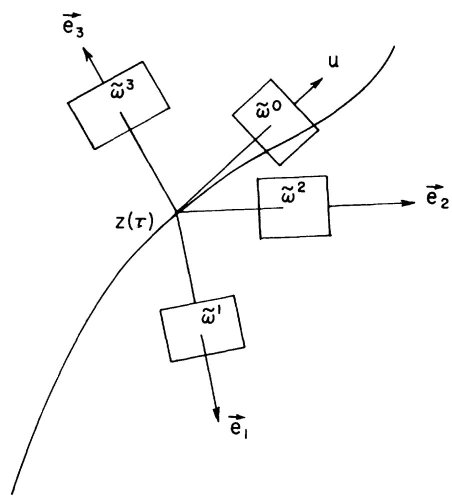

FIG. 1. World line of accelerating observe showing the vector tetrad $\{u, \vec{e}_i\}$ and its corresponding dual $\{\tilde{\omega}^0, \tilde{\omega}^i\}$.

$$\frac{\nabla \vec{e}_i}{\partial \tau} = \langle \Omega, \vec{e}_i \rangle ,$$
$$\frac{\nabla u}{\partial \tau} = \langle \Omega, u \rangle . \qquad (3.3)$$

The definition of Ω in Eq. (3.3) is

$$\Omega = \vec{a} \otimes u - u \otimes \vec{a} + \Omega_s(\vec{\omega}, u) , \qquad (3.4)$$

where Ω_s is a tensor satisfying

$$\langle \Omega_s(\vec{\omega}, u), u \rangle = 0$$

and

$$\langle \Omega_s(\vec{\omega}, u), A \rangle = \vec{\omega} \times \vec{A} .$$

The cross product is the usual one in $S_{z(\tau)} \cong R^3$ and $\vec{a}$ is the *invariant* acceleration of $z(\tau)$ given by

$$\vec{a} = \frac{\nabla^2 z(\tau)}{\partial \tau^2} = \frac{\nabla u}{\partial \tau} . \qquad (3.5)$$

$\vec{\omega}$ is an angular velocity of $z(\tau)$ and both $\vec{a}$ and $\vec{\omega}$ are contained in $S_{z(\tau)}$. $\vec{a} \in S_{z(\nu)}$ follows from

$$0 = \frac{d}{d\tau} \langle u, u \rangle = 2 \langle u, \nabla u / \partial \tau \rangle = 2 \langle u, \vec{a} \rangle .$$

The propagation equations (3.3) preserve the othrogonality relations in Eq. (3.2) since Ω is antisymmetric. Also, Ω acting on the basis set $\{u, \vec{e}_i\}$ is an active infinitesimal Lorentz transformation. This gives the well-known classical equation

$$\frac{d\vec{e}_i}{dt} = \vec{\omega} \times \vec{e}_i = \Omega_s \cdot \vec{e}_i , \qquad (3.6)$$

with

$$\Omega_s = \sum_{i,j,k} \omega_j \epsilon_{ijk} \vec{e}_i \otimes \vec{e}_k .$$

Upon expanding Eq. (3.3) the physical content of both parts of that equation is obtained. The second part yields

$$\frac{\nabla u}{\partial \tau} = \vec{a} , \qquad (3.7)$$

which is the definition of $\vec{a}$. The first part becomes

$$\frac{\nabla \vec{a}}{\partial \tau} = -\langle \vec{e}_i, \vec{a} \rangle u + \vec{\omega} \times \vec{e}_i . \qquad (3.8)$$

The first term on the right-hand side is required to preserve the relation $\langle \vec{e}_i, u \rangle = 0$, and the second term shows that the $\vec{e}_i$'s rotate with the angular velocity $\vec{\omega}$ in the rest frame of z.

Although the previous results are obtained in a straightforward manner, they have an unphysical aspect due to time ordering. This suggests that a new definition of the reference frame is needed to eliminate this difficulty. The material that follows is directed toward this end.

Definition 1, $x \in L$ is spatially simultaneous with $z(\tau)$, written as $x \# z(\tau)$, if for some $\vec{r} \in S_{z(\tau)}$,

$$x = z(\tau) + \vec{r} . \qquad (3.9)$$

Remarks.

(i) This definition depends upon the value of u at $z(\tau)$.

(ii) This definition of simultaneity does not define an equivalent equivalence relation because symmetry is not defined and transitivity is false. Thus, $\vec{r}$ in the definition of X is nonunique.
nonunique.

(iii) Finally given an $(\vec{X}, t > 0) \in L$, there does not exist a $\tau_0 < 0$ such that $z(\tau_0)$ and X are simultaneous. This obtains because time ordering "loses" the time-reversed solution. Thus a natural reference frame based upon the ecomposition of Eq. (3.3) is unphysical.
physical.

Definition 2. The simultaneity set $\mathcal{S}_0[z(\tau), L]$ is the interior of a spacelike cone plus the origin, i.e., $\mathcal{S}_0[z(\tau), L] = \{X \in L \mid X \# z(\tau) \text{ for some } \tau_0\}$. Using definition 2, a better choice of reference, termed *null simultaneity*, can be based upon the decomposition

$$X = z(\tau) + y , \qquad (3.10)$$

where $\langle y, u \rangle < 0$, i.e., y is a past-directed null vector. In Fig. 2 the geometry of the null simultaneous reference frame is displayed.

Theorem. The decomposition in definition 2 is unique

Proof. Suppose the contrary is true, thus

$$X = z(\tau_1) + y_1 = z(\tau_2) + y_2 ,$$

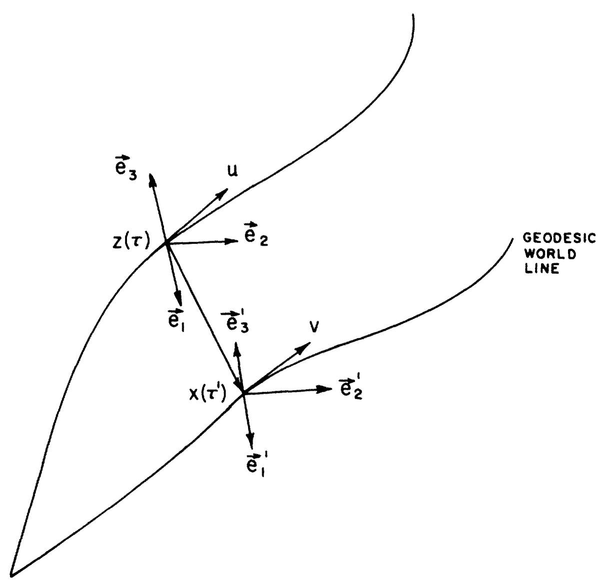

FIG. 2. The geometric concept of null simultaneity in relationship to the accelerating observer.

where $\tau_1 < \tau_2$ and $y_1 \neq y_2$. Then

$$z(\tau_2) - z(\tau_1) = y_1 - y_2$$

is timelike, so that

$$\langle y_1 - y_2,\ y_1 - y_2 \rangle > 0 ,$$

which implies that

$$\langle y_1, y_2 \rangle < 0 . \tag{3.11}$$

However, each past-directed null vector can be written as

$$y_1 = -|\vec{y}_1| u + \vec{y}_1 ,$$

$$y_2 = -|\vec{y}_2| u + \vec{y}_2 .$$

Thus

$$\langle y_1, y_2 \rangle = |\vec{y}_1|\,|\vec{y}_2| - \vec{y}_1 \cdot \vec{y}_2 > 0 ,$$

which is contrary to Eq. (3.11). Q.E.D.

The reference frame based upon Eq. (3.9) is used in this paper because it is the standard frame used and the equations of motion take on a particularly simple and pleasing form. The unphysical effects mentioned above occur only at points for which

$$\langle \vec{a}, \vec{r} \rangle \geq 1 ,$$

so that this frame is valid for most L. We derive the equation of motion for reference frame Eq. (3.10) in the Appendix.

IV. INVARIANT FORMULATION OF FERMI-WALKER TRANSPORT

The equation of motion of a free particle relative to the noninertial frame is now derived. Let $X(\tau')$ be the world line of a free particle with velocity vector

$$V = \frac{\nabla X}{\partial \tau'} , \tag{4.1}$$

see Fig. 2. Then

$$X[\tau'(\tau)] = z(\tau) + \vec{r}(\tau) \tag{4.2}$$

and therefore

$$V = \frac{\nabla X(\tau')}{\partial \tau} = \frac{d\tau}{d\tau'} \frac{\nabla}{\partial \tau} [z(\tau) + \vec{r}(\tau)]$$

$$= \Gamma \left(u + \frac{\nabla \vec{r}(\tau)}{\partial \tau} \right) , \tag{4.3}$$

where $\Gamma = d\tau / d\tau'$. Since $X(\tau')$ is free, its acceleration must vanish,

$$\frac{\nabla V}{\partial \tau'} = 0 . \tag{4.4}$$

Equation (4.4) is an invariant expression of the usual geodesic equation. To rewrite Eq. (4.4) in terms of quantities relative to $z(\tau)$ we introduce the new connection ∇' defined by

$$\frac{\nabla' A}{\partial \tau} = \frac{\nabla A}{\partial \tau} - \langle \Omega, A \rangle . \tag{4.5}$$

Clearly

$$\frac{\nabla' u}{\partial \tau} = \frac{\nabla' \vec{e}_i}{\partial \tau} = 0 . \tag{4.6}$$

$\nabla'/\partial\tau$ is the natural time derivative associated with the noninertial observer since the observer sees the basis $\{u, \vec{e}_i\}$ as constants, as expressed in Eq. (4.6). If $\vec{r} \in S_{z(\tau)}$, then

$$\frac{\nabla' r}{\partial \tau} = \frac{\nabla'}{\partial \tau} r^i \vec{e}_i = \frac{dr^i}{d\tau} \vec{e}_i = \dot{\vec{r}} , \tag{4.7}$$

so that $\dot{\vec{r}} \in S_{z(\tau)}$. Using Eqs. (4.5) through (4.7) one can expand Eq. (4.3) to obtain

$$V = \Gamma(\tau)[u(1 + \vec{a} \cdot \vec{r}) + \dot{\vec{r}} + \vec{\omega} \times \vec{r}] . \tag{4.8}$$

Equation (4.4) can be rewritten as

$$\frac{\nabla V}{\partial \tau'} = \frac{d\tau}{d\tau'} \frac{\nabla V}{\partial \tau} = \Gamma \frac{\nabla V}{\partial \tau} = 0 . \tag{4.9}$$

Therefore from Eq. (4.9) one has

$$0 = \frac{\nabla V}{\partial \tau} = \frac{\nabla' V}{\partial \tau} + \langle \Omega, V \rangle$$

$$= \dot{\Gamma}[(1 + \vec{a} \cdot \vec{r})u + \dot{\vec{r}} + \vec{\omega} \times \vec{r}]$$

$$+ \Gamma[(\dot{\vec{a}} \cdot \vec{r} + \vec{a} \cdot \dot{\vec{r}})u + \ddot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}}]$$

$$+ \Gamma[(1 + \vec{a} \cdot \vec{r})\vec{a} + u(\vec{a} \cdot \dot{\vec{r}} + \vec{a} \cdot \vec{\omega} \times \vec{r})$$

$$+ \vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})] . \tag{4.10}$$

Separating Eq. (4.10) into its components parallel and perpendicular to u gives

$$\dot{\Gamma}(1 + \vec{a} \cdot \vec{r}) + \Gamma(\dot{\vec{a}} \cdot \vec{r} + 2\vec{a} \cdot \dot{\vec{r}} + \vec{a} \cdot \vec{\omega} \times \vec{r}) = 0 \tag{4.11}$$

and

$$\dot{\Gamma}(\dot{\vec{r}} + \vec{\omega} \times \vec{r}) + \Gamma[\ddot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}}$$

$$+ (1 + \vec{a} \cdot \vec{r})\vec{a} + \vec{\omega} \times (\vec{\omega} \times \vec{r})] = 0 . \tag{4.12}$$

From Eq. (4.11) it follows that

$$\dot{\Gamma} = -\Gamma(1 + \vec{a} \cdot \vec{r})^{-1}(\dot{\vec{a}} \cdot \vec{r} + 2\vec{a} \cdot \dot{\vec{r}} + \vec{a} \cdot \vec{\omega} \times \vec{r}) , \tag{4.13}$$

and substituting this Eq. (4.12) gives the acceleration in a noninertial frame as

$$\ddot{\vec{r}} = -\vec{a}\left(1+\frac{\vec{a}\cdot\vec{r}}{c^2}\right) - 2\vec{\omega}\times\dot{\vec{r}} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \dot{\vec{\omega}}\times\vec{r} + \frac{(\dot{\vec{r}}+\vec{\omega}\times\vec{r})}{c^2(1+\vec{a}\cdot\vec{r}/c^2}(\dot{\vec{a}}\cdot\vec{r}+2\vec{a}\cdot\dot{\vec{r}}+\vec{a}\cdot\vec{\omega}\times\vec{r}), \tag{4.14}$$

where the factors of c have been inserted. This equation is *exact* and illustrates the power of invariant methods.[24]

Lastly, Eq. (4.14) is specialized to more familiar cases for ease in interpretation. The quantity $\vec{\omega}_T = \vec{r}\times\vec{a}/2c^2$ is called the Thomas angular velocity. If $\vec{a}$ and $\vec{\omega}$ are constant in time, Eq. (4.14) reduces to

$$\ddot{\vec{r}} = -\vec{a}\left(1+\frac{\vec{a}\cdot\vec{r}}{c^2}\right) - 2\vec{\omega}\times\dot{\vec{r}} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{2(\vec{a}\cdot\dot{\vec{r}})\dot{\vec{r}}}{(1+\vec{a}\cdot\vec{r}/c^2)c^2} + \frac{2(\vec{a}\cdot\dot{\vec{r}})(\vec{\omega}\times\vec{r})}{(1+\vec{a}\cdot\vec{r}/c^2)c^2} + \frac{\vec{a}\cdot\vec{\omega}\times\vec{r}(\dot{\vec{r}}+\vec{\omega}\times\vec{r})}{(1+\vec{a}\cdot\vec{r}/c^2)c^2}. \tag{4.15}$$

The fourth term on the right-hand side of Eq. (4.15) can be rewritten as

$$\frac{2(\vec{a}\cdot\dot{\vec{r}})\dot{\vec{r}}}{(1+\vec{a}\cdot\vec{r}/c^2)c^2} = \frac{2[(\dot{\vec{r}}^2/c^2)\vec{a} - 2\vec{\omega}_T\times\dot{\vec{r}}]}{(1+\vec{a}\cdot\vec{r}/c^2)c^2}. \tag{4.16}$$

Using Eq. (4.16) in Eq. (4.15) and further restricting Eq. (4.15) to a particle at the origin, $\vec{r}=0$, one obtains

$$\ddot{\vec{r}} = -\vec{a}\left(1-\frac{2\dot{\vec{r}}^2}{c^2}\right) - 2(\vec{\omega}+2\vec{\omega}_T)\times\dot{\vec{r}}. \tag{4.17}$$

From Eq. (4.17) it is clear that the Coriolis force is increased by the Thomas precession. Another interesting consequence of Eq. (4.17) occurs if $\dot{\vec{r}}$ is parallel to $\vec{\omega}+2\vec{\omega}_T$. For this situation when $|\dot{\vec{r}}| = c/\sqrt{2}$ the particle is instantaneously unaccelerated, whereas if $|\dot{\vec{r}}| > c/\sqrt{2}$ a particle will "fall up" along $\vec{a}$.

To generalize this to a collection of N free particles one can simply addend a subscript i, $1 \le i \le N$, to $\vec{r}$. If $\vec{r}$ had been restricted to the origin at the very beginning as in Ref. 20, this would not be possible. The expression corresponding to Eq. (4.17) for the null reference frame is included for completeness in Appendix A.

V. CONCLUSIONS

We have used modern differential geometry to formulate Ekstein's presymmetry. This yielded a geometrical construction which is independent of local-inertial-frame considerations, and thereby provided a simpler, and *exact*, treatment of a particle in a flat-space noninertial frame and also of Fermi-Walker transport in special relativity. This provides a non-negligiable improvement over the traditional approximate treatments[20-23] of noninertial frames. The next-to-last term in our Eq. (4.14) has a factor of 2 different from Eq. (20) of Ni and Zimmerman.[23] This difference is due to the differences in our respective calculations: Theirs is approximate and ours is exact. Their method is terribly complicated compared to ours, but we believe that they are correct given their approximation. Since our Eq. (4.14) is exact, it is not necessary to use their formula.

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APPENDIX A

In order to give the equation of motion of a single free particle as it appears in the null reference frame, choose the null vector as

$$y = -ru + \vec{r}. \tag{A1}$$

where $\vec{r} \in Sz(\tau)$ and $r = (\vec{r},\vec{r})^{1/2}$. For $\vec{a}$ and $\vec{\omega}$ uniform, a calculation parallel to that of Sec. III yields

$$\ddot{\vec{r}} = -(1-2\dot{r}+\vec{a}\cdot\dot{\vec{r}}+Br)\vec{a} + r(\vec{\omega}\times\vec{a}) - R(\vec{\omega}\times\dot{\vec{r}}) - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - B(\dot{\vec{r}}+\vec{\omega}\times\vec{r}), \tag{A2}$$

where

$$B = -\frac{R}{r} + \frac{(\vec{r}\times\dot{\vec{r}})^2}{r^3} - 2\vec{a}\cdot\dot{\vec{r}} + r(\vec{a})^2 - \vec{a}\cdot(\vec{\omega}\times\vec{r}) \tag{A3}$$

with

$$R = [2(\vec{\omega}\times\dot{\vec{r}}) + (1-2\dot{r}+\vec{a}\cdot\vec{r})\vec{a} - r(\vec{\omega}\times\vec{a}) + \vec{\omega}\times(\vec{\omega}\times\vec{r})]\cdot\vec{r} \tag{A4}$$

and

$$c = 1.$$

Note that Eq. (A2) is complicated by aberration and the Doppler shift of light emitted from the free particle. It should be emphasized that Eq. (A2) represents the actual motion as seen by the noninertial observer.

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[24]Contrast Eq. (4.13) here with Eq. (13.75) on page 332 of Ref. 17. Also, note that the present derivation is no longer or more difficult than the one on pages 135–149 of Ref. 20 and is more general and is far simpler than the method of Ref. 23.
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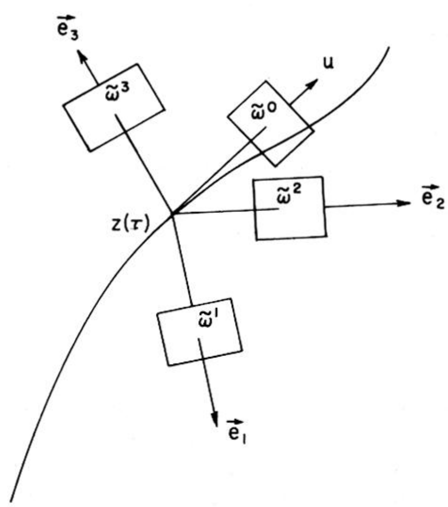

FIG. 1. World line of accelerating observer showing the vector tetrad $\{u, \vec{e}_i\}$ and its corresponding dual $\{\tilde{\omega}^0, \tilde{\omega}^i\}$.

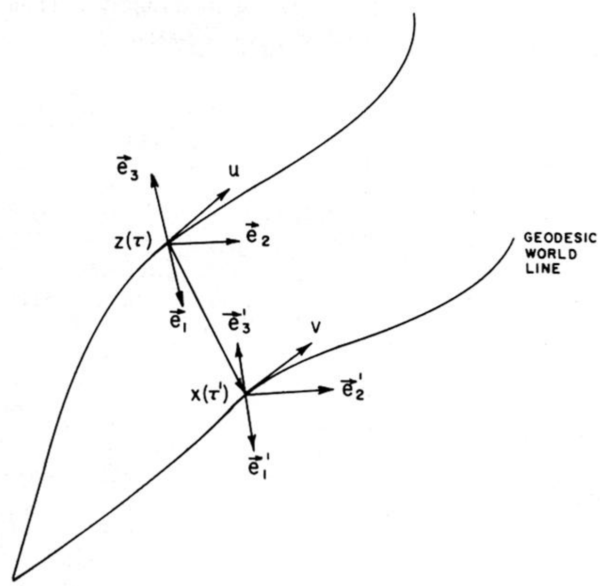

FIG. 2. The geometric concept of null simultaneity in relationship to the accelerating observer.