"No-hair" theorems for the Abelian Higgs and Goldstone models

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We examine the question of whether black holes can have associated external massive vector and/or scalar fields, when the masses are produced by spontaneous symmetry breaking. Working throughout in the spherically symmetric case, we show that "no-hair" theorems can be proved for the vector field in the Abelian Higgs model, for an arbitrary $\xi R|\phi|^2$ term in the Higgs Lagrangian, and for the Goldstone scalar field model with $\xi = 0$. We also show that a Minkowski-space analog problem does have nontrivial screened charge solutions, indicating that the "no-hair" theorems which we prove are consequences of the stringent conditions at the assumed horizon in the general-relativistic case, not of the interacting field or spontaneous-symmetry-breaking aspects of the problem.

I. INTRODUCTION

One of the striking features of the physics of black holes is the existence of "no-hair" theorems, which state that the only external attributes of a black hole (such as its mass M, angular momentum J, and electric charge Q) are those associated with massless fields admitting conserved flux integrals.[1,2] All other types of fields must decouple, under the assumption of a well-behaved geometry at the horizon. These theorems have been proved for a variety of wave equations, including the massless Dirac field, various massive scalar field theories, and the massive spin-1 Proca field. Our purpose in the present paper is to extend this list of equations studied to include classical wave equations in which masses are generated by spontaneous symmetry breaking. This is particularly important in the vector-meson case, since it is widely believed that if massive spin-1 fields exist, they get their masses through a dynamical mechanism of spontaneous symmetry breaking,[3] rather than kinematically as in the Proca equation. The simplest relevant model is the Abelian Higgs model,[3] and so the main focus of this paper is on the question of whether black holes can have Abelian Higgs "hair." We also give some results for the closely related Goldstone scalar-meson model. For simplicity, we assume spherical symmetry throughout, since we expect that if interesting violations of the "no-hair" theorems were to occur, they would be seen in the spherically symmetric case. We find, in fact, no evidence for such violations, and prove "no-hair" theorems for the cases we study. We believe it likely that our proofs will generalize to the nonspherical case.

II. THE ABELIAN HIGGS MODEL

Before writing down the Abelian Higgs model Lagrangian, we begin with some geometric preliminaries.[4] We assume the general time-independent, spherically symmetric line element

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \qquad (1)$$

Using a caret to denote components on the orthonormal basis

$$\tilde{\omega}^{\hat{t}} = e^{\alpha}\tilde{d}t\,, \quad \tilde{\omega}^{\hat{r}} = e^{\beta}\tilde{d}r\,, \quad \tilde{\omega}^{\hat{\theta}} = r\tilde{d}\theta, \quad \tilde{\omega}^{\hat{\phi}} = r\sin\theta\tilde{d}\phi\,, \qquad (2)$$

and using a prime to indicate differentiation d/dr, the Einstein tensor components for this line element are

$$\begin{aligned} G^{\hat{r}\hat{r}} &= \frac{2}{r}e^{-2\beta}\alpha' - r^{-2}(1-e^{-2\beta})\,, \\ G^{\hat{t}\hat{t}} &= \frac{2}{r}e^{-2\beta}\beta' + r^{-2}(1-e^{-2\beta})\,, \\ G^{\hat{\theta}\hat{\theta}} &= G^{\hat{\phi}\hat{\phi}} \\ &= e^{-2\beta}(\alpha'' + \alpha'^2 - \alpha'\beta' + \alpha' r^{-1} - \beta' r^{-1})\,. \end{aligned} \qquad (3)$$

The curvature scalar is

$$R = 2r^{-2}(1-e^{-2\beta}) + 4r^{-1}e^{-2\beta}(\beta'-\alpha') - 2e^{-2\beta}(\alpha'' + \alpha'^2 - \alpha'\beta')\,, \qquad (4)$$

and the Bianchi identity is

$$(G^{\hat{r}\hat{r}})' + \alpha' G^{\hat{t}\hat{t}} - \frac{2}{r}G^{\hat{\theta}\hat{\theta}} + \left(\alpha' + \frac{2}{r}\right)G^{\hat{r}\hat{r}} = 0\,. \qquad (5)$$

The Abelian Higgs model describes a charged scalar field, with a double-well self-interaction, coupled to an initially massless Abelian gauge field. The Lagrangian density for the model, written in generally covariant form, is

$$\mathcal{L} = (-g)^{1/2}[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - d_\mu d^{*\mu} - \xi R|\phi|^2 - h(|\phi|^2 - \phi_\infty{}^2)^2]\,, \qquad (6)$$

with

$$F_{\mu\nu}=\frac{\partial A_\mu}{\partial x^\nu}-\frac{\partial A_\nu}{\partial x^\mu}\,,\qquad d_\mu=\left(\frac{\partial}{\partial x^\mu}-ieA_\mu\right)\phi\,. \tag{7}$$

The parameter ξ is zero for the usual "minimal" scalar wave equation, while $\xi=\frac{1}{6}$ for the "conformal" scalar wave equation which is conformally invariant in the absence of mass terms. Spontaneous symmetry breaking arises because the effective potential

$$V(\phi)=h(|\phi|^2-\phi_\infty{}^2)^2 \tag{8}$$

has its minimum at $|\phi|=\phi_\infty$, rather than at $\phi=0$.

Following the analysis of Bekenstein[2] in the similar case of charged scalar electrodynamics, we use the fact that in the time-independent case of interest in black-hole physics we can choose a gauge in which ϕ is real, $A_i=0$, and A_t is time independent.[5] In this gauge the field equations of motion which follow from the Lagrangian of Eq. (6) are

$$(e^{-(\alpha+\beta)}r^2A_t')'=r^2e^{\beta-\alpha}2e^2A_t\phi^2\,, \tag{9a}$$

$$\begin{aligned}(e^{\alpha-\beta}r^2\phi')'&=r^2e^{\alpha+\beta}[-e^{-2\alpha}e^2A_t{}^2\phi+2h\phi(\phi^2-\phi_\infty{}^2)]\,,\quad\text{"minimal"}\\ (e^{\alpha-\beta}r^2\phi')'&=r^2e^{\alpha+\beta}[-e^{-2\alpha}e^2A_t{}^2\phi+\tfrac{1}{6}R\phi+2h\phi(\phi^2-\phi_\infty{}^2)],\quad\text{"conformal"}\,.\end{aligned} \tag{9b}$$

The stress-energy tensor components in the two cases are the following for the "minimal" model:

$$\begin{aligned}T^{\hat t\hat t}&=\tfrac{1}{2}e^{-2(\alpha+\beta)}(A_t')^2+e^{-2\beta}(\phi')^2+e^{-2\alpha}e^2A_t{}^2\phi^2+h(\phi^2-\phi_\infty{}^2)^2\,,\\ T^{\hat r\hat r}&=-\tfrac{1}{2}e^{-2(\alpha+\beta)}(A_t')^2+e^{-2\beta}(\phi')^2+e^{-2\alpha}e^2A_t{}^2\phi^2-h(\phi^2-\phi_\infty{}^2)^2\,,\\ T^{\hat\theta\hat\theta}&=\tfrac{1}{2}e^{-2(\alpha+\beta)}(A_t')^2-e^{-2\beta}(\phi')^2+e^{-2\alpha}e^2A_t{}^2\phi^2-h(\phi^2-\phi_\infty{}^2)^2\,,\end{aligned} \tag{10a}$$

and the following for the "conformal" model:

$$\begin{aligned}T&=T^\alpha_\alpha=4h\phi_\infty{}^2(\phi^2-\phi_\infty{}^2)\,,\\ T^{\hat t\hat t}&=-\tfrac{1}{4}T+\tfrac{1}{2}e^{-2(\alpha+\beta)}(A_t')^2+\tfrac{1}{3}e^{-2\beta}(\phi')^2+\tfrac{5}{3}e^{-2\alpha}e^2A_t{}^2\phi^2\\ &\quad+\tfrac{2}{3}\alpha'e^{-2\beta}\phi\phi'-\tfrac{1}{9}\phi^2R+\tfrac{1}{3}\phi^2G^{\hat t\hat t}-\tfrac{1}{3}\phi^2h(\phi^2-\phi_\infty{}^2)\,,\\ T^{\hat r\hat r}&=\tfrac{1}{4}T-\tfrac{1}{2}e^{-2(\alpha+\beta)}(A_t')^2+e^{-2\beta}(\phi')^2+\tfrac{1}{3}e^{-2\alpha}e^2A_t{}^2\phi^2\\ &\quad-\tfrac{2}{3}\phi e^{-\beta}(e^{-\beta}\phi')'+\tfrac{1}{9}\phi^2R+\tfrac{1}{3}\phi^2G^{\hat r\hat r}+\tfrac{1}{3}\phi^2h(\phi^2-\phi_\infty{}^2)\,,\\ T^{\hat\theta\hat\theta}&=\tfrac{1}{4}T+\tfrac{1}{2}e^{-2(\alpha+\beta)}(A_t')^2-\tfrac{1}{3}e^{-2\beta}(\phi')^2+\tfrac{1}{3}e^{-2\alpha}e^2A_t{}^2\phi^2\\ &\quad-\tfrac{2}{3}\frac{1}{r}e^{-2\beta}\phi\phi'+\tfrac{1}{9}\phi^2R+\tfrac{1}{3}\phi^2G^{\hat\theta\hat\theta}+\tfrac{1}{3}\phi^2h(\phi^2-\phi_\infty{}^2)\,.\end{aligned} \tag{10b}$$

In both cases these components satisfy the equation of stress-energy conservation

$$(T^{\hat r\hat r})'+\alpha'T^{\hat t\hat t}-\frac{2}{r}T^{\hat\theta\hat\theta}+\left(\alpha'+\frac{2}{r}\right)T^{\hat r\hat r}=0\,, \tag{11}$$

which determines $T^{\hat\theta\hat\theta}$ given $T^{\hat r\hat r}$ and $T^{\hat t\hat t}$. The two independent Einstein equations are then

$$G^{\hat t\hat t}=8\pi T^{\hat t\hat t}\,,\qquad G^{\hat r\hat r}=8\pi T^{\hat r\hat r}\,. \tag{12}$$

We proceed now to prove a "no-hair" theorem for the Abelian Higgs model. We assume that the coupled system consisting of the vector and the Higgs scalar field and the spherically symmetric space-time geometry, described by Eqs. (1) and (2) above, has a horizon at $r=r_H$ at which all physical scalars are finite. We show that these assumptions imply that the vector field A_t vanishes identically outside the horizon. Multiplying Eq. (9a) by A_t and integrating from r_H to ∞ gives, after an integration by parts,

$$\int_{r_H}^{\infty}r^2dr[e^{-(\alpha+\beta)}(A_t')^2+2e^{\beta-\alpha}e^2A_t{}^2\phi^2]=A_tA_t'r^2e^{-(\alpha+\beta)}|_{r_H}^{\infty}\,. \tag{13}$$

The contribution from ∞ to the right-hand side vanishes, since A_t falls off asymptotically at least as $1/r$. The assumption that the physical scalar $F_{\mu\nu}F^{\mu\nu}$ is bounded at $r=r_H$ implies that $e^{-(\alpha+\beta)}A_t'$ is bounded at the horizon. Hence if $A_t=0$ at r_H, the right-hand side of Eq. (13) vanishes, and the fact that the left-hand side is non-negative (note that the metric components $e^{2\alpha}$ and $e^{2\beta}$ are non-negative outside the horizon) then implies $A_t \equiv 0$ for all $r \geq r_H$. So we get a "no-hair" theorem unless[6] $A_t|_H \neq 0$.

The remainder of the argument consists of showing that having $A_t|_H \neq 0$ contradicts the assumption that all physical scalars are finite at the horizon.[7] We do this by examining the behavior of the scalar field equation near the horizon. We note first of all that in the "minimal" model boundedness of $T^{\hat{t}\hat{t}}|_H$, and in the "conformal" model boundedness of $T|_H$, both imply that the scalar field ϕ is bounded on the horizon. Hence when $A_t|_H \neq 0$ we have

$$\frac{(0,\frac{1}{6})R\phi + 2h\phi(\phi^2 - \phi_\infty{}^2)}{-e^{-2\alpha}e^2A_t{}^2\phi} \sim \frac{e^{2\alpha}}{A_t{}^2} \times (\text{bounded}) \xrightarrow[r \to r_H]{} 0 \ , \tag{14}$$

and the scalar field equation can be approximated near the horizon by

$$(e^{\alpha-\beta}r^2\phi')' + r^2 e^{\alpha+\beta}e^{-2\alpha}e^2A_t{}^2\phi = 0 \,. \tag{15}$$

It proves convenient at this point to change the independent variable from r to λ, with λ the affine parameter of an incoming null geodesic. The differential equation relating λ to r is

$$\begin{aligned} ds^2 = 0 &= -e^{2\alpha}\left(\frac{dt}{d\lambda}\right)^2 + e^{2\beta}\left(\frac{dr}{d\lambda}\right)^2 \\ &= -e^{-2\alpha}P_0{}^2 + e^{2\beta}\left(\frac{dr}{d\lambda}\right)^2 . \end{aligned} \tag{16}$$

Since t is a cyclic variable for a time-independent metric, the conjugate momentum P_0 is a constant of the motion,[8] and so after rescaling λ to make $P_0 = 1$, the second line of Eq. (16) gives

$$\frac{dr}{d\lambda} = e^{-(\alpha+\beta)} . \tag{17}$$

Since the horizon must be a finite affine distance away from any $r > r_H$, the value λ_H of λ at the horizon is finite. In terms of λ, and making the definitions

$$\begin{aligned} q &= e^{2\alpha} \ , \\ p &= e^{-2(\alpha+\beta)} \ , \end{aligned} \tag{18}$$

so that $dr/d\lambda = p^{1/2}$, the approximated scalar field equation becomes

$$\frac{d}{d\lambda}\left(qr^2\frac{d\phi}{d\lambda}\right) + r^2e^2A_t{}^2q^{-1}\phi = 0 \ . \tag{19}$$

To proceed, we need some information on the behavior of q and its derivatives near the horizon. This can be obtained by rearranging Eq. (3) for the Einstein tensor components into the form

$$G^{\hat{r}\hat{r}} + \frac{1}{r^2} = \frac{p^{1/2}}{r}\frac{dq}{d\lambda} + \frac{pq}{r^2} \ , \tag{20a}$$

$$G^{\hat{t}\hat{t}} + G^{\hat{r}\hat{r}} = -\frac{2}{r}q\frac{d}{d\lambda}(p^{1/2}) \ , \tag{20b}$$

$$G^{\hat{\theta}\hat{\theta}} = \frac{1}{r}\frac{d}{d\lambda}(p^{1/2}q) + \frac{1}{2}\frac{d^2q}{d\lambda^2} \,. \tag{20c}$$

From the boundedness at the horizon of the left-hand sides of these equations, and the fact that both terms on the right-hand side of Eq. (20a) are non-negative, we deduce the following:

$$\begin{aligned} &\text{Eqs. (20a), (20b)} \Rightarrow pq|_H,\ p^{1/2}\frac{dq}{d\lambda}\Big|_H,\ q\frac{d}{d\lambda}p^{1/2}\Big|_H = \text{bounded} \Rightarrow \frac{d}{d\lambda}(p^{1/2}q)\Big|_H = \text{bounded} \Rightarrow p^{1/2}q|_H = \text{bounded}\ , \\ &\text{Eq. (20c)} \Rightarrow \frac{d^2q}{d\lambda^2}\Big|_H = \text{bounded} \Rightarrow \frac{dq}{d\lambda}\Big|_H = \text{bounded} . \end{aligned} \tag{21}$$

Hence writing

$$\theta = qr^2 \ , \tag{22}$$

we have

$$\begin{aligned} \frac{d\theta}{d\lambda}\Big|_H &= r_H{}^2\frac{dq}{d\lambda}\Big|_H + 2r_Hqp^{1/2}|_H = \text{bounded and} \geq 0, \\ \frac{d^2\theta}{d\lambda^2}\Big|_H &= r_H{}^2\frac{d^2q}{d\lambda^2}\Big|_H + 4r_Hp^{1/2}\frac{dq}{d\lambda}\Big|_H + 2r_Hq\frac{dp^{1/2}}{d\lambda}\Big|_H + 2pq|_H = \text{bounded}\ , \end{aligned} \tag{23}$$

and in terms of θ the scalar field equation near the horizon takes the compact form

$$\begin{aligned} &\theta\frac{d^2\phi}{d\lambda^2} + \frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} + \frac{K\phi}{\theta} = 0 \ , \\ &K = e^2r^4A_t{}^2|_H > 0 \,. \end{aligned} \tag{24}$$

The final ingredient needed for the argument is the fact that boundedness of $T^{\hat{t}\hat{t}}|_H$ requires

$$q^{-1}\phi^2|_H = \text{bounded} \tag{25a}$$

in the "minimal" model (since in this model all terms in $T^{\hat{t}\hat{t}}$ are non-negative), and

$$\left[q^{-1}\phi^2 + K_1\left(\frac{d\phi}{d\lambda}\right)^2 + K_2\phi\frac{d\phi}{d\lambda}\right]_H = \text{bounded} \tag{25b}$$

in the "conformal" model, with

$$K_1 = [q/(5e^2A_t^{\,2})]_H\,, \quad K_2 = [(dq/d\lambda)/(5e^2A_t^{\,2})]_H \tag{25c}$$

two bounded constants. The strategy of the argument now is to show that Eqs. (23)–(25) are inconsistent. We consider separately the two cases where $d\theta/d\lambda\,|_H > 0$ and where $d\theta/d\lambda\,|_H = 0$.

When $d\theta/d\lambda\,|_H = C > 0$, we can approximate $\theta = C(\lambda - \lambda_H)$ near the horizon, and Eq. (24) takes the form

$$\frac{d^2\phi}{d\lambda^2} + \frac{1}{\lambda-\lambda_H}\frac{d\phi}{d\lambda} + \frac{K}{C^2}\frac{\phi}{(\lambda-\lambda_H)^2} = 0\,, \tag{26}$$

which has the general solution

$$\phi = \phi_0\cos(x+\delta), \quad x = \frac{K^{1/2}}{C}\ln(\lambda-\lambda_H)\,. \tag{27}$$

Hence in this case we find near the horizon

$$\begin{aligned} q^{-1}\phi^2 &+ K_1\left(\frac{d\phi}{d\lambda}\right)^2 + K_2\phi\frac{d\phi}{d\lambda} \\ &\propto (\lambda-\lambda_H)^{-1}(\cos^2 x + C_1\sin^2 x \\ &\qquad + C_2\sin x\cos x)\,, \end{aligned} \tag{28}$$

which is unbounded at λ_H for all values of the constants $C_{1,2}$. In the second case, when $d\theta/d\lambda\,|_H = 0$, we make an exponential substitution $\phi = e^f$ in Eq. (24), giving

$$\theta\left[\frac{d^2f}{d\lambda^2} + \left(\frac{df}{d\lambda}\right)^2\right] + \frac{d\theta}{d\lambda}\frac{df}{d\lambda} + \frac{K}{\theta} = 0\,. \tag{29}$$

Assuming

$$\left.\frac{d^2f/d\lambda^2}{(df/d\lambda)^2}\right|_H = 0\,, \tag{30}$$

then Eq. (29) is simply a quadratic equation for $df/d\lambda$, which can be solved to give

$$\frac{df}{d\lambda} \approx \frac{1}{\theta}\left(-\frac{1}{2}\frac{d\theta}{d\lambda} \pm iK^{1/2}\right)\,. \tag{31}$$

From Eq. (31) we get

$$\frac{d^2f/d\lambda^2}{(df/d\lambda)^2} \approx \pm\frac{i\,d\theta/d\lambda}{K^{1/2}} + \frac{1}{2}\frac{\theta d^2\theta/d\lambda^2 - (d\theta/d\lambda)^2}{K}\,, \tag{32}$$

which vanishes at the horizon, justifying the assumption of Eq. (30). So we find in the second case that the two linearly independent solutions of Eq. (24) have the following approximate form near the horizon,

$$\phi_\pm = \frac{\text{const}}{\theta^{1/2}} \times \exp\left(\pm iK^{1/2}\int d\lambda/\theta\right)\,. \tag{33}$$

Both solutions are singular at the horizon, again giving a contradiction with our initial assumptions. The conclusion of this somewhat lengthy analysis is that $A_t\,|_H \neq 0$ is not allowed, and thus by our earlier arguments, A_t must vanish identically outside the horizon. That is, a black hole cannot support an exterior massive vector-meson field, even when the mass is generated by spontaneous symmetry breaking.

III. THE GOLDSTONE MODEL ["MINIMAL" ($\xi=0$) CASE]

With $A_t \equiv 0$, Eqs. (1)–(12) of Sec. II describe the Goldstone model of a self-interacting scalar field, as generalized to curved space-time. We will now show that for this model in the "minimal" ($\xi = 0$) case, a further "no-hair" theorem can be proved, stating that $\phi \equiv \phi_\infty$ for all $r \geq r_H$. That is, outside the horizon the scalar field reduces to an unobservable constant, and [cf. Eq. (10a)] the scalar field stress-energy tensor vanishes identically. Our argument does not apply to the "conformal" ($\xi = \frac{1}{6}$) case, where the scalar field stress-energy tensor has a considerably more complicated structure than in the "minimal" case.

The argument proceeds from the scalar field equation, which with $A_t \equiv 0$ takes the form

$$\begin{aligned} &(p^{1/2}qr^2\phi')' = r^2p^{-1/2}\tfrac{1}{2}V(\phi)\,, \\ &V(\phi) = h(\phi^2 - \phi_\infty^{\,2})^2\,, \end{aligned} \tag{34}$$

and from the Einstein equations, which with $A_t \equiv 0$ may be rearranged to give

$$\begin{aligned} &p' = -16\pi r p(\phi')^2\,, \\ &(p^{1/2}qr)' = p^{-1/2} - 8\pi r^2 p^{-1/2}V(\phi)\,. \end{aligned} \tag{35}$$

Multiplying Eq. (34) by ϕ' and integrating from r_H to ∞ gives, after use of Eq. (35) and an integration by parts,

$$\begin{aligned} 0 = \int_{r_H}^{\infty} dr\,[&\tfrac{1}{2}(\phi')^2p^{1/2}qr + \tfrac{1}{2}(\phi')^2p^{-1/2}r \\ &+ rp^{-1/2}V(\phi)] \\ &+ \tfrac{1}{2}[r^2p^{-1/2}V(\phi)]_H - [p^{1/2}qr^2\tfrac{1}{2}(\phi')^2]_H\,. \end{aligned} \tag{36}$$

Since all terms in Eq. (36) are non-negative except for the final one, we see that if $[p^{1/2}q(\phi')^2]_H$

$=0$, then we can conclude that $\phi\equiv\phi_\infty$ for $r\geq r_H$, and the desired "no-hair" theorem follows.

To complete the proof, we must exclude the possibility $[p^{1/2}q(\phi')^2]_H\neq 0$. Just as in the preceding section, this is done by a local analysis in the vicinity of the horizon. We begin by noting that since $dq/d\lambda\,|_H\geq 0$, Eq. (20a) implies

$$G^{\hat r\hat r}+\frac{1}{r^2}=\text{bounded}$$
$$=\frac{p^{1/2}}{r}\left(\frac{dq}{d\lambda}+p^{1/2}\frac{q}{r}\right)\geq\frac{1}{r^2}\frac{1}{q}(p^{1/2}q)^2$$
$$\Rightarrow q\times\text{bounded}\geq(p^{1/2}q)^2$$
$$\Rightarrow p^{1/4}q^{1/2}|_H=0\,. \tag{37}$$

Furthermore, since $T^{\hat r\hat r}\,|_H$ is bounded, and since both terms in $T^{\hat r\hat r}$ are positive semidefinite, we have that $p^{1/2}q^{1/2}\phi'\,|_H$ is bounded. Since the first equation in Eq. (35) implies that $dp/d(-r)\geq 0$, and since $p(\infty)=1$, we have $p\geq 1$, which puts the boundedness of $p^{1/2}q^{1/2}\phi'$ into the form

$$p^{1/4}q^{1/2}\phi'|_H=\frac{p^{1/2}q^{1/2}\phi'\,|_H}{p^{1/4}\,|_H}=\text{bounded}\,. \tag{38}$$

Suppose now that $p^{1/4}q^{1/2}\phi'\,|_H=K\neq 0$. Then

$$\phi\,|_H=\text{bounded}\Rightarrow\int_{r_H}dr\frac{d\phi}{dr}=\text{bounded}$$
$$\Rightarrow\int_{r_H}\frac{dr}{a(r)}=\text{convergent}\,, \tag{39}$$

with

$$a(r)\equiv p^{1/4}q^{1/2}r^2,\quad a|_H=0\,. \tag{40}$$

But on the other hand, the differential equation for ϕ in Eq. (34) gives

$$(p^{1/2}qr^2\phi')'|_H=\text{bounded}, \tag{41}$$

which on substituting $\phi'\approx K/(p^{1/4}q^{1/2})$ gives

$$a'|_H=\text{bounded}$$
$$\Rightarrow\lim_{r\to r_H}\frac{a(r)}{r-r_H}=\text{bounded}$$
$$\Rightarrow\int_{r_H}\frac{dr}{a(r)}=\text{divergent}\,, \tag{42}$$

in contradiction with Eq. (39). Hence we must have $K=0$, which completes the proof.

IV. AN ABELIAN HIGGS ANALOG MODEL IN MINKOWSKI SPACE-TIME

As our final topic we briefly investigate a Minkowski space-time analog of the Abelian Higgs model analyzed in Sec. II. We consider a sphere of radius r_H, impenetrable to the Higgs field, and carrying charge Q, surrounded by the Higgs scalar medium. The differential equations and boundary conditions describing the time-independent behavior of this system are

$$(r^2A_t')'=r^2 2e^2A_t\phi^2\,,$$
$$(r^2\phi')'=r^2[-e^2A_t^{\,2}\phi+2h\phi(\phi^2-\phi_\infty^{\,2})]\,, \tag{43}$$
$$\phi(r_H)=0,\quad A_t'(r_H)=-\frac{Q}{r_H^{\,2}}\,,$$

which apart from the absence of the metric factors e^α, e^β have essentially the same structure as the system of equations analyzed in Sec. II. However, unlike the situation found in the general relativistic case, the Minkowski model of Eq. (43) has a nontrivial screened-charge solution.[9] To prove this, we consider the energy functional

$$E(r_H,Q)=4\pi\int_{r_H}^\infty r^2dr[\tfrac{1}{2}(A_t')^2+e^2A_t^{\,2}\phi^2+h(\phi^2-\phi_\infty^{\,2})^2] \tag{44}$$

and use the differential equation for A_t (the charge conservation constraint equation) and its associated boundary condition to write

$$A_t'=\frac{1}{r^2}\left[\int_{r_H}^r dr'r'^2 2e^2A_t(r')\phi^2(r')-Q\right]\,, \tag{45}$$

which when substituted into Eq. (45) gives the new functional,

$$*E(r_H,Q)=4\pi\int_{r_H}^\infty r^2dr\left\{\frac{1}{2r^4}\left[\int_{r_H}^r dr'r'^2 2e^2A_t(r')\phi^2(r')-Q\right]^2+e^2A_t^{\,2}\phi^2+h(\phi^2-\phi_\infty^{\,2})^2\right\}\,. \tag{46}$$

Extremizing $*E$ with respect to variations in A_t and ϕ [with an endpoint condition $\delta\phi(r_H)=0$] is easily verified to lead to the differential equations of Eq. (43). Hence these equations describe the field configuration which minimizes the field energy, subject to the constraint that the inaccessible region $r\leq r_H$ contains total charge Q.

Since the functional $*E$ is positive semidefinite, and since there is a nonempty class of functions A_t, ϕ for which $*E$ is bounded from above, functions A_t, ϕ which minimize $*E$ must exist, and thus the coupled equations in Eq. (43) have a solution.[10] Near $r=\infty$, the solution has the behavior

$$\phi \approx \phi_\infty$$
$$A_t \approx r^{-1}\exp[-r/(2e^2\phi_\infty^{\,2})^{1/2}], \tag{47}$$

and as expected, the Higgs mechanism results in screening of the charge Q from view at infinity. The conclusion from this analysis is that the absence of screened-charge black-hole solutions in the general-relativistic case is a result of the stringent conditions for the existence of a horizon, not of the interacting field or spontaneous-symmetry-breaking aspects of the problem.

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[3]For a pedagogical review and references, see J. Bernstein, Rev. Mod. Phys. 46, 7 (1974).

[4]See, e.g., C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 14.

[5]Note that A_t is the time component on a coordinate basis.

[6]In the massless vector case, an exterior vector field is present precisely because the possibility $A_t|_H \neq 0$ can be realized.

[7]At this point in his analysis of the charged scalar-meson case, Bekenstein [J. D. Bekenstein, Phys. Rev. D 5, 1239 (1972)] introduces the assumption that the "charge per meson" which he defines by $(-j^\mu j_\mu)^{1/2}/\phi^2 \propto (A_t\phi^2 A^t\phi^2)^{1/2}/\phi^2 \propto e^{-\alpha}A_t$ is bounded at the horizon, which would imply the vanishing of $A_t|_H$ with no further detailed analysis. However, it is not clear to us that the requirement of boundedness at the horizon should apply to physical scalars formed as the *quotients* of other scalars, when the denominator is a physical quantity (such as ϕ) which can develop nodes. In our analysis, we only assume boundedness at the horizon of $F_{\mu\nu}F^{\mu\nu}$ and of $G_{\mu\nu}G^{\mu\nu}=(8\pi)^2T_{\mu\nu}T^{\mu\nu}$. Since $G^{\hat\mu\hat\nu}$ is diagonal, boundedness of $G_{\mu\nu}G^{\mu\nu}$ implies that all components of $G^{\hat\mu\hat\nu}$ are individually bounded at the horizon.

[8]See C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Ref. 4), Chap. 25.

[9]Our variational argument for existence of a screened charge solution applies when the boundary condition $\phi(r_H)=0$ is generalized to $\phi(r_H)=\phi_H$, with ϕ_H any specified constant.

[10]The proof does not extend to the limit $r_H\to 0$ because a point charge has infinite Coulomb self-energy, as a result of which the functional $*E$ is not bounded from above. In the point charge case, an analysis of the indicial equation for ϕ around $r=0$ suggests that, for $(eQ)^2<\frac{1}{4}$, there may be a solution which would behave as $\phi\sim\hat\phi r^\lambda$, $\lambda=-\frac{1}{2}+[\frac{1}{4}-(eQ)^2]^{1/2}$ near $r=0$, and which joins on to an exponentially decreasing asymptotic solution at $r=\infty$. However, we do not have an existence proof in this case.