Einstein-Yang-Mills pseudoparticles and electric charge quantization

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Solutions of the Euclidean-space Einstein-Yang-Mills equations are presented which describe a dyon black hole of mass M, angular momentum J, electric charge Q_e, and magnetic charge Q_m. In addition to the usual magnetic-charge quantization rule $eQ_m = m$ (m = integer), the space-time topology of the Euclidean black hole, in particular the periodicity in the time coordinate, leads to a new *electric*-charge quantization $eQ_e = n(1 - r_-/r_+)$, where n = integer and $r_\pm = M \pm [M^2 + J^2M^{-2} + 4\pi(Q_e^2 - Q_m^2)]^{1/2}$. Regarded as a (non-self-dual) Einstein-Yang-Mills pseudoparticle, the solution has finite action, a gravitational Euler number $\chi = 2$, and an SU(2) Pontryagin number $P = 2mn$.

I. INTRODUCTION

In two previous publications,[1,2] finite-action pseudoparticle[3,4] solutions to the coupled Einstein-Yang-Mills field equations were presented which were characterized by a nonvanishing gravitational Euler number, χ, and an SU(2) Pontryagin number, P. Having no flat-space analog, these solutions served to illustrate how different families of Yang-Mills pseudoparticles exist in space-times of different topology.

In Refs. 1 and 2, attention was focused mainly on solutions for which the Yang-Mills field strengths were self-dual and for which, therefore, the energy-momentum tensor vanished identically. The problem of solving the coupled Einstein-Yang-Mills field equations was thus reduced to first solving the vacuum Einstein equations and then finding field strengths which were self-dual in the given geometry. In particular, one could take the geometry to be that of a Schwarzschild black hole continued to Euclidean space in the way described by Hawking[4] and by Gibbons and Hawking.[5] Two such solutions were provided. The first had Pontryagin number $P = \pm 1$ and the second, with $P = \pm 2n^2$ (n = integer), corresponded to a dyon with equal or opposite electric and magnetic charges.

It was also pointed out, however, that there exist non-self-dual pseudoparticles of the dyon type, with nonvanishing stress tensor, in the Reissner-Nordström (or more generally Kerr-Newman) geometry. The purpose of the present paper is to examine these solutions in greater detail with particular reference to the electric- and magnetic-charge quantization rules.

Our results may be summarized as follows. The Einstein-Yang-Mills field equations admit real solutions in Euclidean space (i.e., a Riemannian space with a metric signature +4) with parameters M, J, Q_e, and Q_m which we refer to by their "Minkowski" titles of mass, angular momentum, electric charge, and magnetic charge of the dyon black hole. The finite total action is given, in natural units $\hbar = c = G = 1$, by

$$S = \frac{2\pi}{r_+ - r_-}\left[M(r_+^2 - J^2M^{-2}) + 4\pi(Q_e^2 + Q_m^2)r_+\right], \tag{1.1}$$

where

$$r_\pm = M \pm [M^2 + J^2M^{-2} + 4\pi(Q_e^2 - Q_m^2)]^{1/2}. \tag{1.2}$$

[The reader may recognize Eq. (1.2) as describing the location of the event horizon but, as discussed in Sec. II, J^2 and Q_e^2 enter in Euclidean space with the opposite sign to Minkowski space.] The magnetic and electric charges are quantized according to

$$eQ_m = m \tag{1.3}$$

and

$$eQ_e = n(1 - r_-/r_+), \tag{1.4}$$

where m and n are integers and e is the Yang-Mills coupling strength. We note from Eq. (1.2) that the electric-charge quantization rule (1.4) is highly nonlinear and involves both M and J. The solution carries a nonvanishing Euler number

$$\chi = 2, \tag{1.5}$$

and an SU(2) Pontryagin number

$$P = 2e^2Q_eQ_m\frac{r_+}{r_+ - r_-} = 2mn = \text{even integer}. \tag{1.6}$$

The paper is divided into five sections. Before embarking on the Yang-Mills problem and before discussing the effects of rotating black holes, we consider in Sec. II the simpler case of spherically symmetric ($J = 0$) solutions of the field equations for an Abelian U(1) gauge field coupled to gravity. In other words, we consider the Reissner-Nordström solution of Einstein-Maxwell equations (generalized to include magnetic charge) but paying particular attention to the unfamiliar features of Euclidean space. Although dyons and magnetic monopoles in curved space have been considered previously by a number of authors,[6] our Euclidean approach will be very different. By transforming to Kruskal-type coordinates, one is able to see how the real Reissner-Nordström black-hole manifold is everywhere free of singularities (including the curvature singularity at $r=0$ encountered in Minkowski space) provided the Euclidean time coordinate is periodic. This periodicity means that the dyon potentials exhibit not only the familiar Dirac magnetic string singularity but also an "electric string" singularity. Consequently, both the magnetic *and* the electric charges of the dyon obey a Dirac quantization rule.

In Sec. III, we convert these Abelian solutions into solutions of the SU(2) Yang-Mills-Einstein equations. Although we are concerned with four-dimensional pseudoparticles, the time independence of these solutions means that we make contact with the work on three-dimensional solitons. Since there are no Higgs fields present, the resulting solutions are akin to the Wu-Yang monopole[7] rather than 't Hooft-Polyakov monopole[8] or Julia-Zee dyon.[9] Indeed, one of the remarkable features of the Euclidean Reissner-Nordström topology is that the singularity at $r=0$ normally encountered in the Wu-Yang solutions is avoided without the need for Higgs fields. And in sharp contrast to the Julia-Zee dyon, the *electric* as well as the magnetic charge is again quantized. In this non-Abelian case, the quantization rules are of the Schwinger type, rather than the Dirac type. In this section we also discuss the topological invariant of the SU(2) fields, relating it to the electric- and magnetic-charge quantum numbers of the U(1) fields, and discussing its effect as an obstruction to the removal of the electric and magnetic string singularities.

In Sec. IV we compute the total action of the combined gravitational and gauge field system and show it to be finite as a consequence of the space-time topology. The quantization laws (1.3) and (1.4) allow us to eliminate the dependence on Q_m and Q_e in favor of the integers m and n; and this involves us in a closer examination of the nonlinear nature of the quantization law for the electric charge.

Finally, in Sec. V, after examining the effects of endowing the black hole with angular momentum J, we touch briefly on some further questions of topological interest. These include (a) an understanding of Eq. (1.6) in terms of the Whitney sum formula,[10] (b) how our pseudoparticles exhibit "charge without charge" in the Wheeler[11] sense, (c) the Euclidean version of some "no hair" theorems,[11] (d) the existence of more general solutions requiring, in the self-dual limit at least, not less than $8P$-6 parameters to specify the gauge fields (as distinct from $8P$-3 in flat space), and (e) the problems associated with the physical interpretation of the solutions, in particular how the periodicity in the time obscures the usual vacuum-tunneling interpretation and favors a viewpoint of Yang-Mills pseudoparticles at finite temperature.

II. U(1) GAUGE FIELDS

Consider the coupled Einstein-Maxwell field equations in Euclidean space:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}\,, \tag{2.1}$$

where

$$T_{\mu\nu} = f_{\mu\rho} f_\nu{}^\rho - \tfrac{1}{4} g_{\mu\nu} f^{\rho\sigma} f_{\rho\sigma}\,, \tag{2.2}$$

and

$$\nabla^\mu f_{\mu\nu} = 0\,, \tag{2.3}$$

where

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu\,. \tag{2.4}$$

In coordinates $x^\mu = (\tau, r, \theta, \phi)$, the most general spherically symmetric asymptotically flat solution is given by the Reissner-Nordström line element

$$ds^2 = \frac{(r-r_+)(r-r_-)}{r^2}\, d\tau^2 + \frac{r^2}{(r-r_+)(r-r_-)}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{2.5}$$

where

$$r_\pm = M \pm [M^2 + 4\pi(Q_e^2 - Q_m^2)]^{1/2}\,, \tag{2.6}$$

and by the dyon potential

$$a \equiv a_\mu dx^\mu = Q_e\left(\frac{1}{r} - \frac{k_e}{r_+}\right) d\tau + Q_m(\cos\theta + k_m)d\phi\,. \tag{2.7}$$

The quantities M, Q_e, and Q_m are the dyon's mass, electric charge, and magnetic charge, respectively, while k_e, and k_m are arbitrary constants. The Maxwell field strength,

$$f \equiv \tfrac{1}{2} f_{\mu\nu}\, dx^\mu \wedge dx^\nu\,, \tag{2.8}$$

is given by

$$f = da = \frac{Q_e}{r^2}\, d\tau\wedge dr - Q_m \sin\theta\, d\theta\wedge d\phi\,. \tag{2.9}$$

We note that Q_e^2 enters the line element with the opposite sign to Minkowski space. This is because the imaginary-time replacement $t \to \tau = it$ must be accompanied by $a_t \to a_\tau = -i a_t$ in order to obtain a real potential. One consequence of this is that the field strength becomes self-dual:

$$f=\pm{}^*f\,,\quad {}^*f_{\mu\nu}\equiv\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}f^{\rho\sigma}\,, \tag{2.10}$$

when

$$Q_e=\mp Q_m\,. \tag{2.11}$$

In this case the line element (2.5) reduces to the Schwarzschild solution with $r_+=2M$ and $r_-=0$, since the stress tensor of Eq. (2.2) vanishes when $f=\pm{}^*f$.

Similarly, in the case of the Kerr solution, the squared angular momentum J^2 would enter the real Euclidean line element with a sign opposite to the Minkowski metric. In this respect, our approach differs from that of Gibbons and Hawking,[5] who start with real Minkowski solutions and perform only the Wick rotation, thereby obtaining a complex potential when $Q_e\neq 0$ and a complex metric when $J\neq 0$. Here we are seeking real solutions to the field equations directly in Euclidean space in the spirit of Belavin, Polyakov, Schwartz, and Tyupkin.[3] The criterion for the existence of a black hole, rather than a naked singularity, is that $r_\pm$ be real. In Euclidean space this now becomes

$$M^2+J^2M^{-2}+4\pi Q_e^{\,2}>4\pi Q_m^{\,2}\,, \tag{2.12}$$

and hence r_- obeys the inequality

$$-r_+<r_-<r_+\,. \tag{2.13}$$

For the moment we shall confine our attention to the case $J=0$. Rotating black holes are mentioned in Sec. V.

Our difference from Gibbons and Hawking, however, and our inclusion of a magnetic charge, do not prevent us from repeating their analysis and thereby finding solutions for which the metric is positive-definite and everywhere free of singularities, but for which both the potential and the metric are everywhere real. To make this explicit, we transform to the Kruskal-type coordinates $x^\mu=(\xi,\eta,\theta,\phi)$ since in (τ,r) coordinates it is not yet apparent that the Reissner-Nordström solution (2.5) satisfies all these criteria. We define

$$\xi=\exp(\kappa r)(\kappa r-\kappa r_+)^{1/2}(\kappa r-\kappa r_-)^{-r_-^{\,2}/2r_+^{\,2}}\sin\kappa\tau\,,$$
$$\eta=\exp(\kappa r)(\kappa r-\kappa r_+)^{1/2}(\kappa r-\kappa r_-)^{-r_-^{\,2}/2r_+^{\,2}}\cos\kappa\tau\,. \tag{2.14}$$

where the quantity κ, usually referred to as the "surface gravity," is given by

$$\kappa=\frac{r_+-r_-}{2r_+^{\,2}}\,. \tag{2.15}$$

The line element (2.5) now becomes

$$ds^2=\frac{(\kappa r-\kappa r_-)^{1+r_-^{\,2}/r_+^{\,2}}}{\kappa^4r^2}\exp(-2\kappa r)(d\xi^2+d\eta^2)$$
$$+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\,. \tag{2.16}$$

From (2.14), it follows that $\mathfrak{M}$, the region $r\geq r_+$, is a complete smooth manifold without singularities provided τ is periodic with period $2\pi\kappa^{-1}$. Thus τ behaves like an angular coordinate about the "axis" $r=r_+$ $(\eta=\xi=0)$. The topology of $\mathfrak{M}$ is $R^2\times S^2$, and the Euler-Poincaré characteristic of the manifold is given by

$$\chi[\mathfrak{M}]=\frac{1}{32\pi^2}\int_{\mathfrak{M}}d^4x\sqrt{g}\;{}^*R^*_{\alpha\beta\lambda\mu}R^{\alpha\beta\lambda\mu}$$
$$=2\,. \tag{2.17}$$

The important point is that we have succeeded in removing not only the coordinate singularities at $r=r_\pm$, but also the curvature singularity at $r=0$ encountered in Minkowski space. This will have far-reaching consequences not only for the gravitational field itself but also for the gauge fields defined on $\mathfrak{M}$. In particular, the field strengths (though not the potentials) will be everywhere nonsingular. [We note that the metric (2.16) is also real, but singular, in the region $r\leq r_-$. However, this region is disjoint from $r\geq r_+$ and we need not consider it further; it can have no influence on the functions or fields defined on $\mathfrak{M}$.]

From Eq. (2.9), it follows that the scalar

$$g^{\mu\rho}g^{\nu\sigma}f_{\mu\nu}f_{\rho\sigma}=\frac{2(Q_e^{\,2}+Q_m^{\,2})}{r^4} \tag{2.18}$$

tends to zero as $r\to\infty$. This boundary condition means that, as far as the dyon solutions are concerned, the (ξ,η) plane is equivalent to the sphere S^2, and we shall shortly discuss the implications of this for the dyon potentials.

Before doing so, however, it will prove instructive to introduce a third set of coordinates $x^\mu=(\theta_e,\phi_e,\theta_m,\phi_m)$ defined by

$$\frac{1-\cos\theta_e}{1+\cos\theta_e}=\xi^2+\eta^2\,,\quad \phi_e=\kappa\tau=\tan^{-1}\frac{\xi}{\eta}\,,$$
$$\theta_m=\theta\,,\quad \phi_m=\phi\,, \tag{2.19}$$

so that $\theta_e=0$ corresponds to $r=r_+$ and $\theta_e=\pi$ to $r=\infty$ and such that ϕ_e has period 2π. From Eq. (2.16), it follows that the induced metric on the surface with constant θ and ϕ is conformal to the standard metric on the (θ_e,ϕ_e) sphere $S_e^{\,2}$:

$$d\theta_e^{\,2}+\sin^2\theta_e\,d\phi_e^{\,2}\,,$$

just as the induced metric on the surface with constant ξ and η is conformal to the standard metric on the (θ_m,ϕ_m) sphere $S_m^{\,2}$:

$$d\theta_m^2 + \sin^2\theta_m d\phi_m^2 ,$$

but that the conformal factors are different in each case. (Were they the same, $\mathfrak{M}$ would be conformally equivalent to $S_e^2 \times S_m^2$.)

Now consider the dyon potential (2.7). For arbitrary k_m is exhibits a magnetic string singularity along the entire z axis (the Dirac string) since

$$d\phi_m = d\phi = \frac{1}{x^2+y^2}(x\,dy - y\,dx) . \tag{2.20}$$

The novel feature of Euclidean space is that, for arbitrary k_e, the potential also exhibits an "electric string" along the "axis" $r = r_+$ ($\xi = \eta = 0$) since

$$d\phi_e = \kappa d\tau = \frac{1}{\xi^2+\eta^2}(\eta d\xi - \xi d\eta) . \tag{2.21}$$

Nor does the potential vanish as $r=\infty$. We refer to the vanishing of a as r becomes large as "regularity at infinity" (this statement can be made in a coordinate-independent fashion, i.e., $g^{\mu\nu}a_\mu a_\nu$ is not regular at infinity). In summary, the potential ceases to be regular at

$$\theta_m = 0 , \quad \theta_m = \pi ,$$

$$\theta_e = 0 \ \ (r = r_+) , \quad \theta_e = \pi \ \ (r = \infty) .$$

Although there exists no choice of gauge for which the potential is everywhere regular, it is nevertheless possible to cover the manifold $\mathfrak{M}$ with four overlapping regions such that there exists a regular potential in each region and such that in the intersection of any two regions the corresponding potentials are connected by a gauge transformation. To see this define the regions

1. $\theta_m > 0 , \quad \theta_e > 0 \ \ (r > r_+)$,
2. $\theta_m < \pi , \quad \theta_e > 0$,
3. $\theta_m > 0 , \quad \theta_e < \pi \ \ (r < \infty)$,
4. $\theta_m < \pi , \quad \theta_e < \pi$,

and the corresponding gauges

(1) $k_e = 0 , \quad k_m = +1$,

(2) $k_e = 0 , \quad k_m = -1$,

(3) $k_e = 1 , \quad k_m = +1$,

(4) $k_e = 1 , \quad k_m = -1$,

In the gauges (1), (2), (3), and (4) the potential is regular in the regions 1, 2, 3, and 4, respectively. As an example, consider the gauge transformation which connects 1 and 2:

$$a_1 = a_2 + 2Q_m d\phi . \tag{2.22}$$

In the presence of a spinor field ψ with charge e transforming under (2.22) as

$$\psi_1 = \Omega_{12}\psi_2 , \quad \Omega_{12} = \exp(ieQ_m\phi) , \tag{2.23}$$

the requirement that Ω be single-valued,

$$\Omega(\phi) = \Omega(\phi + 2\pi) , \tag{2.24}$$

leads to the well-known Dirac quantization condition

$$2eQ_m = \text{integer} . \tag{2.25}$$

In an analogous fashion, consider regions 1 and 3; then

$$a_1 = a_3 + \frac{Q_e}{r_+} d\tau , \tag{2.26}$$

and

$$\Omega_{13} = \exp(ieQ_e\tau/r_+) . \tag{2.27}$$

Periodicity in τ,

$$\Omega(\tau) = \Omega(\tau + 2\pi\kappa^{-1}) , \tag{2.28}$$

leads to a new *electric*-charge quantization rule:

$$\frac{eQ_e}{\kappa r_+} = \text{integer} . \tag{2.29}$$

The two integers in Eqs. (2.25) and (2.29) are in general distinct and coincide only in the self-dual limit $Q_e = Q_m$ for which $\kappa r_+ = \frac{1}{2}$. These Dirac rules, valid in the presence of spinors transforming according to the U(1) gauge group, are not the ones with which we are primarily concerned. Our purpose in discussing Abelian gauge fields was to pave the way for the treatment of pure non-Abelian gauge fields, to which we now turn.

III. SU(2) GAUGE FIELDS

In this section we shall consider the SU(2) Yang-Mills potentials

$$A = A_\mu dx^\mu , \quad A_\mu = A_\mu^i T^i (i = 1, 2, 3) , \tag{3.1}$$

where

$$[T^i, T^j] = \epsilon^{ijk} T^k , \quad T^i = \frac{\sigma^i}{2i} , \tag{3.2}$$

and the corresponding field strengths

$$F = dA + A\wedge A = \tfrac{1}{2} F^i_{\mu\nu} T^i dx^\mu \wedge dx^\nu . \tag{3.3}$$

We have absorbed the Yang-Mills coupling constant, e, into the definition of the potentials, so that the action takes the form

$$S_{\text{YM}} = \frac{-1}{e^2}\,\text{Tr}\int F\wedge {}^*F . \tag{3.4}$$

We may repeat in curved space a procedure well known in flat space. A solution of the coupled Einstein-Yang-Mills field equations ($g_{\mu\nu}, A$) may be obtained from a solution of the Einstein-Maxwell equations ($g_{\mu\nu}, a$) by keeping $g_{\mu\nu}$ unchanged but setting

$$A = eaT^3, \tag{3.5}$$

so that

$$F = efT^3. \tag{3.6}$$

The solution so obtained will, of course, exhibit the same singularities as those of the Maxwell field. In particular, it will possess electric and magnetic string singularities when the Maxwell field it taken to be the dyon potential discussed in Sec. II. Now, however, the SU(2) gauge transformations which connect the different regions, e.g., $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$, are [cf. Eqs. (2.23) and (2.27)]

$$\Omega_{12} = \exp(-ieQ_m\phi\sigma_3) \tag{3.7}$$

and

$$\Omega_{13} = \exp(-i\,eQ_e\tau\sigma_3/2r_+). \tag{3.8}$$

Therefore the Ω's will not be smooth functions with values in SU(2) unless

$$eQ_m = m = \text{integer} \tag{3.9}$$

and

$$\frac{eQ_e}{2\kappa r_+} = n = \text{integer}. \tag{3.10}$$

These quantization rules are of the Schwinger type rather than Dirac type. Hence only half the Maxwell solutions provide SU(2) solutions. [Note that we have identified the Yang-Mills coupling constant with the Abelian charge e of Sec. II. As pointed out by 't Hooft, however, the group SU(2) admits isospinor fields with an elementary charge equal to one-half the Yang-Mills coupling strength. In this case the rules (3.9) and (3.10) would deserve the Dirac title.]

The gauge fields carry a topological number (the second Chern number) often referred to as the Pontryagin number, given by

$$P = -\frac{-1}{8\pi^2}\int_{\mathfrak{M}} (\mathrm{Tr}F\wedge F - \mathrm{Tr}F\wedge\mathrm{Tr}F) \tag{3.11}$$

$$= \frac{e^2}{16\pi^2}\int_{\mathfrak{M}} f\wedge f, \tag{3.12}$$

on using Eq. (3.6). [Usually the term $\mathrm{Tr}F\wedge\mathrm{Tr}F$ is omitted since it vanishes identically in the case of SU(2). Here we exhibit it explicitly in order to emphasize that P itself would vanish for a strictly Abelian field.] Substituting for f the dyon field strength of Eq. (2.9), and remembering the range of integration $r_+ < r < \infty$ and $0 < \tau < 2\pi\kappa^{-1}$, we find from Eq. (3.12) that

$$P = \frac{e^2 Q_e Q_m}{\kappa r_+}, \tag{3.13}$$

$$P = 2mn, \tag{3.14}$$

on invoking the quantization rules (3.9) and (3.10). Hence, the Pontryagin number of the dyon solutions is given by twice the product of the electric- and magnetic-charge quantum numbers of U(1) dyon and is thus an even integer. (An Einstein-Yang-Mills pseudoparticle with odd P is discussed Refs. 1 and 2.) This may be understood as follows. For a potential of the form (3.5) we can decompose the integration over $\mathfrak{M}$, which is effectively $S_e^2 \times S_m^2$, into the product form

$$P = 2\left(\frac{e}{4\pi}\int_{S_e^2} f\right)\left(\frac{e}{4\pi}\int_{S_m^2} f\right). \tag{3.15}$$

But

$$\frac{e}{4\pi}\int_{S_e^2} f = \frac{e}{4\pi}\int_0^{2\pi\kappa^{-1}} d\tau \int_{r_+}^{\infty} dr\,\frac{Q_e}{r^2} = \frac{eQ_e}{2\kappa r_+} = n \tag{3.16}$$

and

$$\frac{e}{4\pi}\int_{S_m^2} f = \frac{e}{4\pi}\int_0^{\pi} d\theta\,\sin\theta\int_0^{2\pi} d\phi\,Q_m = eQ_m = m. \tag{3.17}$$

Therefore, we have (3.14). The nonvanishing of P, that is to say, the nontrivial topology of the SU(2) field, means that the singularities in the solution which are manifest in the gauge of Eq. (3.5) cannot all be removed by means of an SU(2) gauge transformation even if this transformation is itself singular. However, in the case where the solution is of the pure electric type ($Q_m = 0$), or the pure magnetic type ($Q_e = 0$), it ought to be possible to remove the singularities completely since P vanishes. We shall now demonstrate that this is indeed the case.

The removal of the Dirac string is by now familiar,[8,12] but we repeat it here to aid the understanding of the electric string. Consider the solution with $Q_e = 0$ and $k_m = 1$ so that the magnetic singularity is situated along the positive z axis. In the notation of Eq. (2.19) we have

$$A = eaT^3 = m(\cos\theta_m + 1)d\phi_m T^3 \tag{3.18}$$

and

$$F = efT^3. \tag{3.19}$$

Next we make the gauge transformation

$$A' = \Omega A\Omega^{-1} + \Omega\, d\Omega^{-1}, \quad \Omega^{+}\Omega = 1, \tag{3.20}$$

under which

$$F' = \Omega F\Omega^{-1}, \tag{3.21}$$

with

$$\Omega(\alpha,\beta)=\begin{pmatrix}\cos\frac{1}{2}\alpha e^{-i\beta} & -\sin\frac{1}{2}\alpha\\ \sin\frac{1}{2}\alpha & \cos\frac{1}{2}\alpha e^{i\beta}\end{pmatrix}. \tag{3.22}$$

Then

$$A'=[m(\cos\theta_m+1)d\phi_m-(\cos\alpha+1)d\beta]\,\hat{\phi}-[\hat{\phi},d\hat{\phi}] \tag{3.23}$$

and

$$F=ef\,\hat{\phi}, \tag{3.24}$$

where

$$\hat{\phi}=\hat{\phi}^i T^i, \tag{3.25}$$

and $\hat{\phi}^i$ is the unit vector

$$\hat{\phi}^i=(\sin\alpha\cos\beta,\sin\alpha\sin\beta,\cos\alpha). \tag{3.26}$$

The choice

$$\alpha=\theta_m,\quad \beta=m\phi_m, \tag{3.27}$$

will not only remove the string at $\theta_m=0$ but will remove it completely since $[\hat{\phi},d\hat{\phi}]$ will be everywhere regular including $\theta_m=\pi$.

We now repeat the analysis with $Q_m=0$ and $k_e=0$ so that the potential is singular at $r=r_+$, i.e., at $\theta_e=0$ in the notation of Eq. (2.19). We have

$$A=\frac{eQ_e}{r_+}d\tau T^3=2n\frac{r_+}{r}d\phi_e T^3. \tag{3.28}$$

Under the transformation (3.22),

$$A'=\left[2n\frac{r_+}{r}d\phi_e-(\cos\alpha+1)d\beta\right]\hat{\phi}-[\hat{\phi},d\hat{\phi}]. \tag{3.29}$$

The choice

$$\alpha=\theta_e,\quad \beta=n\phi_e \tag{3.30}$$

will remove the singularity since $\cos\theta_e+1\to 2$ as $r\to r_+$. Moreover, $[\hat{\phi},d\hat{\phi}]$ will be everywhere regular including $\theta_e=\pi$. In other words, the potential will not only be regular at $r=r_+$, but will also vanish as $r\to\infty$. Note that the term in (3.29) along the direction $\hat{\phi}(\theta_e\phi_e)$, though regular, does not vanish everywhere in contrast to the term in (3.23) along the direction $\hat{\phi}(\theta_m,\phi_m)$. This merely reflects the fact, mentioned in Sec. II, that although the (ξ,η) plane is conformal to the (θ_e,ϕ_e) sphere, the conformal factor differs from the (θ_m,ϕ_m) sphere, and the topology is really $R^2\times S^2$ rather than $S^2\times S^2$.

In summary, when both Q_e and Q_m are nonzero, it is possible to remove either the magnetic string or the electric string but not both simultaneously. It is, however, possible to obtain Yang-Mills solutions of the dyon type with no singularities at all by enlarging the group to SU(2)× SU(2) and assigning to one SU(2) subgroup a purely electric potential, a_e, and to the other a purely magnetic potential, a_m:

$$A=e\begin{bmatrix}a_eT^3 & 0\\ 0 & a_mT^3\end{bmatrix}. \tag{3.31}$$

There will then exist an SU(2)× SU(2) gauge in which the potential is everywhere regular. Such a topologically trivial solution would have vanishing topological invariants.

IV. FINITENESS OF THE ACTION

In sharp contrast to flat space where the Yang-Mills action (3.4) of the dyon solution would diverge, in the Reissner-Nordström geometry it takes the finite value

$$S_{\rm YM}=4\pi^2(Q_e^2+Q_m^2)/\kappa r_+, \tag{4.1}$$

which is consistent with the inequality

$$S_{\rm YM}\geq\frac{8\pi^2}{e^2}|P|=\frac{8\pi^2}{\kappa r_+}|Q_eQ_m| =\frac{16\pi^2}{e^2}|nm|, \tag{4.2}$$

the equality holding in the self-dual limit $Q_e=\pm Q_m$. As shown by Gibbons and Hawking the gravitational action is also finite, and nonvanishing, even though the curvature scalar vanishes. It is given by

$$S_G=\pi M/\kappa. \tag{4.3}$$

Hence the total action of the Einstein-Yang-Mills system is

$$S=S_G+S_{\rm YM} =\frac{\pi}{\kappa}\left[M+\frac{4\pi(Q_e^2+Q_m^2)}{r_+}\right]. \tag{4.4}$$

(In the absence of a magnetic monopole, $Q_m=0$, S reduces to

$$S=\pi r_+^2, \tag{4.5}$$

which is just one-quarter the area of the event horizon.) The action is the same, of course, whether we regard the Yang-Mills field to be U(1), SU(2), or SU(2)× SU(2), and its dependence on the parameters M, Q_e, and Q_m is reasonably straightforward.

However, if we wish to eliminate Q_e and Q_m in favor of the integers n and m, then the action looks rather complicated because the electric-charge quantization rule is highly nonlinear:

$$eQ_e=n2\kappa r_+=n(1-r_-/r_+),\quad eQ_m=m, \tag{4.6}$$

with $r_\pm$, as given by Eq. (2.6), themselves depending on M, Q_e, and Q_m.

The novelty of this quantization rule justifies a closer examination of its implications. Define the parameter α by

$$eQ_e = n\alpha , \tag{4.7}$$

then

$$\alpha = 1 - r_-/r_+ , \tag{4.8}$$

and from Eqs. (4.6) and (2.6) we see that α takes on discrete values according to the equation

$$\alpha = 1 - \left\{ \frac{1-[1+a^2(n^2\alpha^2 - m^2)]^{1/2}}{1+[1+a^2(n^2\alpha^2-m^2)]^{1/2}} \right\} , \tag{4.9}$$

where the dimensionless parameter a^2 is given by

$$a^2 = \frac{4\pi}{e^2 M^2} = 4\pi \frac{\hbar c}{e^2} \frac{\hbar c}{G} \frac{1}{M^2} , \tag{4.10}$$

upon restoring the units of charge and mass to e and M. Furthermore, α lies in the regions

$$\begin{aligned} &0 < \alpha < 1, \quad Q_e^2 < Q_m^2 , \\ &\alpha = 1 , \quad Q_e^2 = Q_m^2 , \\ &1 < \alpha < 2 , \quad Q_e^2 > Q_m^2 , \end{aligned} \tag{4.11}$$

The actions may now be rewritten in terms of n, m, and α. For the gravitational action, we have

$$S_G = \frac{16\pi^2}{e^2 a^2} \frac{1}{\alpha(2-\alpha)} \tag{4.12}$$

$$= \frac{4\pi^2}{e^2} \frac{(2-\alpha)(n^2\alpha^2 - m^2)}{\alpha(\alpha-1)} , \quad \alpha \neq 1 \tag{4.13}$$

upon using Eq. (4.9), and for the Yang-Mills action

$$S_{\rm YM} = \frac{8\pi^2}{e^2} \frac{n^2\alpha^2 + m^2}{\alpha} . \tag{4.14}$$

Note that the self-dual limit $Q_e^2 = Q_m^2$, for which $\alpha = 1$ and $n^2 = m^2$, is a special case in that α is independent of M. In this case, therefore, the total action becomes

$$S = \frac{16\pi^2}{e^2} \left(\frac{1}{a^2} + n^2 \right) , \quad P = \pm 2n^2 \tag{4.15}$$

while for $\alpha \neq 1$, we have

$$S = \frac{4\pi^2}{e^2} \left[\frac{n^2\alpha^3 + m^2(3\alpha - 4)}{\alpha(\alpha-1)} \right] , \quad P = 2mn \tag{4.16}$$

with α given by Eq. (4.9), an equation which is difficult to solve explicitly except for special values of a^2. (For example, when $a^2 m^2 = 1$, we have a nontrivial solution $\alpha = 2 - n/m$.) Note also that as $M \to 0$, $\alpha \to 2$ and from (4.14)

$$S \to \frac{4\pi^2}{e^2} (4n^2 + m^2) ,$$

which, when $n = m$, differs from the $M \to 0$ limit of Eq. (4.15). Hence the value of the action as we approach flat space (but still with nontrivial topology $R^2 \times S^2$) differs according to which of the two routes we adopt, but remains finite in both cases.

We may also ask, for fixed n and m, what value of the mass minimizes the action (4.16). For a pure electric charge ($m = 0$), the minimum occurs at $\alpha = 2$, i.e., $M = 0$. For a pure magnetic monopole, however, the minimum occurs at $\alpha = \frac{2}{3}$. This corresponds to $r_+ = 3r_-$, i.e., to the nonvanishing mass value

$$M^2 = \frac{16\pi}{3} Q_m^2 = \frac{16\pi m^2}{3} \frac{\hbar c}{e^2} \frac{\hbar c}{G} .$$

In general, the action is minimized by $M = 0$ only when the electric charge exceeds the magnetic charge.

We conclude this section with a brief comment on duality transformations in Euclidean space. The Einstein-Maxwell *field equations* (2.1) to (2.3) are invariant under the transformation

$$f \to f' = f \cosh\gamma + {}^*f \sinh\gamma , \quad \gamma = \text{constant}, \tag{4.17}$$

For our solutions, this means that a dyon with charges

$$\begin{aligned} Q_e' &= Q_e \cosh\gamma + Q_m \sinh\gamma , \quad eQ_e' = n'\alpha , \\ Q_m' &= Q_m \cosh\gamma + Q_e \sinh\gamma , \quad eQ_m' = m' \end{aligned} \tag{4.18}$$

will yield the same geometry as a dyon with charges Q_e and Q_m, since $Q_e^2 - Q_m^2$ (and hence the parameter α) are left invariant. The solutions are topologically distinct, however, since the action and the Pontryagin number of the dyon are not invariant but transform as

$$\begin{aligned} S'_{\rm YM} &= S_{\rm YM} \cosh 2\gamma + \frac{8\pi^2}{e^2} P \sinh 2\gamma , \\ \frac{8\pi^2}{e^2} P' &= \frac{8\pi^2}{e^2} P \cosh 2\gamma + S_{\rm YM} \sinh 2\gamma . \end{aligned} \tag{4.19}$$

S' and P' will again be given by Eq. (4.16), but with the integers n and m replaced by the integers n' and m'.

V. FURTHER ASPECTS

In this final section we address ourselves to some further questions. First of all, we shall summarize the effects of endowing the black hole with an angular momentum J. Again, we may refer to Gibbons and Hawking, but again we must replace their J by iJ in order to obtain a real solution in Euclidean space. As in Sec. II, the regularity of the metric on the horizon means that the radial coordinate is constrained by $r \geq r_+$ where r_+ are now given by

$$r_{\pm}=M\pm[M^2+J^2M^{-2}+4\pi(Q_e^{\,2}-Q_m^{\,2})]^{1/2}. \quad (5.1)$$

The analysis is most easily carried out in a coordinate system rotating with angular velocity

$$\omega=\frac{JM^{-1}}{r_+^{\,2}-J^2M^{-2}}. \quad (5.2)$$

Regularity of the metric on the horizon then requires that we again identify τ with $\tau+2\pi\kappa^{-1}$, and $\tilde{\phi}$ with $\tilde{\phi}+2\pi$, where

$$\tilde{\phi}=\phi+\omega\tau. \quad (5.3)$$

In this frame, the arguments proceed along similar lines to those of Secs. II and III. The quantization laws are

$$eQ_m=m$$

and

$$\frac{eQ_e r_+}{2\kappa(r_+^{\,2}-J^2M^{-2})}=n=\frac{eQ_e r_+}{r_+-r_-}. \quad (5.4)$$

We note that when $J\neq 0$, the self-dual limit $Q_e=\pm Q_m$ no longer implies that $n=\pm m$. Rather, as can be seen from Eqs. (5.1) and (5.4), it imposes a relation between M and J:

$$(2n\mp m)^2J^2=4n(\pm m-n)M^4. \quad (5.5)$$

In general, the action is given by Eq. (1.1), and the Pontryagin number is

$$P=\frac{e^2Q_mQ_e r_+}{\kappa(r_+^{\,2}-J^2M^{-2})}, \quad (5.6)$$

and hence

$$P=2mn \quad (5.7)$$

as before. The generality of this result may be understood from the point of view of the theory of characteristic classes. We refer to Milnor and Strasheff[10] for the relevant definitions. The U(1) bundles we have constructed (for Reissner-Nordström or Kerr-Newman geometries) have in general a nonvanishing first Chern class c_1. We have designated by n and m the integral of c_1 over the electric and magnetic spheres, respectively. The SU(2) bundles, by construction, split as the Whitney sum of a U(1) bundles and its conjugate. In general they have a nonvanishing second Chern class c_2. We have designated by P the integral of $-c_2$ over $\mathfrak{M}$. Equation (5.7) follows from the Whitney sum formula and the definition of the cup product, quite independent of the functional dependence of the field strength $f_{\mu\nu}$ on the coordinates. The SU(2) bundles are not "characterized" by P since different values of n and m can yield the same value of P. It is, however, necessary for P to vanish if the bundle is to be trivial, and this will not be the case unless either Q_e or Q_m vanishes.

One other point of topological interest is that throughout the paper we have referred to the quantity Q_e as the electric charge, since this is consistent with the Q_e/r behavior of the electromagnetic potential as r becomes large. Similarly we have identified Q_m with the magnetic charge. Yet the field strengths are completely free of singularities; the Maxwell equations are quite literally source-free:

$$d{*}f=0,\quad df=0. \quad (5.8)$$

This situation is reminiscent of Wheeler's "charge without charge," and is simply a consequence of the unusual topology of the space-time manifold: $\mathfrak{M}$ has a nonvanishing second Betti number. (See Misner, Thorne, and Wheeler, Ref. 11, p. 1200, for a discussion of "wormholes"; but note that in contrast to the wormhold, our pseudoparticle *does* obey an electric-charge quantization condition.)

In our treatment of gauge theories so far, we have not considered the possibility of spontaneous symmetry breakdown in the presence of Higgs fields. An isotriplet of Higgs scalars $\phi=\phi^iT^i$ could be introduced without changing the gauge field solutions or the total action, provided that

$$D\phi\equiv d\phi+[A,\phi]=0,$$

which implies

$$[F,\phi]=0,$$

so that in the unprimed gauge of Eq. (3.20) the solution is the trivial one $\phi=cT^3$, or $\phi=c\hat{\phi}$ in the primed gauge, where the constant c is chosen so that ϕ corresponds to a zero of the effective potential. In other words, the solution satisfies everywhere the boundary condition at infinity imposed by 't Hooft.[8] In this case, both the Yang-Mills source and the Higgs stress tensor would vanish. It may be that the only solutions compatible with the existence of an event horizon are these trivial $D\phi=0$ solutions. Unfortunately, we have been unable to prove this since the usual "no hair" theorems[11] do not appear to go through in the same way for theories of scalar fields which suffer spontaneous symmetry breakdown. On the other hand, one can prove Euclidean versions of the "no hair" theorem for a single scalar field which satisfies the coupled Einstein-Klein-Gordon equation and which vanishes on the boundary so that the effective topology of the manifold remains $S^2\times S^2$. In this case, one can easily see by integrating

$$\frac{1}{\sqrt{g}}\,\partial_\mu(\phi\sqrt{g}\,g^{\mu\nu}\partial_\nu\phi)=\phi\Box\phi+\partial_\mu\phi\partial^\mu\phi \quad (5.9)$$

over $\mathfrak{M}$ and using Stokes's theorem that this implies $\partial_\mu \phi = 0$. Nontrivial Maxwell fields, of course, are permitted since $S^2 \times S^2$ does support nontrivial harmonic 2 forms.

In the particular case $Q_e = \pm Q_m$, the Yang-Mills solutions we have found satisfy the self-duality condition

$$F = \pm {}^*F ,$$

on the Euclidean Kerr manifold with even topological number P. Because of the quantization laws, our gauge fields have zero parameters [J is no longer a parameter since it is determined by Eq. (5.5) in terms of M, and, in any case M is to be regarded as a parameter of the base space rather than the gauge field]. However, one knows from recent work on the Atiyah-Singer index theorem[13] that there exists a family of self-dual solutions with at least $8P - 3\chi'/2$ parameters, where χ' is the Euler number of the effective topology. In the present context the effective topology of the base space is $S^2 \times S^2$ and hence $\chi' = 4$. (Contrast with flat space where the effective topology is S^4 and $\chi' = 2$.) Therefore we have in general at least $8P - 6$ parameters. The particular solutions we have found happen to have no parameters at all. Since the Kerr-Newman solution is believed to be the unique black-hole solution of the Einstein-Maxwell equations, these more general Yang-Mills potentials are presumably not simple SU(2) versions of the Maxwell potentials but rather of the intrinsically non-Abelian type, like the $P = \pm 1$ solution discussed in Refs. 1 and 2.

We note that our solutions, though carrying a nonvanishing gravitational Euler number χ, do not possess a nonvanishing *gravitational* Pontryagin number and will hence make no gravitational contribution to symmetry-breaking via the axial-vector current anomaly.[3,4] However, the Yang-Mills pseudoparticles will presumably contribute via *their* Pontryagin number in a way quite independent of the strength of gravitational coupling, whose presence was nevertheless responsible for their very existence. We also note that when we relax the self-duality constraint and allow a nonvanishing stress tensor, our pseudoparticles are no longer "free," in the sense that the Yang-Mills part of the action for P pseudoparticles is no longer P times the action of a single pseudoparticle.

There is one other vital difference from the flat-space solutions discussed in Ref. 3. In flat space, the Yang-Mills pseudoparticle interpolates between inequivalent pure gauge solutions as the imaginary time tends to plus or minus infinity, and hence gives rise to the quantum interpretation of tunneling between topologically distinct vacuums with an amplitude of order e^{-S}. For the solutions discussed in this paper, such an interpretation is obscured by the periodic nature of the time coordinate. Our Yang-Mills potentials are either periodic in time with period $2\pi n\kappa^{-1}$, an integral multiple of the black-hole period, or else, depending on the choice of gauge, completely time independent.

This periodicity does, however, lend itself to finite-temperature interpretation. Indeed, from the historical viewpoint, it was Hawking's treatment of the thermodynamic[14] properties of black holes which first led him and others to consider an imaginary-time analysis. The connection with the parallel developments in Euclidean-space pseudoparticles did not come until later. While in no way claiming that the desired grand synthesis of these ideas has yet been achieved (indeed our sign difference from Hawking and Gibbons would seem to require further investigation), we do hope that the Einstein-Yang-Mills solutions presented in this paper will throw more light on the fascinating interplay between recent developments in quantum field theory, algebraic topology, general relativity, and thermodynamics.

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