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Functional measures for the Feynman-quantized Einstein-Yang-Mills system

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A natural formulation of the Hamiltonian null dynamics of the Einstein-Yang-Mills system is given. In order to overcome difficulties which prevent the straightforward generalization of the previous results obtained by Fradkin and Vilkovisky and by Kaku for pure gravity a new null-gauge condition is introduced for the Einstein field. Using this gauge, both the action measure and the Lagrangian measure of the Einstein-Yang Mills system are calculated. In the absence of the Yang-Mills boson vector field, both measures coincide with previous results of Fradkin and Vilkovisky.

I. INTRODUCTION

After the analysis made by Aragone and Chela[1] of the dynamics of pure gravity in a family of algebraic null gauges, Kaku[2] chose a subfamily of them and was able to prove that their Faddeev-Popov ghosts do not contribute to the S-matrix element.

Therefore, through a mathematical Hamiltonian formulation of the null Lagrangian in such a way that second-class constraints appear, Kaku and Senjanović[3] asserted that the Fradkin-Vilkovisky[4] Lagrangian measure of quantum gravity is the right one and the Faddeev-Popov[5] proposal has to be abandoned.

In the present article we present a systematic Feynman-quantization approach for the Einstein-Yang-Mills system in such a way that we can obtain the Hamiltonian null dynamics with a very compact expression for the second-class constraints.

This allows us to use the heuristic results of Senjanović[6] in order to calculate directly the action measure of the system and then, as a consequence, the Lagrangian measure.

Throughout this article we use a gauge condition which can be considered as a generalization of both the Bondi gauge and the Kaku family, and we adopt the point of view stated by the author in previous work,[7] and later supported in their analysis of the initial-value problem by Gambini and Restuccia,[8] that the scalar constraint on the null coordinates has to be regarded as a first-order differential equation for g^{ur} instead of a second-order differential equation for obtaining the two-dimensional volume ${}_2g$.

II. THE FIRST-ORDER COVARIANT ACTION PLUS THE FEYNMAN PATH QUANTIZATION AS THE FUNDAMENTAL GUIDING PRINCIPLE FOR THE PHYSICS OF RELATIVISTIC QUANTUM SYSTEMS

Our aim is to evaluate the S-matrix element for the present tensor-multivector system:

$$\langle f|S(TUV)|i\rangle = \int \exp\left[\frac{i}{\hbar}\int_{|i(TUV)\rangle}^{|f(TUV)\rangle} dA(\mathfrak{g};\Gamma;A;f)\right] d\mu(TUV)\,, \tag{1a}$$

where the structure of the functional action measure $d\mu(TUV)$ for this gauge system is

$$d\mu(TUV) = \delta(G^A)\Delta(G^A)p_A(\text{fields})D\mathfrak{g}^{\mu\nu} \times D\Gamma^\sigma_{\mu\nu}DA^a_\mu Df^{b\mu\nu}. \tag{1b}$$

In this expression, G^A are the gauge conditions, $\Delta(G^A)$ is their corresponding Faddeev-Popov determinant, $\mathfrak{g}^{\mu\nu}$ is a contravariant symmetric density, $\Gamma^\sigma_{\mu\nu}$ is the affinity, A^a_μ is a covariant vector multiplet, $f^{b\mu\nu}$ is a contravariant antisymmetric density, and p_A (fields) is the weight we attach to the Cartesian measure $D\mathfrak{g}D\Gamma DADf$ in the overall mathematical field space.

The action functional $A(\mathfrak{g};\Gamma;A;f)$ has to be, by definition, a functional depending upon the fields $(\mathfrak{g};\Gamma;A;f)$ and, at most, of their first-order derivatives.

However, one could also try to make all the calculations by using a Lagrangian or second-order approach. In this case, variations of the Γ and f fields determine their values in terms of the coordinates ($\mathfrak{g}$ and A for this system) which, after substitution in the first-order action $A(\mathfrak{g};\Gamma;A;f)$, yield the second-order Lagrangian functional $L(\mathfrak{g};A) \equiv A(\mathfrak{g};\Gamma=\Gamma(\mathfrak{g});A;f=f(A;\mathfrak{g}))$. If one thinks that the matrix element (1a) is independent of whether we develop the dynamics in the first-order fashion or in the second-order way, one would like to evaluate instead of (1a) the similar expression

$$\langle f|S(TUV)|i\rangle = \int \exp\left[\frac{i}{\hbar}\int_{|i(TUV)\rangle}^{|f(TUV)\rangle} dL(\mathfrak{g};A)\right] d\nu(TUV)\,, \tag{2a}$$

where the structure of the functional Lagrangian measure $d\nu(TUV)$ for this gauge system shall be given by

$$d\nu(TUV)=\delta(G^A)\Delta(G^A)p_L(\mathfrak{g};A)D\mathfrak{g}^{\mu\nu}DA^a_\mu . \quad (2b)$$

The function $p_L(\mathfrak{g};A)$ is the weight assigned to the mathematical unphysical Cartesian functional measure $D\mathfrak{g}\wedge DA$.

Physically, if both methods are thought to be equally relevant, one expects (and this is one of the main points of this work) that the matrix elements (1a) and (2a) must be the same complex number

$$\langle f|S(TUV)|i\rangle'=\langle f|S(TUV)|i\rangle . \quad (3)$$

Conversely, we can obtain one of the two weights p_A, p_L in terms of the remaining one, assuming that Eq. (3) holds as a complementary law.

III. GEOMETRY OF THE SYSTEM ON THE LIGHT-FRONT (NULL) COORDINATES

Our treatment is based upon the choice of null coordinates for the dynamical description of the system. The possibility of these types of coordinates is an intrinsically relativistic feature which brings considerable mathematical simplicity in most of the usually difficult points encountered in the Newtonian 3×1 approach. For instance, the null-coordinates dynamics suggests gauges which in general are determined by algebraic equations (instead of the partial differential equations one is used to in the 3×1 approach) which, therefore, lead to simpler estimates of the Faddeev-Popov determinant.

Using null coordinates, the metric and the gradient squared have the form

$$ds^2=g_{ij}(dx^i+N^i du)(dx^j+N^j du)-2m^2du(dr-ndu), \quad (4a)$$

$$(\partial f)^2=g^{ij}\partial_i f\partial_j f-2m^{-2}(\partial_u f+n\partial_r f-N^i\partial_i f)\partial_r f , \quad (4b)$$

where

$$\begin{gathered} {}_4g_{ij}\equiv{}_2g_{ij},\quad g_{iu}\equiv N_i\equiv g_{ij}N^j,\quad g_{ir}=0 ,\\ g_{uu}\equiv 2nm^2+g_{ij}N^iN^j,\quad g_{ur}=-m^2,\quad g_{rr}=0 , \end{gathered} \quad (5a)$$

and

$$\begin{gathered} {}_4g^{ij}={}_2g^{ij},\quad g^{iu}=g^{uu}=0 ,\\ g^{ir}=m^{-2}N^i,\quad g^{ur}=-m^{-2},\quad g^{rr}=-2nm^{-2} . \end{gathered} \quad (5b)$$

In these coordinates, using the variables defined in Eqs. (4) and (5), the four-dimensional volume element $d^4\tau$ has the value

$$d^4\tau=m^2{}_2g^{1/2}d^2\vec{x}dudr\equiv m^2hd^2\vec{x}dudr , \quad (6)$$

and the contravariant density $\mathfrak{g}^{\mu\nu}\equiv g^{\mu\nu}(-_4g)^{1/2}$ becomes

$$\begin{gathered} \mathfrak{g}^{ij}=m^2e^{ij},\quad \mathfrak{g}^{iu}=\mathfrak{g}^{uu}=0,\quad \det e^{ij}=+1 ,\\ \mathfrak{g}^{ir}=hN^i=e^{il}N_l=\hat{N}^i,\quad \mathfrak{g}^{ur}=-h,\quad \mathfrak{g}^{rr}=-2nh\equiv-2\hat{n} , \end{gathered} \quad (7)$$

showing that m can be obtained by

$$m^2=(\det \mathfrak{g}^{ij})^{1/2} . \quad (8)$$

Note that we have also introduced the two-dimensional unimodular symmetric tensor e^{ij}, whose inverse e_{ij} satisfies

$$e_{ij}e^{jl}=\delta^l_i,\quad \det e_{ij}=\det e^{jl}=+1,\quad {}_4g_{ij}={}_2g_{ij}=he_{ij} . \quad (9)$$

For the description of the Maxwell tensor density $f^{a\mu\nu}$, we shall use the variables $(e^a, b^a, c^{ai}, d^{ai})$ defined by

$$\begin{gathered} e^a\equiv f^{aru},\quad f^a_{ij}=g_{i\alpha}g_{j\beta}f^{a\alpha\beta}\equiv\epsilon_{ij}hm^2b^a ,\\ c^{ai}\equiv f^{aiu},\quad d^{ai}\equiv f^{ari}+nc^{ai}+N^ie^a , \end{gathered} \quad (10)$$

while A^a_μ are still useful for representing the isovector potential.[9]

Then the action of Eq. (1a) is given by

$$dA(\mathfrak{g};\Gamma;A;f)\equiv dA^T(\mathfrak{g};\Gamma)+\kappa^2dA^V(A;f;\mathfrak{g}) , \quad (11a)$$

where the pure gravity first-order action is ($\mathfrak{g}_{\mu\alpha}$: $\mathfrak{g}_{\mu\alpha}\mathfrak{g}^{\alpha\nu}=\delta^\nu_\mu$)

$$\begin{aligned} dA^T(\mathfrak{g};\Gamma)\equiv d^4x(&\Gamma^\alpha_{\nu\alpha}\partial_\mu\mathfrak{g}^{\mu\nu}-\Gamma^\alpha_{\mu\nu}\partial_\alpha\mathfrak{g}^{\mu\nu}+\mathfrak{g}^{\mu\nu}\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}\\ &-\mathfrak{g}^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}) \end{aligned} \quad (11b)$$

and

$$\begin{aligned} dA^V(A;f;\mathfrak{g})\equiv d^4x[&\tfrac{1}{2}f^{a\mu\nu}(\partial_\nu A^a_\mu-\partial_\mu A^a_\nu+gf^{abc}A^b_\nu A^c_\mu)\\ &+\tfrac{1}{4}f^{a\mu\nu}f^{a\alpha\beta}\mathfrak{g}_{\mu\alpha}\mathfrak{g}_{\nu\beta}(-\mathfrak{g})^{1/2}] . \end{aligned} \quad (11c)$$

On a curved space it is convenient to distinguish between $f^{a\mu\nu}$ (which shall be a metric-dependent quantity) and its metric-independent part $A^a_{\mu\nu}$:

$$A^a_{\mu\nu}\equiv\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu\equiv(-_4g)^{-1/2}f^a_{\mu\nu} . \quad (12a)$$

For instance, the second of Eqs. (10) can be written

$$b^a=A^a_{12}=\tfrac{1}{2}\epsilon^{ij}A^a_{ij} . \quad (12b)$$

The n^2+3 gauge conditions we shall use are $G^A\equiv(G^\mu_\infty;G^a)=0$, where

$$G^\mu_\infty\equiv\mathfrak{g}^{u\mu}-h_0\eta^{u\mu},\quad G^a\equiv A^a_r,\quad h_0>0 , \quad (13)$$

and $\eta^{\mu\nu}$ is the flat metric tensor in the null-plane coordinates: $\eta^{ij}=\delta^{ij}$, $\eta^{ur}=-1$, $\eta^{iu}=\eta^{ir}=\eta^{uu}=\eta^{rr}=0$. $G^a\equiv A^a_r=0$ is the Tomboulis gauge condition.[10]

In this purpose, note that one could also take into account gauge conditions of the two related types[11]

$$q,l \text{ real},\quad \hat{G}^\mu_q\equiv(-_4g)^q g^{u\mu}-\eta^{u\mu}=0 , \quad (14a)$$

$$G^{u}_{l} \equiv {}_2 g^{l} g^{uu} - \eta^{uu} = 0 \,. \qquad (14b)$$

Actually, for $q \neq l$, $\hat{G}^{u}_{q} = 0$ is equivalent to $G^{u}_{l(q)} = 0$, with $2l(q) = q(1-q)^{-1}$; $q = 1^{\pm}$ corresponds to a finite representation of $l = \mp\infty$ and $q = \pm\infty$ is represented finitely by $2l = -1$. In this sense, one can state that the gauge choice (13) is not represented by any finite value of the Kaku real parameter l.[12]

IV. THE INTEGRATION PROCESS

We start from the matrix element (1a):

$$\langle f|S(TUV)|i\rangle = \int \exp\left\{\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}[dA^{T}(\mathfrak{g};\Gamma) + \kappa^2 dA(A;f;\mathfrak{g})]\right\}\delta(\mathfrak{g}^{ui})$$
$$\times\,\delta(\mathfrak{g}^{uu})\delta(\mathfrak{g}^{ur}+h_0)\delta(A^{a}_{r})\Delta(G^{A})p_A(\mathfrak{g};\Gamma;A;f)D\mathfrak{g}^{\mu\nu}\wedge D\Gamma^{\tau}_{\rho\sigma}\wedge DA^{a}_{\lambda}\wedge Df^{b\beta\gamma}\,. \qquad (15)$$

In order to take advantage of the simplicity of the generalized Bondi gauge, we first integrate this expression with respect to $\mathfrak{g}^{ui}$ and $\mathfrak{g}^{uu}$. This gives us ($\partial_r f \equiv f'$; $\partial_u f \equiv \dot{f}$)

$$\langle f|S(TUV)|i\rangle = \int \exp\Big(\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}\{\kappa^2 dA^{V}(A;f;\mathfrak{g}) + d^4x[(\mathfrak{g}^{ij}_{,j}+\hat{N}^{i\prime})\Gamma^{l}_{il} - \mathfrak{g}^{ij}_{,l}\Gamma^{l}_{ij} - \dot{\mathfrak{g}}^{ij}\Gamma^{u}_{ij} - \mathfrak{g}^{ij\prime}\Gamma^{r}_{ij} - h'\Gamma^{l}_{ul} - 2\hat{N}^{i}_{,j}\Gamma^{j}_{ri}$$
$$+ (\hat{N}^{i}_{,i} - \dot{h} - 2\hat{n}')\Gamma^{j}_{rj} + (\mathfrak{g}^{ij}_{,j}+\hat{N}^{i\prime})\Gamma^{u}_{iu} - h'\Gamma^{u}_{uu} + (\hat{N}^{i}_{,i} - 2\hat{n}' + \dot{h})\Gamma^{u}_{ur} - 2\dot{\hat{N}}^{i}\Gamma^{u}_{ri}$$
$$+ 2h_{,j}\Gamma^{j}_{ur} + (\mathfrak{g}^{ij}_{,j} - \hat{N}^{i\prime})\Gamma^{r}_{ir} + 2\hat{n}_{,i}\Gamma^{i}_{rr} + h'\Gamma^{r}_{ur} + 2\dot{\hat{n}}\Gamma^{u}_{rr} + (\hat{N}^{i}_{,i} - \dot{h})\Gamma^{r}_{rr} + Q(\Gamma)]\}\Big)$$
$$\times\,\delta(h-h_0)\delta(A^{a}_{r})\Delta(G^{A})p_A D\mathfrak{g}^{ij}\wedge D\hat{N}^{l}\wedge Dh\wedge D\hat{n}\wedge D\Gamma^{\alpha}_{\mu\nu}\wedge DA^{a}_{\lambda}\wedge Df^{b\beta\gamma}\,, \qquad \textbf{(16a)}$$

where $Q(\Gamma)$ is the quadratic (in Γ) part of the gravity action

$$Q(\Gamma) \equiv \mathfrak{g}^{\mu\nu}\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \mathfrak{g}^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\,. \qquad (16b)$$

Now, in order to spare the reader the details of the lengthy calculations one has to perform, we shall roughly summarize them giving what we feel are the more relevant steps.

We observe that the quantities

$$\hat{\Gamma}^{j}_{ri} \equiv \Gamma^{j}_{ri} - \tfrac{1}{2}\delta^{j}_{i}\Gamma^{p}_{rp},\quad \Gamma^{i}_{ur},\quad \Gamma^{i}_{rr},$$
$$\hat{\Gamma}^{j}_{ui} \equiv \Gamma^{j}_{ui} - \tfrac{1}{2}\delta^{j}_{i}\Gamma^{p}_{up},\quad \Gamma^{r}_{ui},\quad \Gamma^{i}_{uu}, \Gamma^{r}_{uu}$$

either do not appear in the linear part in Γ in Eq. (16a) (for instance $\Gamma^{i}_{uu}, \Gamma^{r}_{uu}$) or they do not have factors containing a "time" derivative ∂_u or a ∂_r derivative. At most they have factors which contain two-space derivatives ∂_i (as is the case for $\hat{\Gamma}^{j}_{ri}$).

We start the process integrating Γ^{i}_{uu}, Γ^{r}_{uu}, Γ^{u}_{ir}, and Γ^{u}_{rr}. Then the integrations with respect to $D\Gamma^{r}_{ui}$ and thereafter with respect to $D\Gamma^{i}_{rr}$ are immediate. Then, in order to proceed with the integration process it is convenient to define the quantities Γ_i and M_p,

$$\Gamma_i \equiv \Gamma^{j}_{ij},\qquad M_p \equiv \mathfrak{g}^{ij}\Gamma^{l}_{ij}\mathfrak{g}_{lp}\,. \qquad (17a)$$

These quantities are useful to decompose algebraically the three index object Γ^{l}_{ij} in the form

$$\Gamma^{l}_{ij} \equiv \hat{\Gamma}^{l}_{ij} + \tfrac{1}{2}\delta^{l}_{i}\Gamma_j + \tfrac{1}{2}\delta^{l}_{j}\Gamma_i - \tfrac{1}{2}\mathfrak{g}_{ij}\Gamma^{l}$$
$$-\tfrac{1}{4}\delta^{l}_{i}M_j - \tfrac{1}{4}\delta^{l}_{j}M_i + \tfrac{3}{4}\mathfrak{g}_{ij}M^{l}\,, \qquad (17b)$$

where $\hat{\Gamma}^{l}_{ij}$ is symmetric and pure traceless, i.e.,

$$\hat{\Gamma}^{l}_{ij} = \hat{\Gamma}^{l}_{ji},\quad \mathfrak{g}^{ij}\hat{\Gamma}^{l}_{ij} = 0 = \hat{\Gamma}^{j}_{ij}\,, \qquad (17c)$$

and we understand that Γ^{l} and M^{l}, respectively, mean $\mathfrak{g}^{lj}\Gamma_j$ and $\mathfrak{g}^{lp}M_p$. [Therefore, each term of Eq. (17b) depends only upon Γ^{l}_{ij} and $e^{ij} \equiv \mathfrak{g}^{ij}(\det\mathfrak{g}^{ij})^{-1/2}$, the unimodular part of the two-dimensional contravariant density $\mathfrak{g}^{ij}$.]

It is also convenient to split Γ^{u}_{ij}, Γ^{r}_{ij}, Γ^{j}_{ui}, and Γ^{j}_{ri} into the addition of their traceless parts plus the corresponding multiple of the unit tensor

$$\Gamma^{u}_{ij} \equiv \hat{\Gamma}^{u}_{ij} + \tfrac{1}{2}\mathfrak{g}_{ij}\mu,\quad \Gamma^{r}_{ij} \equiv \hat{\Gamma}^{r}_{ij} + \tfrac{1}{2}\mathfrak{g}_{ij}\rho\,, \qquad (18a)$$

$$\Gamma^{j}_{ui} \equiv \hat{\Gamma}^{j}_{ui} + \tfrac{1}{2}\delta^{j}_{i}\nu,\quad \Gamma^{j}_{ri} \equiv \hat{\Gamma}^{j}_{ri} + \tfrac{1}{2}\delta^{j}_{i}\sigma\,, \qquad (18b)$$

where

$$\mu \equiv \mathfrak{g}^{ij}\Gamma^{u}_{ij},\quad \nu \equiv \Gamma^{j}_{uj},\quad \rho \equiv \mathfrak{g}^{ij}\Gamma^{r}_{ij},\ \text{and}\ \sigma \equiv \Gamma^{j}_{rj}\,.$$

Through Eqs. (17) and (18) we go from the initial set (Γ^{l}_{ij}, Γ^{u}_{ij}, Γ^{r}_{ij}, Γ^{j}_{iu}, Γ^{j}_{ir}) to the new independent variables ($\hat{\Gamma}^{1}_{11}$, $\hat{\Gamma}^{2}_{22}$, Γ_i, M_l, $\hat{\Gamma}^{u}_{11}$, $\hat{\Gamma}^{u}_{22}$, $\hat{\Gamma}^{r}_{11}$, $\hat{\Gamma}^{r}_{22}$, $\hat{\Gamma}^{1}_{u1}$, $\hat{\Gamma}^{2}_{u1}$, $\hat{\Gamma}^{1}_{u2}$, $\hat{\Gamma}^{1}_{1r}$, $\hat{\Gamma}^{2}_{r1}$, $\hat{\Gamma}^{1}_{r2}$, μ, ν, ρ, and σ), where we have to take into account the five different functional Jacobians (the $\mathfrak{g}^{ij}$ can be considered constants because their exterior product $\wedge D\mathfrak{g}^{ij}$ already appears in the functional volume element):

$$\wedge D\Gamma^{l}_{ij} = \frac{(\det\mathfrak{g}^{ij})}{\mathfrak{g}^{11}\mathfrak{g}^{22}}D\hat{\Gamma}^{1}_{11}\wedge D\hat{\Gamma}^{2}_{22}\wedge D\Gamma_i\wedge DM_l$$
$$\equiv \frac{(\det\mathfrak{g}^{ij})}{\mathfrak{g}^{11}\mathfrak{g}^{22}}D\hat{\Gamma}^{l}_{ij}D\Gamma_p DM_q\,, \qquad (19a)$$

$$\wedge D\Gamma^{u;r}_{ij} = \frac{1}{2|g^{12}|} D\hat{\Gamma}^{u;r}_{11} \wedge D\hat{\Gamma}^{u;r}_{22} \wedge D(\mu;\rho)$$

$$\equiv \frac{1}{2|g^{12}|} D\hat{\Gamma}^{u;r}_{ij} D(\mu;\rho)\ , \tag{19b}$$

and

$$\wedge D\Gamma^{j}_{u;ri} = D\hat{\Gamma}^{1}_{u1} \wedge D\hat{\Gamma}^{2}_{u1} \wedge D\hat{\Gamma}^{1}_{u2} \wedge D(\nu;\sigma)$$
$$\equiv D\hat{\Gamma}^{j}_{u;ri}\ D(\nu;\sigma)\ . \tag{19c}$$

Introduction of the new independent variables in Eq. (16) makes terms appear less coupled, and some immediate integrations can be performed.[13]

Integration with respect to Γ^{u}_{uu}, ν, μ, and σ yields

$$\int (16\text{a}) D\Gamma^{u}_{uu} D\nu D\mu D\sigma = \hbar^2 h^{-1} \times (16\text{a})\,[\,\mu = h',\ \ \sigma = (nh)' + 2\Gamma^{u}_{ur}\,]\ . \tag{20}$$

Next we integrate with respect to Γ^{r}_{ur}, Γ^{u}_{ur}, ρ, and Γ^{r}_{rr}.
Symbolically, the result of these four integrations can be written in the form

$$\int (20)\, D\Gamma^{r}_{ur} D\Gamma^{u}_{ur} D\rho D\Gamma^{r}_{rr} = \hbar^4 h^{-2} \times (20)\,[\,\mu = h',\ \ \sigma = (nh)',\ \ \Gamma^{u}_{ur} = 0,\ \ \Gamma^{r}_{rr} = (\ln m^2)'\,]\ . \tag{21a}$$

Furthermore, as the variables $\hat{\Gamma}^{j}_{ui} \equiv (\hat{\Gamma}^{1}_{u1},\ \Gamma^{2}_{u1},\ \Gamma^{1}_{u2},\ \hat{\Gamma}^{2}_{u2} = -\hat{\Gamma}^{1}_{u1})$ enter in the amplitude (16) as Lagrange multipliers, the functional integration can be easily performed with respect to them and later on with respect to the $\hat{\Gamma}^{j}_{ri}$:

$$\int (21\text{a}) D\hat{\Gamma}^{j}_{ui} D\hat{\Gamma}^{m}_{rl} = \hbar^3 h^{-3} \times (21\text{a})\ \ (\hat{\Gamma}^{m}_{rl} = h^{-1} g^{mp} \hat{\Gamma}^{u}_{pl}\,)\ . \tag{21b}$$

Incidentally, it is worthwhile to observe that this value (21b) obtained for $\hat{\Gamma}^{m}_{rl}$ ensures its "symmetry" (in the sense that $\hat{\Gamma}^{m}_{rl} = g^{mi} g_{lj} \hat{\Gamma}^{j}_{ri}$).

After this process let us write in full detail the matrix element (16). One finds the reduced expression

$$\begin{aligned}
\langle f|\,S(TUV)|\,i\,\rangle = \int \exp\Big[\frac{i}{\hbar} \int_{|i\rangle}^{|f\rangle} \big(\, \kappa^2 dA^V + d^4x \{ & \hat{N}^{i\prime}\,\Gamma_i - g^{ij}{}_{,l}\,\hat{\Gamma}^{l}_{ij} + (\ln m^2)_{,l}\,\Gamma^l - \hat{\Gamma}^{u}_{ij}\,\dot{g}^{ij} - \hat{\Gamma}^{r}_{ij}\,g^{ij} - h'(\ln m^2)^{\cdot} \\
& - (\ln h)'\,\dot{h} - 2\hat{N}^{l}{}_{,j}\,h^{-1} g^{jl}\,\hat{\Gamma}^{u}_{li} + (g^{ij}{}_{,j} + \hat{N}^{i\prime})\Gamma^{u}_{iu} + (g^{ij}{}_{,j} - \hat{N}^{i\prime})\Gamma^{r}_{ir} \\
& + (\hat{N}^{i}{}_{,i} - \dot{h})(\ln m^2)' + 2\hat{N}^{i} h^{-1} g^{jl}\,\hat{\Gamma}^{u}_{il}\,\Gamma^{u}_{ju} + (\ln h)'\,\hat{N}^{i}\,\Gamma^{u}_{iu} + 2\hat{N}^{i}(\ln h)'\,\Gamma^{r}_{ir} \\
& - 2\hat{n}'(\ln h)' - 2\hat{n}(\ln h)'(\ln m^2)' + 2\hat{n} h^{-2} g^{jl} g^{ip}\,\hat{\Gamma}^{u}_{ij}\,\hat{\Gamma}^{u}_{pl} + \hat{n}(\ln h)'^2 + \tfrac{1}{4} g^{ij} M_i M_j \\
& - g^{ij} M_i\,[\,h^{-1}\hat{N}^{l}\,\hat{\Gamma}^{u}_{jl} + \tfrac{3}{2}(\ln m^2)_{,j} - \tfrac{1}{2} g_{jl}\, g^{lp}{}_{,p} - \Gamma^{u}_{ju} - \Gamma^{r}_{jr}\,] - g^{ij}\,\hat{\Gamma}^{p}_{il}\,\hat{\Gamma}^{l}_{jp} \\
& - g^{ij}\,\Gamma^{u}_{iu}\Gamma^{u}_{ju} - g^{ij}\,\Gamma^{r}_{ir}\Gamma^{r}_{jr} - 2 g^{ij} g^{lp} h^{-1}\hat{\Gamma}^{r}_{il}\,\hat{\Gamma}^{u}_{jp} - 2h^{-1}\hat{N}^{i}\,\hat{\Gamma}^{l}_{ij}\, g^{jp}\,\hat{\Gamma}^{u}_{lp} \\
& + 2\hat{N}^{i}\,\Gamma^{j} h^{-1}\hat{\Gamma}^{u}_{ij} - \Gamma^{i}_{ur}\,(2h\Gamma_i - 2h_{,i} + 2\hat{N}^{j}\,\hat{\Gamma}^{u}_{ij} + g_{ij}\hat{N}^{j} h')\}\big)\Big]
\end{aligned}$$
$$\times\, \delta(h - h_0)\,\delta(A^{a}_{r})\,\Delta p_A\, \frac{m^4 h^{-10}\hbar^{12}}{g^{11} g^{22} (g^{12})^2}\, DA^{a}_{\mu} Df^{b\beta\gamma} D\hat{\Gamma}^{l}_{ij} D\Gamma_i\, DM_i\, D\hat{\Gamma}^{u}_{ij} D\hat{\Gamma}^{r}_{ij} D\Gamma^{u}_{iu} D\Gamma^{r}_{lr} D\Gamma^{j}_{ur} Dg^{ij} D\hat{N}^{l} Dh D\hat{n}\ . \tag{22}$$

Fortunately, it is still possible to make other integrations.

The integration with respect to Γ^{i}_{ur} provides another δ functional, which suggests that we proceed to integrate Γ_i. That gives (up to a numerical factor)

$$\int (22) D\Gamma^{i}_{ur} D\Gamma_i = \hbar^2 h^{-2} (22)\ \ [\,\Gamma_i = (\ln h)_{,i} - N^j\,\hat{\Gamma}^{u}_{ij} - \tfrac{1}{2} g_{ij} N^j h'\,]\ . \tag{23a}$$

Then we observe the quadratic structure of the reduced action (23a) in the variables M_i:

$$\langle f|\,S(TUV)|\,i\,\rangle \sim \int C \exp\Big[\frac{i}{\hbar} \int_{|i\rangle}^{|f\rangle} (\tfrac{1}{4} g^{ij} M_i M_j - g^{ij} M_i\, a_j)\, d^4x\Big] DM_i\ , \tag{23b}$$

where C means the remaining part of (23a), not containing terms in these variables, and a_j means

$$a_j \equiv h^{-1}\hat{N}^{l}\,\hat{\Gamma}^{u}_{jl} + \tfrac{3}{2}(\ln m^2)_{,j} - \tfrac{1}{2} g_{jl}\, g^{lp}{}_{,p} - \Gamma^{u}_{ju} - \Gamma^{r}_{jr} \equiv b_j - \Gamma^{u}_{ju} - \Gamma^{r}_{jr}\ . \tag{23c}$$

As this expression is equivalent to

$$\int C \exp\left\{\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}\left[g^{ij}(\tfrac{1}{2}M_i - a_i)(\tfrac{1}{2}M_j - a_j) - g^{ij}a_i a_j\right]d^4x\right\} D M_i\,, \tag{23d}$$

we can perform the integration, since its first part is a typical Gaussian expression[14]

$$(23\text{b}) = C(\det g^{ij})^{-1/2}\exp\left(-\frac{i}{\hbar}\,g^{ij}a_i a_j\right) = C\hbar\, m^{-2}\exp\left(-\frac{i}{\hbar}\,g^{ij}a_i a_j\right)\,. \tag{23e}$$

To proceed we call $\mathfrak{C}_T$ the functional, giving rise to the null-coordinates scalar constraint[15]

$$2\mathfrak{C}_T \equiv 2\hat{n}h^{-2}g^{il}g^{jp}\hat{\Gamma}^u_{il}\hat{\Gamma}^u_{jp} + \hat{n}(\ln h)'^2 - 2\hat{n}'(\ln h)' - 2\hat{n}(\ln h)'(\ln m^2)'\,, \tag{24}$$

and integrate $\hat{\Gamma}^l_{ij}$. Details of this integration are given in Appendix A. Taking into account Eq. (A15), the matrix element (23e) can be cast in the form

$$\begin{aligned}\langle f|S(TUV)|i\rangle = \int \exp\Big[\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}\big(\kappa^2 dA^V + d^4x\{&-\hat{\Gamma}^u_{ij}\dot{g}^{ij} - h'(\ln m^2)^{\cdot} - (\ln h)'\dot{h} + g^{ij}(\ln m^2)_{,i}(\ln h)_{,j} - (\ln m^2)'\dot{h}\\
&-\hat{\Gamma}^r_{ij}(g^{ij\prime} + 2h^{-1}g^{il}g^{jp}\hat{\Gamma}^u_{lp}) - \hat{N}^{i\prime}N^j\hat{\Gamma}^u_{ij} - 3g^{ij}(\ln m^2)_{,i}N^l\hat{\Gamma}^u_{lj}\\
&-h'\hat{\Gamma}^u_{ij}N^iN^j - g^{ij}(\ln m^2)_{,i}(\ln m^2)_{,j} - 2[N^i{}_{,j} + \Gamma_{lj}{}^{i(e)}N^l]g^{jl}\,\hat{\Gamma}^u_{il}\\
&+m^2e^{ij}\Gamma^p_{il}(e)\Gamma^l_{jp}(e) + \hat{N}^i{}_{,i}(\ln m^2)' - m^2(\ln m^2)_{,i}\lambda^i - 2g^{ij}\hat{\Gamma}^u_{il}\hat{\Gamma}^u_{jp}N^lN^p\\
&+\hat{N}^{i\prime}(\ln h)_{,i} - \tfrac{1}{2}h'g_{ij}\hat{N}^{i\prime}N^j - \tfrac{1}{2}h'(\ln m^2)_{,i}N^i - 2g^{ij}\Gamma^u_{iu}\Gamma^u_{ju} + h'N^i\Gamma^u_{iu}\\
&+\hat{N}^{i\prime}\Gamma^u_{iu} + 4N^lg^{ji}\hat{\Gamma}^u_{li}\Gamma^u_{ju} - 2g^{ij}\Gamma^u_{iu}\Gamma^r_{jr} + 3g^{ij}(\ln m^2)_{,j}\Gamma^u_{iu} - 2g^{ij}\Gamma^r_{ir}\Gamma^r_{jr}\\
&+(h'N^i - hN^{u})\Gamma^r_{ir} + 3g^{ij}(\ln m^2)_{,j}\Gamma^r_{ir} + 2g^{ij}N^l\hat{\Gamma}^u_{lj}\Gamma^r_{ir} + 2\mathfrak{C}_T\}\big)\Big]\\
\times\delta(h-h_0)\delta(A^a_r)\Delta p_A\, m^{-8}h^{-12}\hbar^{16}(e^{12})^{-2}DA^a_\mu Df^{b\beta\gamma}D\hat{\Gamma}^u_{ij}D\hat{\Gamma}^r_{ij}\,&D\Gamma^u_{iu}D\Gamma^r_{jr}Dg^{ij}D\hat{N}^l DhD\hat{n}\,.\end{aligned} \tag{25}$$

Our next step is to introduce the new independent variables

$$\hat{p}_{ij} \equiv -m^2\hat{\Gamma}^u_{ij}\,,\quad \hat{\sigma}_{ij} \equiv -m^2\hat{\Gamma}^r_{ij}\,, \tag{26a}$$

$$N^i \equiv h^{-1}\hat{N}^i\,, \tag{26b}$$

$$g^{11} \equiv m^2e^{11},\ g^{12} \equiv m^2e^{12},\ g^{22} \equiv m^2[1 + (e^{12})^2]/e^{11}\,, \tag{27}$$

$$\Gamma^u_{iu} \equiv \gamma_i + \beta_i\,,\quad \Gamma^r_{ir} \equiv \gamma_i - \beta_i\,, \tag{28}$$

where γ_i and β_i have been chosen in order to decouple Γ^u_{iu} from Γ^r_{rr}. This change of variables has the Jacobian

$$D\hat{\Gamma}^u_{ij}D\hat{\Gamma}^r_{ij}D\hat{N}^lDg^{ij}D\Gamma^u_{iu}D\Gamma^r_{ir} = m^{-4}h^2(e^{11})^{-1}D\hat{p}_{ij}D\hat{\sigma}_{ij}DN^iDe^{11}De^{12}Dm^2D\gamma_jD\beta_l\,. \tag{29}$$

In this stage we can get rid of the two exact Gaussians by integration with respect to γ_i and β_i. We obtain two residual phases and a factor $\sim m^{-4}$ stemming from the existence of the nonunit matrices g^{ij} in the exact Gaussians.

The matrix element (25) and the scalar constraint reach, respectively, the reduced forms

$$\begin{aligned}\langle f|S(TUV)|i\rangle = \int \exp\Big(\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}\{\kappa^2dA^V + d^4x[&\hat{p}_{ij}\dot{e}^{ij} - h'(\ln m^2)^{\cdot} - (\ln m^2)'\dot{h} - (\ln h)'\dot{h} + e^{ij}m^2_{,i}(\ln h)_{,j} + \hat{\sigma}_{ij}(e^{ij\prime} - 2h^{-1}\hat{p}^{ij})\\
&+2\mathfrak{C}_T + m^2e^{ij}\Gamma^p_{il}\Gamma^l_{jp}(e) - (m^2)_{,i}\lambda^i(e) + \tfrac{1}{2}e^{ij}(m^2)_{,i}(\ln m^2)_{,j}\\
&-2N^iD^{(e)}_j\hat{p}^j_i - N^ih(\ln m^2)'_{,i} + h'N^i(\ln m^2)_{,i} - N^ih(\ln h)'_{,i}\\
&+\frac{h^2}{2m^2}e_{ij}N^{i\prime}N^{j\prime}]\}\Big)\\
\times m^{-16}h^{-10}\hbar^{18}(e^{11})^{-1}(e^{12})^{-2}\,p_A\Delta\delta(h-h_0)\delta(A^a_r)&DA^a_\mu Df^{b\beta\gamma}D\hat{p}_{ij}D\hat{\sigma}_{ip}Dm^2De^{11}De^{12}DN^lDhD\hat{n}\end{aligned} \tag{30}$$

and

$$2\mathfrak{C}_T = 2\hbar h^{-2}\hat{p}^i_j\hat{p}^j_i + \hbar(\ln h)'^2 - 2\hbar(\ln h)'(\ln m^2)' - 2\hbar'(\ln h)' . \tag{31}$$

Then we start to consider the contribution of the vector action dA^V to the matrix element $\langle fS(TUV)|i\rangle$ in terms of the 2×2 intrinsic variables defined in Eqs. (10).

[We are considering the non-Abelian vector multiplet linked by the local gauge group SU(n); therefore the internal indices run from 1 to $n^2-1\equiv d$, the dimension of the gauge group.] We denote by $\mathfrak{C}_V$ the generator of the Coulomb constraint of the vector fields

$$\mathfrak{C}_V \equiv -A^{a\prime}_u e^a + A^a_u(\partial_i c^{ai} + gf^{abc}A^b_r e^c + gf^{abd}A^b_i c^{di}) . \tag{32}$$

The integration of (30) with respect to h, A^a_r, and b^a is completely straightforward. One obtains

$$\begin{aligned}\langle f|S(TUV)|i\rangle = \int \exp\Big(\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle} d^4x\Big\{&\hat{p}_{ij}\dot{e}^{ij} - h'(\ln m^2) - (\ln m^2)'\dot{h} - (\ln h)'\dot{h} + \hat{\sigma}_{ij}(e^{ij\prime} - 2h^{-1}\hat{p}^{ij}) + \kappa^2 c^{ai}\dot{A}^a_i \\ &+ 2\mathfrak{C}_T + \hbar\kappa^2 h^{-1}c^{ai}A^{a\prime}_i + \kappa^2 d^a_i(c^{ai} - e^{\,ij}A^{a\prime}_j) + \tfrac{1}{2}e^{ij}m^2_{,i}(\ln h^2m^2)_{,j} \\ &+ m^2\Gamma^p_{il}(e)\Gamma^l_{jp}(e)e^{ij} - m^2_{,i}\lambda^i(e) + \frac{\hbar^2}{2m^2}e_{ij}N^{i\prime}N^{j\prime} \\ &+ N^i[-2D_j\hat{p}^j_i + h'(\ln m^2)_{,i} - h(\ln hm^2)'_{,i} + \kappa^2 e^a A^{a\prime}_i + \kappa^2 c^{aj}\epsilon_{ji}A^a_{12}] + \kappa^2\mathfrak{C}_V \\ &- \tfrac{1}{2}\kappa^2 m^2 h^{-1}e^{a2} - \tfrac{1}{2}\kappa^2 m^2 h^{-1}(A^a_{12})^2\Big\}\Big) \\ \times\, \Delta\phi_A\,\kappa^{1-n^2}m^{n^2-17}&h^{-(n^2+19)/2}(e^{11})^{-1}(e^{12})^{-2}DA^a_i DA^a_u De^a \\ \times\, Dc^{ai}Dd^{aj}Dm^2De^{11}&De^{12}DN^iD\hbar D\hat{p}_{ij}D\hat{\sigma}_{lp}\hbar^{18+(n^2-1)/2} ,\end{aligned} \tag{33a}$$

where the vectorial constraint $\mathfrak{C}_V$ has been reduced to the expression

$$\mathfrak{C}_V = -A^{a\prime}_u e^a + A^a_u(\partial_i c^{ai} + gf^{abd}A^b_i c^{di}) \equiv A^a_u(e^{a\prime} + \epsilon^a) - (A^a_u e^a)' . \tag{33b}$$

V. CONNECTION BETWEEN THE LAGRANGIAN MEASURE AND THE ACTION MEASURE

If we go on in this way, we have to integrate the remaining components of $f^{a\mu\nu}$ and $\Gamma^\alpha_{\mu\nu}$ in the expression (33): $\{e^a, c^{ai}, d^{aj}, \hat{p}_{ij}, \hat{\sigma}_{lp}\}$. This can easily be done because each of these integrations involves only either quadratic or linear exponentials in the integrating variables.

More precisely, integrations with respect to d^a_i and $\hat{\sigma}_{ij}$ induce us to proceed with the integration of c^{ai} and $\hat{p}^{ij}$, and thereafter one can integrate with respect to e^a, obtaining a Gaussian residue.

Symbolically, we have

$$\int (33\text{a})\, Dd^a_i D c^{aj} = \kappa^{-4(n^2-1)}\hbar^{2(n^2-1)}(33\text{a})(c^{ai} = e^{ij}A^{a\prime}_j) . \tag{34}$$

Taking into account the symmetric traceless character of both $\hat{\sigma}_{ij}$ and $(e^{ij\prime} - 2h^{-1}\hat{p}^{ij}) \equiv \hat{\alpha}^{ij}$, we find for the bilinear expression $\hat{\sigma}\cdot(e' - 2h^{-1}\hat{p})$ the explicit form

$$\hat{\sigma}\cdot(e - 2h^{-1}p) \equiv \hat{\sigma}_{ij}\hat{\alpha}^{ij} = -\frac{\hat{\sigma}_{11}}{2(e^{12})^2}\{\hat{\alpha}^{11}[1-(e^{12})^2] + \hat{\alpha}^{22}(e^{11})^2\} - \frac{\hat{\sigma}_{22}}{2(e^{12})^2}\{\hat{\alpha}^{11}(e^{22})^2 + \hat{\alpha}^{22}[1-(e^{12})^2]\} . \tag{35a}$$

Thus, the integration with respect to $\hat{\sigma}_{11}, \hat{\sigma}_{22}$ gives two δ functionals

$$\int (34) D\hat{\sigma}_{11}D\hat{\sigma}_{22} \cong (34)(e^{12})^2\hbar^2\delta(e^{11\prime} - 2h^{-1}\hat{p}^{11})\delta(e^{22\prime} - 2h^{-1}\hat{p}^{22}) , \tag{35b}$$

which suggest following the process with the integration of $\hat{p}^{ij}$, which in fact can be easily performed

$$\int (35\text{b}) D\hat{p}^{ij} \cong h^2(35\text{b}) \quad (\hat{p}^{ij} = 2^{-1}h\,e^{ij\prime}) . \tag{36}$$

In order to reach a second-order (Lagrangian) formulation for the matrix element $\langle f|S|i\rangle$, we have still to perform the integration with respect to e^a, which appears quadratically in the matrix element (33a), and then leads us to integrate an exact Gaussian plus a residual phase.

In this way we found the second-order expression

$$\langle f|S(TUV)|i\rangle=\int \exp\Big(\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle} d^4x\Big\{\tfrac{1}{2}e_{il}e_{jm}h e'^{lm}\dot{e}^{ij}-h'(\ln m^2)^{\cdot}-(\ln hm^2)'\dot{h}$$
$$+\kappa^2 e^{ij}A_i^{a\prime}A^{\cdot a}_j+2\mathfrak{C}_T+\kappa^2\hat{n}h^{-1}e^{ij}A_i^{a\prime}A_j^{a\prime}+\tfrac{1}{2}e^{ij}m^2_{,i}(\ln h^2m^2)_{,j}$$
$$+m^2\Gamma^p_{il}(e)\Gamma^l_{jp}(e)e^{ij}-m^2_{,i}\lambda^i(e)+\frac{h^2}{2m^2}e_{ij}N^{i\prime}N^{j\prime}$$
$$+N^i[-2D_j\hat{p}^j_i+h'(\ln m^2)_{,i}-h(\ln hm^2)'_{,i}+\kappa^2e^{jl}A_l^{a\prime}\epsilon_{ji}A^a_{12}]$$
$$+\kappa^2A^a_u(\partial_i c^{ai}+gf^{abd}A^b_ic^{di})+\tfrac{1}{2}\kappa^2m^{-2}h(A_u^{a\prime}-N^iA_i^a)^2-\tfrac{1}{2}\kappa^2m^2h^{-1}(A^a_{12})^2\Big\}\Big)$$
$$\times\Delta p_A\kappa^{-6(n^2-1)}\hbar^{3(n^2-1)+20}(e^{11})^{-1}(hm^2)^{-8}DA^a_iDA^b_uDm^2De^{11}De^{12}DN^iD\hat{n}\,. \quad (37)$$

Should we have started with the Lagrangian (second order)

$$\langle f|S(TUV)|i\rangle=\int\exp\left[\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}dL(\mathfrak{g};A)\right]d\nu(TUV)\,, \quad (38a)$$

with the Lagrangian measure having the form

$$d\nu(TUV)=\delta(G^A)\Delta'(G^A)p_L(\text{fields})D\mathfrak{g}^{\mu\nu}DA^a_\mu\,, \quad (38b)$$

we should have found the same phase, with the same reduced density as we have in Eq. (37), times an apparently slightly different functional factor stemming from the fact that in this case we had only to perform integrations with respect to A^a_r and $\mathfrak{g}^{u\mu}$.

The gauge conditions are the same already considered for the first-order calculation; they are described by Eqs. (13). As they depend only upon the coordinate fields $(\mathfrak{g}^{\mu\nu},A^a_0)$, their Faddeev-Popov determinant $\Delta'(G^A)=\Delta(G^A)$.

That means that Eq. (38a), after integration with respect to DA^a_r, $D\mathfrak{g}^{u\mu}$ shall achieve the form

$$\langle f|S(TUV)|i\rangle'\cong\int\exp\left[\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}dL(e^{ij};m^2;N^i;\hat{n},A^a_j,A^a_u)\right]\Delta p_L\frac{h^2(m^2)^2}{e^{11}}De^{11}De^{12}Dm^2DN^iD\hat{n}DA^a_iDA^a_u\,. \quad (39)$$

As expressions (37) and (39) must give the same value, p_L and p_A are linked through this condition. We obtain

$$p_L=\left(\frac{\hbar^2}{hm^2}\right)^{10}p_A\left(\frac{\hbar}{\kappa^2}\right)^{3(n^2-1)}$$
$$=[\hbar^2(-\mathfrak{g})^{-1/2}]^{10}(\hbar\kappa^{-2})^{3(n^2-1)}p_A\,. \quad (40)$$

VI. THE EVALUATION OF THE MEASURE

In this section we shall perform the direct evaluation of the action measure, following the work of Senjanovic[6] for systems which only have second-class constraints.

In the first step, we integrate with respect to N^i. Using the standard definition of $(\partial_r^{-n})_{xy}$,[3]

$$(\partial_r^{-n})xy\equiv[(n-1)!]^{-1}(x-y)^{n-1}\tfrac{1}{2}\epsilon(x-y)\,, \quad (41)$$

we introduce the vector π_i,

$$\pi_i\equiv\partial^{-1}[-2D_j^{(e)}\hat{p}^j_i+h'(\ln m^2)_{,i}-h(\ln hm^2)'_{,i}+\kappa^2e^aA_i^{a\prime}+\kappa^2c^{aj}\epsilon_{ji}A^a_{12}]\,. \quad (42)$$

Then we obtain the result that the terms depending upon N^i can be transformed into the addition of a quadratic form plus a "constant" complement, allowing us to carry out the integration we wanted:

$$\int\exp\left[\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}d^4x\left(\frac{h^2}{2m^2}e_{ij}N^{i\prime}N^{j\prime}+N^i\pi_i'\right)\right]DN^i=\int\exp\left\{\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}d^4x\left[\frac{h^2}{2m^2}e_{ij}N^{i\prime}N^{j\prime}-\pi_iN^{i\prime}+(\pi_iN^i)'\right]\right\}DN^i$$
$$=|\partial_r^{-1}|^2\frac{\hbar m^2}{h^2}\exp\left[\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}d^4x\left(-\frac{1}{2}\frac{m^2}{h^2}e^{ij}\pi_i\pi_j\right)\right]$$
$$\times\exp\left[\frac{i}{\hbar}(\pi_iN^i)|^{|f\rangle}_{|i\rangle}\right]. \quad (43)$$

One can also perform the integrations with respect to A_u^a, e^a, $\hat{n}$, and m^2. In fact, $\hat{n}$ and A_u^a are essentially Lagrange multipliers linked with the scalar-tensor constraint $\mathfrak{C}_{TUV}$ and the Coulomb vectorlike constraint $\mathfrak{C}_V$, respectively:

$$\hat{\mathfrak{C}}_{TUV} \equiv (\ln m^2)'(\ln h)' - \tfrac{1}{2}(\ln h)'^2 - h^{-2}\hat{p}^i{}_j\hat{p}^j{}_i - (\ln h)'' - \tfrac{1}{2}\kappa^2 h^{-1}c^{ai}A_i^{a\prime} = 0\,, \tag{44}$$

$$\hat{\mathfrak{C}}_V \equiv e^{a\prime} + \epsilon^a(A_i^a; c^{bj}) = 0\,. \tag{45}$$

Therefore, the integration with respect to e^a and m^2 provides for the matrix element (33a) the value $[Dm^2 = m^2 D(\ln m^2)]$

$$\begin{aligned}\langle f|S(TUV)|i\rangle = \int \exp\Big(\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle} d^4x\,[\hat{p}_{ij}\dot{e}^{ij} - h'(\ln m^2)^{\cdot} - (\ln m^2)'\dot{h} \\ -(\ln h)'\dot{h} + \tfrac{1}{2}e^{ij}m^2_{,i}(\ln h^2 m^2)_{,j} + \hat{\sigma}_{ij}(e^{ij\prime} - 2h^{-1}\hat{p}^{ij}) + \kappa^2 c^{ai}\dot{A}_i^a \\ +\kappa^2 d_i^a(c^{ai} - e^{ij}A_j^{a\prime}) + m^2 e^{ij}\Gamma^p_{li}(e)\Gamma^l_{jp}(e) - m^2_{,i}\lambda^i(e) - \tfrac{1}{2}(m^2/h^2)e^{ij}\pi_i\pi_j \\ -\tfrac{1}{2}\kappa^2 m^2 h^{-1}(\partial_r^{-1}\epsilon^a)^2 - \tfrac{1}{2}\kappa^2 m^2 h^{-1}(A_{12}^a)^2]\Big) \\ \times \exp\Big(\frac{i}{\hbar}[\pi_i N^i - A_u^a e^a - 2\hat{n}(\ln h)']\Big|_{|i\rangle}^{|f\rangle}\Big)\Delta p_A\, \kappa^{-3(n^2-1)}\hbar^{20+(3/2)(n^2-1)}|\partial_r^{-1}|^{2+n^2} m^{n^2-13}\frac{h^{-(n^2-1)/2-12}}{(\ln h)'} \\ \times (e^{11})^{-1}(e^{12})^{-2}DA_i^a Dc^{bj}Dd_i^c De^{11}De^{12}D\hat{p}_{ij}D\hat{\sigma}_{lp}\,,\end{aligned} \tag{46a}$$

where $\ln m^2$ means

$$\ln m^2 = (\partial_r^{-1})\{[\ln(\ln h)']' + \tfrac{1}{2}(\ln h)' + h^{-1}(h')^{-1}\hat{p}^i_j\hat{p}^j_i + \tfrac{1}{2}\kappa^2(h')^{-1}c^{ai}A_i^{a\prime}\}\,. \tag{46b}$$

To reach the typical structure of a second-class constrained Hamiltonian system, we parametrize the unit contravariant tensor e^{ij} in the following way:

$$e^{ij} = \begin{bmatrix} 1+\kappa q_1 & \kappa q_2 \\ \kappa q_2 & (1+\kappa^2 q_2^2)(1+\kappa q_1)^{-1}\end{bmatrix}\,, \tag{47a}$$

$$e_{ij} = \begin{bmatrix} (1+\kappa^2 q_2^2)(1+\kappa q_1)^{-1} & -\kappa q_2 \\ -\kappa q_2 & 1+\kappa q_1\end{bmatrix}\,. \tag{47b}$$

Then we explicitly calculate $\hat{p}_{ij}\dot{e}^{ij} + \hat{\sigma}_{ij}(e^{ij\prime} - 2h^{-1}e^{il}e^{jm}\hat{p}_{lm})$. In order to attain the canonical structure for the dynamical germ $\hat{p}_{ij}\dot{e}^{ij}$, we have to change the independent variables $\hat{p}_{11}, \hat{p}_{22}$ and define new momenta p_1, p_2 associated to q_1 and q_2. This redefinition of the momenta suggests the introduction of new variables σ_1, σ_2 to replace $\hat{\sigma}_{11}$ and $\hat{\sigma}_{12}$ (or $\hat{\sigma}_{22}$) in order to ease the expression of the gravitational second-class constraints.

More precisely, the expression

$$\int \exp\Big\{\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}[\hat{p}_{ij}\dot{e}^{ij} + \hat{\sigma}_{ij}(e^{ij\prime} - 2h^{-1}e^{il}e^{jm}\hat{p}_{lm})]d^4x\Big\}D\hat{p}_{11}D\hat{p}_{22}D\hat{\sigma}_{11}D\hat{\sigma}_{22} \tag{48a}$$

is transformed into

$$\begin{aligned}(48a) \equiv \int \exp\Big(\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}d^4x\Big\{p_1\dot{q}_1 + p_2\dot{q}_2 + \sigma_1\Big[p_1 - \frac{\kappa^2 h(1+\kappa^2 q_2^2)q_1'}{(1+\kappa q_1)^2} + \frac{\kappa^3 h q_2}{(1+\kappa q_1)}q'_2\Big] \\ +\sigma_2\Big[p_2 + \frac{h\kappa^3 q_2 q'_1}{1+\kappa q_1} - h\kappa^2 q'_2\Big]\Big\}\Big)\, h^2(e^{12})^2 Dp_1 Dp_2 D\sigma_1 D\sigma_2\,,\end{aligned} \tag{48b}$$

where p_1, p_2, σ_1, and σ_2 are given by

$$p_1 \equiv 2h^{-1}\kappa\hat{p}_{11} + \frac{2\kappa^2 h^{-1} q_2}{1+\kappa q_1}\hat{p}_{12}, \quad p_2 \equiv -2\kappa^2 h^{-1} q_2 \frac{(1+\kappa q_1)}{1+\kappa^2 q_2^2}\hat{p}_{11} + 2\kappa h^{-1}\frac{(1-\kappa^2 q_2^2)}{1+\kappa^2 q_2^2}\hat{p}_{12},$$
$$\hat{\sigma}_{11} \equiv -\tfrac{1}{2}h\kappa\frac{(1+\kappa^2 q_2^2)}{(1+\kappa q_1)^2}\sigma_1 + h\kappa^2\frac{q_2}{1+\kappa q_1}\sigma_2, \quad \sigma_{12} \equiv -\tfrac{1}{2}h\kappa\sigma_2. \tag{48c}$$

It is also convenient to absorb $h^{-1}\kappa^2$ into c^{ai} (keeping the same notation for the new variables $h^{-1}\kappa^2 c^{ai}$) and transform the term $\sim h'(\ln m^2)^{\cdot}$ appearing in (46a) in a way that clearly shows the nondynamical character of the variable m^2, as it has been discussed in Ref. 1.[16]

In terms of these new variables, the expression (46a) becomes

$$\begin{aligned}
\langle f|S(TUV)|i\rangle = \int \exp\Bigg(i\int_{|i\rangle}^{|f\rangle} d^4x\Bigg\{ & p_1\mathring{q}_1 + p_2\mathring{q}_2 + \sigma_1\left[p_1 - \frac{h\kappa^2}{\hbar}\frac{(1+\kappa^2 q_2^2)}{(1+\kappa q_1)^2}q_1' + h\frac{\kappa^3}{\hbar}\frac{q_2 q'_2}{(1+\kappa q_1)}\right] \\
& + \sigma_2\left[p_2 + h\frac{\kappa^3}{\hbar}\frac{q_2}{1+\kappa q_1}q'_1 - h\frac{\kappa^2}{\hbar}q'_2\right] \\
& -2\hbar^{-1}\mathring{h}(\ln m^2)' - \hbar^{-1}\mathring{h}(\ln h)' + c^{ai}A_i^{a\cdot} + d_i^a\left(c^{ai} - \frac{\kappa^2}{\hbar}e^{ij}A_j^{a\prime}\right) \\
& + \frac{1}{2\hbar}e^{ij}m^2_{,i}(\ln h^2 m^2)_{,j} + \hbar^{-1}m^2 e^{ij}\Gamma^p_{il}\Gamma^l_{jp}(e) - \hbar^{-1}m^2_{,i}\lambda^i(e) \\
& - \frac{1}{2}\hbar^{-1}\frac{m^2}{h^2}e^{ij}\pi_i\pi_j - \frac{1}{2}\frac{\kappa^2}{\hbar}m^2 h^{-1}(\partial_r^{-1}\epsilon^a)^2 - \frac{1}{2}\frac{\kappa^2}{\hbar}m^2 h^{-1}(A^a_{12})^2\Bigg\}\Bigg) \\
\times \exp&\left\{\frac{i}{\hbar}\int_{\Sigma_r}[\pi_i N^i - A^a_u e^a - 2\hat{n}(\ln h)' - h(\ln m^2)^{\cdot}]\,d^2x\,du\Big|_{r_1}^{r_2}\right\} \\
\times \exp&\left[\frac{i}{\hbar}\int_{\Sigma_u} h(\ln m^2)' d^2x\,dr\Big|_{u_1}^{u_2}\right]\Delta p_A \kappa^{2-5(n^2-1)}\hbar^{22+(5/2)(n^2-1)} \\
\times |\partial_r^{-1}&|^{n^2+2} m^{n^2-13}\frac{h^{-10-(n^2-1)/2}}{(\ln h)'}(e^{11})^{-1}DA_i^a Dc^{bj}Dd^{cl}Dq_1 Dq_2 Dp_1 Dp_2 D\sigma_1 D\sigma_2,
\end{aligned} \tag{49}$$

where the dynamical germ has the canonical bilinear structure $p_i\dot{q}_i + c^{ai}A_i^{a\cdot}$ in terms of the proper degrees of freedom (q_i, A_j^a) and the corresponding momenta (p_i, c^{aj}). As always happens in the light-front dynamics, the whole set of momenta enter associated with the coordinates through second-class constraints S_i, D^{ai} whose multipliers are, respectively, σ_i and d_i^a:

$$D^{ai} \equiv c^{ai} - \kappa^2\hbar^{-1}e^{ij}A_j^{a\prime} = 0, \tag{50a}$$

$$S_i \equiv p_i - \frac{\kappa^2 h}{\hbar}\frac{e_{ij}}{(1+\kappa q_1)}q_j' = 0. \tag{50b}$$

At this point let us recall that the Poisson brackets of two functions f_1, f_2 of the canonical variables (q^A, p_A) is usually defined by

$$\{f_1, f_2\} \equiv \frac{\partial f_1}{\partial q^A}\frac{\partial f_2}{\partial p_A} - \frac{\partial f_1}{\partial p_A}\frac{\partial f_2}{\partial q^A}, \tag{51a}$$

where the generalized index A runs over the gravitational and isovector coordinates

$$(q^A; p_A) \equiv (q^{ix,r'}(u);\ A^a_{jyr''}(u)/p_{jxr'}(u),\ c^a_{jyr''}(u)). \tag{51b}$$

Moreover, we shall take into account that the one-dimensional δ function satisfies

$$\partial_x\delta(x-y) = -\partial_y\delta(y-x) = -\partial_y\delta(x-y) = \partial_x\delta(y-x). \tag{52}$$

We need these two results in order to compute the Poisson brackets between pairs of second-class constraints which constitute the entries of the functional determinant whose value determines the measure we are looking for.

They turn out to be

$$\begin{aligned}
&\{S_i(xr',u)D^{bj}(yr'',u)\} \equiv C_i^{bj}(xr',yr'',u), \\
&\{S_i(xr',u), S_j(yr'',u)\} \equiv C_{ij}(xr',yr'',u), \\
&\{D^{ai}(xr',u), D^{bj}(yr'',u)\} \equiv C^{aibj}(xr',yr'',u).
\end{aligned} \tag{53a}$$

$$\{S_1(xr',u),S_1(yr'',u)\}=-[\kappa^2h\hbar^{-1}(1+\kappa q_1)^{-1}e_{11}](xr,u)\delta(x-y)\partial_{r'}\delta(r'-r'')$$
$$+[\kappa^2h\hbar^{-1}(1+\kappa q_1)^{-1}e_{11}](yr'',u')\partial_{r''}\delta(r''-r')\delta(x-y),$$
$$\{S_1(xr',u),S_2(yr'',u)\}=[-\kappa^4h\hbar^{-1}(1+\kappa q_1)^{-2}q_2q_1'+\kappa^3h\hbar^{-1}(1+\kappa q_1)^{-1}q'_2](xr',u)\delta(x-y)\delta(r'-r'')$$
$$+\{[\kappa^3h\hbar^{-1}q_2(1+\kappa q_1)^{-1}](xr',u)\partial_{r'}\delta(r'-r'')-[\kappa^3h\hbar^{-1}q_2(1+\kappa q_1)^{-1}](yr'',u)\partial_{r''}\delta(r'-r'')\}\,\delta(x-y),$$
$$\{S_2(xr',u),S_1(yr'',u)\}=[\kappa^4h\hbar^{-1}q_2(1+\kappa q_1)^{-2}q'_1-\kappa^3h\hbar^{-1}(1+\kappa q_1)^{-1}q'_2](xr',u)\delta(x-y)\delta(r'-r'')$$
$$+\{[\kappa^3h\hbar^{-1}q_2(1+\kappa q_1)^{-1}](xr',u)\partial_{r'}\delta(r'-r'')-[\kappa^3h\hbar^{-1}q_2(1+\kappa q_1)^{-1}](yr'',u)\partial_{r''}\delta(r''-r')\}\delta(x-y),$$
$$\{S_2(xr',u),S_2(yr'',u)\}=-(\kappa^2h\hbar^{-1})(xr',u)\,\partial_{r'}\delta(r'-r'')\delta(x-y)\ +(\kappa^2h\hbar^{-1})(yr'',u)\partial_{r''}\delta(r''-r')\,\delta(x-y),\tag{53b}$$
$$\{S_1(xr',u),D^{b1}(yr'',u)\}=\kappa^3\hbar^{-1}A_1^{b\prime}(xr',u)\delta(y-y)\delta(r'-r''),$$
$$\{S_1(xr',u),D^{b2}(yr'',u)\}=-\kappa^3\hbar^{-1}(1+\kappa^2q_2^2)(1+\kappa q_1)^{-1}A_2^{b\prime}(xr',u)\delta(x-y)\delta(r'-r''),$$
$$\{S_2(xr',u),D^{b1}(yr'',u)\}=\kappa^3\hbar^{-1}A_2^{b\prime}(xr',u)\delta(x-y)\delta(r'-r''),$$
$$\{S_2(xr',u),D^{b2}(yr'',u)\}=\kappa^3\hbar^{-1}[A_1^{b\prime}(xr',u)+2\kappa q_2(1+\kappa q_1)^{-1}A_2^{b\prime}]\delta(x-y)\delta(r'-r''),$$
$$\{D^{ai}(xr'',u),D^{bj}(yr'',u)\}=\kappa^2\hbar^{-1}[e^{ij}(yr'',u)\partial_{r''}\delta(r''-r')-e^{ij}(xr',u)\partial_{r'}\delta(r'-r'')]\delta^{ab}\delta_2(x-y).$$

After the analysis given by Yabuki[17] and Senjanović[6] for Hamiltonian systems with second-class constraints, and in accordance with the calculations recently performed by Nakano[18] for nonlinear models quantized along null planes, the functional measure corresponding to the second-class constrained action $\sim p_A\dot{q}^A+\sigma_A D^A-H$ is given by the square root of the determinant of the Poisson brackets $\{D^A(xr',u), D^B(yr'',u)\}$.

In the specific problem we are concerned with here, the second-class constraints are Eqs. (50). We want to compute

$$d\mu(p_i;q^i;c^{ai};A_i^a)=\left\{\det\begin{bmatrix}C_{ij}(xr',yr'',u) & C_i^{bj}(xr',yr'',u)\\ C_j^{ai}(xr',yr'',u) & C^{aibj}(xr',yr'',u)\end{bmatrix}\right\}^{1/2}\delta(S_i)\delta(D^{ai})Dp_iD_q^iDc^{ai}DA_j^b.\tag{54}$$

After the values obtained in (53b) for the elements of this determinant, and owing to the simplicity of the constraints (50), the measure (54) turns out to be

$$d\mu(p_i;q^i;c^{ai};A_j^a)=\det(\kappa^2\hbar^{-1}e^{ij})^{(n^2-1)/2}[\det(\kappa^2\hbar^{-1}h)e_{ij}(1+\kappa q_1)^{-1}]^{1/2}\delta(S_i)\delta(D^{ai})Dp_iDq^iDc^{ai}DA_j^b$$
$$=(\kappa^2\hbar^{-1})^{n^2}h(1+\kappa q_1)^{-1}\delta(S_i)\delta(D^{ai})Dp_iDq^iDc^{ai}DA_j^b.\tag{55}$$

This result determines p_A. In fact, the functional measure appearing in Eq. (49) has to be identified with the "canonical" measure (55). Then the action measure introduced in the matrix element (1) will have the form

$$\Delta p_A=h^{-23}(\kappa^2\hbar^{-1})^{(7/2)(n^2-1)}(hm^{-2})^{(1/2)(n^2-1)-6}h^{16}h'.\tag{56}$$

Using the fact (detailed in Appendix B) that

$$\Delta=h^3h'|\partial_r|^{n^2+2},\tag{57}$$

we obtain for p_A

$$p_A=\hbar^{-23}(\kappa^2\hbar^{-1})^{(7/2)(n^2-1)}(hm^2)^{6-(1/2)(n^2-1)}h^{n^2}.\tag{58}$$

Therefore, taking into account Eq. (40), one immediately obtains the corresponding Lagrangian measure p_L:

$$p_L=(\hbar hm^2)^{-3+(n^2-1)/2}(\kappa^2\hbar^{-1})^{(n^2-1)/2}m^{-2n^2},\tag{59}$$

which, for $n^2-1=0$, exactly coincides with the result first given by Fradkin and Vilkovisky[19] for the pure gravity system and thereafter confirmed by the calculations of Kaku and Senjanović in Ref. 3.

The measures p_A and p_L can also be expressed showing their covariant structure. In fact, $\det\ g^{\mu\nu}\equiv g=-h^2m^4=\det g_{\mu\nu}$, and introducing a pair of associate null vectors (in the momentum space) $\eta_{(u;r)\mu}\equiv-\delta_{(r;u)\mu}$ we have ($n^2-1\equiv d$)

$$p_A=\hbar^{-23}(\kappa^2\hbar^{-1})^{(7/2)d}(-g)^{3-(1/4)d}(-g^{\mu\nu}\eta_{\mu\nu}\eta_{r\nu})^{d+1}\tag{60}$$

and

$$p_L=\hbar^{-3}(-g^{\mu\nu}\eta_{u\mu}\eta_{r\nu})\,g^{-2}[\kappa(-g^{\mu\nu}\eta_{u\mu}\eta_{r\nu})\times(-g)^{-1/4}]^d.\tag{61}$$

VII. DISCUSSION AND COMMENTS

After defining the action measure p_A and the Lagrangian measure p_L we reviewed the geometry of both the tensor-Einstein field and the vector Yang-Mills field on the null coordinates.

The connection between the new gauge $\mathfrak{g}^{u\mu} = h_0\eta^{u\mu}$ used in this article and both the Kaku family and Bondi choice was established. Later on, dynamical reasons were also given.

Then the Feynman integration process was performed, starting from the first-order covariant action of the system.

Through convenient choices of 2×2 variables, the system was considerably decoupled and a lot of functional integrations easily performed, leading us to establish the connection between the measures p_A and p_L.

Proceeding along slightly different lines, a reduced first-order canonical null structure for the action of the system appeared, with the whole set of second-class constraints expressed in a closed and natural form.

This allowed us to make use of the heuristic results of Senjanović in order to compute the Poisson brackets of the Hamiltonian system and thereafter the measure.

The Einstein field was parametrized using rational functions of its dynamical coordinates and their respective momenta defined in a natural way.

The ghosts' contribution to the S-matrix element was explicitly given, and contrary to what happens in the case of pure gravity in the Kaku subfamily of gauges, it was found not to be constant.

Finally, the results of p_A and p_L were explicitly given, and it was shown that they agree with the results given by Fradkin and Vilkovisky when the isovector field vanishes.

APPENDIX A

In this part we shall calculate the integral

$$\int \exp\left\{\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}[\mathfrak{g}^{ij}\hat{\Gamma}^p_{il}\hat{\Gamma}^l_{jp}+\mathfrak{g}^{ij}_{,l}\hat{\Gamma}^l_{ij} + (N^i\mathfrak{g}^{jp}+N^j\mathfrak{g}^{ip})\hat{\Gamma}^u_{lp}\hat{\Gamma}^l_{ij}]d^4x\right\}D\hat{\Gamma}^l_{ij}. \tag{A1}$$

We want to transform this expression in the product of an exact Gaussian integral (which we know how to evaluate) times a factor which will likely depend upon the coefficients $\mathfrak{g}^{ij}_{,l}+(N^i\mathfrak{g}^{jp}+N^j\mathfrak{g}^{ip})\hat{\Gamma}^u_{pl}$ of the term linear in $\hat{\Gamma}^l_{ij}$, the variables we are integrating. The quadratic exponent appearing in Eq. (A1) has the form

$$\mathfrak{g}^{ij}\hat{\Gamma}^p_{il}\hat{\Gamma}^l_{jp}+\Lambda^{il}_p\hat{\Gamma}^p_{il}\equiv\mathfrak{g}^{ij}\hat{\Gamma}^p_{il}\hat{\Gamma}^l_{jp}+\hat{\Lambda}^{il}_p\hat{\Gamma}^p_{il}, \tag{A2}$$

where

$$\Lambda^{il}_p\equiv\mathfrak{g}^{il}_{,p}+(N^i\mathfrak{g}^{lq}+N^l\mathfrak{g}^{iq})\hat{\Gamma}^u_{pq}, \tag{A3}$$

and $\hat{\Lambda}^{il}_p$ is the symmetric, pure traceless, third-rank object

$$\hat{\Lambda}^{il}_p\equiv\Lambda^{il}_p-\tfrac{1}{2}\delta^i_p\Lambda^l-\tfrac{1}{2}\delta^l_p\Lambda^i+\tfrac{1}{2}\mathfrak{g}^{il}\Lambda_p+\tfrac{1}{4}\delta^i_p\mathfrak{g}^{lq}\mu_q + \tfrac{1}{4}\delta^l_p\mathfrak{g}^{iq}\mu_q-\tfrac{3}{4}\mathfrak{g}^{il}\mu_p, \tag{A4}$$

with

$$\Lambda^i\equiv\Lambda^{il}_l,\ \mu_q\equiv\mathfrak{g}_{ij}\Lambda^{ij}q. \tag{A5}$$

Introducing the factorization of $\mathfrak{g}^{ij}$ as $\mathfrak{g}^{ij}\equiv m^2e^{ij}$, $\det e^{ij}=+1$, one finds that $\hat{\Lambda}^{il}_p$ is given by

$$\begin{aligned}\hat{\Lambda}^{il}_p&=m^2e^{il}_{,p}+\tfrac{1}{2}m^2\delta^i_p\lambda^l+\tfrac{1}{2}m^2\delta^l_p\lambda^i\\&-\tfrac{1}{2}m^2e^{il}e_{pq}\lambda^q+m^2N^ie^{lq}\hat{\Gamma}^u_{qp}\\&+m^2N^le^{iq}\hat{\Gamma}^u_{qp}-m^2e^{il}N^s\hat{\Gamma}^u_{sp},\end{aligned} \tag{A6}$$

with λ^i defined by

$$\lambda^i\equiv-e^{iq}_{,q}=+e^{pq}\Gamma^i_{pq}(e). \tag{A7}$$

The first derivatives $e^{ij}_{,l}$ can also be expressed through the Riemannian affinities $\Gamma(e)$

$$e^{ij}_{,l}=-e^{ip}\Gamma^j_{pl}(e)-e^{jp}\Gamma^i_{pl}(e). \tag{A8}$$

The classical value ${}_0\hat{\Gamma}^l_{ij}$ for the affinity $\hat{\Gamma}^l_{ij}$ can easily be expressed in terms of $\hat{\Lambda}^{il}_p$:

$$2{}_0\hat{\Gamma}^l_{ij}\equiv\mathfrak{g}_{ip}\mathfrak{g}_{jq}\mathfrak{g}^{lr}\hat{\Lambda}^{pq}_r-\mathfrak{g}_{ip}\hat{\Lambda}^{pl}_j-\mathfrak{g}_{jp}\hat{\Lambda}^{pl}_i. \tag{A9}$$

Then the nonexact quadratic expression (A2) can be shown to be identical to the addition of two exact quadratic forms

$$\begin{aligned}(\text{A2})&\equiv\mathfrak{g}^{ij}(\hat{\Gamma}-{}_0\hat{\Gamma})^p_{il}(\Gamma-{}_0\hat{\Gamma})^l_{jp}-\mathfrak{g}^{ij}{}_0\hat{\Gamma}^p_{il0}\hat{\Gamma}^l_{jp}\\&\equiv\mathfrak{g}^{ij}(\hat{\Gamma}-{}_0\hat{\Gamma})^p_{il}(\hat{\Gamma}-{}_0\hat{\Gamma})^l_{jp}-\tfrac{1}{2}\hat{\Lambda}^{ij}{}_{..l}\hat{\Lambda}_{i}{}^{.l}{}_{.j}\\&+\tfrac{1}{4}\hat{\Lambda}^{.ij}{}_{.,l}\hat{\Lambda}_{ij}{}^{.l}_{.}.\end{aligned} \tag{A10}$$

Substituting in this equation the value of $\hat{\Lambda}^{ij}_l$ found in (A6), we have a new form for Eq. (A2):

$$\begin{aligned}(\text{A2})\equiv{}&\mathfrak{g}^{ij}(\hat{\Gamma}-{}_0\hat{\Gamma})^p_{il}(\hat{\Gamma}-{}_0\hat{\Gamma})^l_{jp}-m^2e^{ij}\Gamma^p_{il}(e)\Gamma^l_{jp}(e)-\tfrac{1}{4}m^2e_{ij}\lambda^i(e)\lambda^j(e)\\&-m^2\hat{\Gamma}^u_{ij}\lambda^i(e)N^j+2m^2e^{il}\hat{\Gamma}^u_{ij}\Gamma^j_{lp}N^p-m^2\hat{\Gamma}^u_{il}e^{lp}\hat{\Gamma}^u_{jp}N^iN^j.\end{aligned} \tag{A11}$$

Now we can perform the integration (A1). In fact, because of the symmetric, double traceless structure of both $\hat{\Gamma}^l_{ij}$ and ${}_0\hat{\Gamma}^l_{ij}$, if we introduce the new variables X_i,

$$X_1\equiv{}_0\hat{\Gamma}^1_{11}-\hat{\Gamma}^1_{11},\quad X_2\equiv\hat{\Gamma}^2_{22}-{}_0\hat{\Gamma}^2_{22}, \tag{A12}$$

expression (A11) becomes

$$(\mathrm{A11})=\frac{4m^2}{e^{11}e^{22}}\;e^{ij}X_iX_j-m^2e^{ij}\Gamma^p_{il}\Gamma^l_{jp}(e)-\tfrac{1}{4}m^2e_{ij}\lambda^i\lambda^j(e)-m^2\hat{\Gamma}^u_{ij}\lambda^i(e)N^j$$
$$+2m^2e^{il}\hat{\Gamma}^u_{lj}\Gamma^j_{ip}(e)N^p-m^2\hat{\Gamma}^u_{il}e^{lp}\hat{\Gamma}^u_{jp}N^iN^j\,. \tag{A13}$$

Therefore, the integration of (A1) with respect to $\hat{\Gamma}^l_{ij}=(\hat{\Gamma}^1_{11};\hat{\Gamma}^2_{22})$ gives a phase [stemming from the "constant" part of (A13)] times a factor arising from the exact Gaussian, i.e.,

$$\int\exp\left\{\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}d^4x\left[\frac{4m^2}{e^{11}e^{22}}\,e^{ij}X_iX_j\right]\right\}DX_i=\left[\det\left(\frac{4m^2}{\hbar e^{11}e^{22}}\,e^{ij}\right)\right]^{1/2}=\hbar\,\frac{e^{11}e^{22}}{4m^2}\,, \tag{A14}$$

and, consequently, we arrived at the result

$$(\mathrm{A1})=\hbar\,\frac{e^{11}e^{22}}{4m^2}\,\exp\left\{\frac{i}{\hbar}\int_{|i\rangle}^{|f\rangle}d^4x\,[-m^2e^{ij}\Gamma^p_{il}\Gamma^l_{jp}(e)-\tfrac{1}{4}m^2e_{ij}\lambda^i(e)\lambda^j(l)\right.$$
$$\left.-m^2\hat{\Gamma}^u_{ij}\lambda^i(e)N^j+2m^2e^{il}\hat{\Gamma}^u_{lj}\Gamma^j_{ip}(e)N^p-m^2\hat{\Gamma}^u_{il}e^{lp}\hat{\Gamma}^u_{jp}N^iN^j]\right\}. \tag{A15}$$

APPENDIX B

The value of $\Delta(G^A)$ can be obtained by direct computation after its definition.

In fact,

$$\Delta(G^A)\equiv\det[\delta G^u_\infty(x)/\delta\xi^\alpha(y),\;\delta G^a/\delta\xi^\alpha(y)\,,\;\delta G^a/\delta\eta^b(y)] \tag{B1}$$

These functional derivatives can be easily derived after the Lie derivatives of all the gauge conditions $G^A=0$, with respect to the infinitesimal vector $[\xi^\alpha(y),\eta^a(y)]$. It is completely straightforward to make these derivatives

$$\begin{aligned}F^\nu&\equiv\mathcal{L}_{(\xi;\eta)}\mathfrak{g}^{u\nu}=\mathcal{L}_\xi\mathfrak{g}^{u\nu}=-\xi^\sigma{}_{,\sigma}\mathfrak{g}^{u\nu}-\xi^\beta\partial_\beta\mathfrak{g}^{u\nu}\\&\quad+\mathfrak{g}^{u\beta}\xi^\nu{}_{,\beta}+\mathfrak{g}^{\beta\nu}\xi^u{}_{,\beta}\,,\\F^a&\equiv\mathcal{L}_{(\xi;\eta)}A^a=-\xi^\beta\partial_\beta A^a_u-A^a_\beta\,\xi^\beta{}_{,u}\\&\quad+gf^{abc}\eta^bA^c_u-\partial_u\eta^a(x)\,.\end{aligned} \tag{B2}$$

Introducing the values given in Eqs. (7), (B2) becomes

$$\begin{aligned}&F^i=-h\xi^{i\prime}+m^2e^{ij}\xi^u{}_{,j}\,,\quad F^u=-2h\xi^{u\prime}\,,\\&F^r=(h\xi^j)_{,j}+\dot{h}\xi^u-2nh\xi^{u\prime}+N^j\xi^u{}_{,j}+h'\xi^r\,,\\&F^a=-A^a_i\,\xi^{i\prime}-\eta^{a\prime}\,.\end{aligned} \tag{B3}$$

From this set of expressions one directly sees that (up to constants)

$$\Delta(G^A)=h^3h'\,|\partial_r|^{n^2+2}\,. \tag{B4}$$

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[9] The more natural (intrinsic) variable $A^a_{\mp}\equiv A^a_u+nA^a_r-N^NA^a_i$ was not particularly useful.

[10] E. Tomboulis used $A^a_r=0$ for quantization of the Yang-Mills field on a flat space: E. Tomboulis, Phys. Rev. D 8, 3382 (1973).

[11] $G^u=0$ is Kaku's choice, which generalizes the Robinson-Trautman ray gauge $g^{uu}=\eta^{uu}$. See Ref. 2.

[12] Using l to label the gauge conditions $G^u_l\equiv(g^{ui}=g^{uu}=0,\;m^2=h^{2l})$, $l_1=0$ gives the ray gauge of Robinson, Trautman, Newman, and Penrose: $l_2=-\frac{1}{4}$, $l_3=+\frac{1}{4}$, $l_4=-\frac{1}{2}$, and $l_5=\infty$ seems to be the more relevant to physics.

[13] Disregarding numerical factors.

[14] Disregarding numerical factors.

[15] See Ref. 1.

[16] Actually, variations of m^2 in the classical action generate the constraint equation $G_{ur}=0$, which we regard as the hypersurface equation determining the variable $\hat{n}$.

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[19] In order to compare results, one has to use their equation (2.27), second row, for $p=\frac{1}{2}$, and adopting the arguments given by Kaku, substitute g^{00} by $g^{ur}=m^{-2}$.