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Charged dust distributions in equilibrium in Brans-Dicke theory

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This paper shows that, unlike the case in general relativity, the ratio of charge density to mass density does not have a constant value for equilibrium distributions of charged dust in Brans-Dicke theory.

I. INTRODUCTION

In both the classical and the general relativity theories, one has the simple result that for static equilibrium of a charged dust distribution the matter density and the charge density must be equal (in units $G=c=1$).[1] This result, although independent of any symmetry requirement, is nevertheless subject to some restrictions, such as the absence of any singularity or of "any hole or pocket of alien matter" in the charged dust distribution.[1]

For an equilibrium which is stationary rather than static, one may have arbitrary values of the ratio of charge density to mass density as shown by Som and Raychaudhuri[2] by considering a cylindrically symmetric charged dust distribution in rigid rotation. However, in their solutions there were closed timelike lines.

It would be interesting to study analogous situations in the background of the Brans-Dicke theory as the problem of charge to mass ratio is of obvious importance in building up a model of the electron. However, it turns out that the results are much more complicated in the Brans-Dicke theory.

II. THE STATIC DISTRIBUTION

The static line element may be written in the form

$$ds^2=g_{00}dt^2+g_{ik}dx^i dx^k \ , \tag{1}$$

with the Latin indices running from 1 to 3. The Brans-Dicke-Maxwell equations are

$$R^\mu_\nu=\frac{8\pi}{\psi}(T^\mu_\nu-\tfrac{1}{2}\delta^\mu_\nu T)+\frac{\omega}{\psi^2}\psi^{,\mu}\psi_{,\nu}+\frac{\psi^{,\mu}{}_{;\nu}}{\psi}+\tfrac{1}{2}\delta^\mu_\nu\frac{\Box\psi}{\psi}\ , \tag{2}$$

$$F^{\mu\nu}{}_{;\nu}=4\pi\sigma v^\mu \ , \tag{3}$$

$$F_{[\mu\nu;\alpha]}=0\ , \tag{4}$$

$$\Box\psi=\frac{8\pi}{3+2\omega}T=\frac{8\pi\rho}{3+2\omega}\ , \tag{5}$$

with

$$T^\mu_\nu=\rho v^\mu v_\nu-\tfrac{1}{4}(F^{\mu\alpha}F_{\nu\alpha}-\tfrac{1}{4}\delta^\mu_\nu F_{\alpha\beta}F^{\alpha\beta})\ , \tag{6}$$

where ρ,σ are the mass and charge densities, respectively, v^μ is the velocity vector of matter, and ψ is the Brans-Dicke scalar. For the static case, $v^\mu=g_{00}{}^{-1/2}\delta^\mu_0$, and we may write $F_{0i}=\phi_{,i}, F_{ik}=0$, where ϕ is the electrostatic potential. The equations of motion,

$$v^\mu{}_{;\nu}v^\nu=-\frac{\sigma}{\rho}F^\mu_\alpha v^\alpha\ , \tag{7}$$

then give

$$(g_{00}{}^{1/2})_{,i}=-(\sigma/\rho)\phi_{,i}\ , \tag{8}$$

showing that g_{00}, ϕ, and σ/ρ are functionally related. Writing $g_{00}=F(\phi)$, Eq. (8) reads as follows:

$$\sigma/\rho=-\tfrac{1}{2}F'F^{-1/2}\ , \tag{9}$$

where a prime denotes differentiation with respect to ϕ.

Now if in the Raychaudhuri identity

$$-R_{\mu\nu}v^\mu v^\nu=2(\omega^2-\Sigma^2)+\tfrac{1}{3}\theta^2+\theta_{,\alpha}v^\alpha+(v^\mu{}_{;\nu}v^\nu)_{;\mu} \tag{10}$$

we substitute for $R_{\mu\nu}$ from Eq. (2) and use Eqs. (3), (5), and (7), and remember that in the present static case the vorticity ω, the shear Σ, and the expansion θ all vanish, we get

$$(4\pi\rho+E^2)\left(\frac{4+2\omega}{3+2\omega}\frac{1}{\psi}-\frac{\sigma^2}{\rho^2}\right)=\frac{E^2}{\psi(3+2\omega)}-(\sigma/\rho)_{,i}E^i-\frac{1}{2\psi}g^{00}g_{00,i}g^{ik}\psi_{,k}\ , \tag{11}$$

$$E_\mu\equiv F_{\mu\alpha}v^\alpha\ ,\quad E^2=-E_\mu E^\nu\ .$$

With the help of equation Eq. (9) this may be further reduced to

$$4\pi\rho\left(\frac{4+2\omega}{3+2\omega}\frac{1}{\psi}-\frac{\sigma^2}{\rho^2}\right)=F^{-1}g^{ik}\phi_{,i}\left(\frac{\phi_{,k}}{\psi}-\frac{F''}{2}\phi_{,k}-\frac{F'}{2\psi}\psi_{,k}\right)\ . \tag{12}$$

Using $\omega\to\infty$ and $\psi_{,k}\to 0$, the above equation leads to the results obtained by De and Raychaudhuri.[1] One may be tempted to think that in the present case Eq. (12) would be satisfied with

$$\frac{\sigma^2}{\rho^2}=\frac{4+2\omega}{3+2\omega}\frac{1}{\psi},$$

as indeed was conjectured by Nayak.[3] However, such a relation is not consistent with other field equations, as may be easily verified. To proceed further we assume that the distribution is spherically symmetric.

III. SPHERICALLY SYMMETRIC STATIC DISTRIBUTION

We can now write $\psi=\psi(\phi)$, as both are functions of the radial coordinate alone. Using Eq. (3), we have

$$\begin{aligned}(g^{ik}\sqrt{-g}\psi_{,k})_{,i} &= (g^{ik}\sqrt{-g}\psi'\phi_{,k})_{,i}\\ &= F^{1/2}\sqrt{-g}\psi' 4\pi\sigma + F^{-1}\sqrt{-g}\,g^{ik}\phi_{,k}\phi_{,i}(F\psi')'.\end{aligned} \tag{13}$$

Also, Eq. (12) can be rewritten:

$$4\pi\rho\left(\frac{4+2\omega}{3+2\omega}-\tfrac{1}{4}F^{-1}F'^2\psi\right)=F^{-1}g^{ik}\phi_{,i}\phi_{,k}\left(\phi-\frac{F'\psi}{2}\right)'. \tag{14}$$

Eliminating σ,ρ from Eqs. (13) and (14) with the help of Eqs. (5) and (9), we get

$$\left(F'\psi'+\frac{4}{3+2\omega}\right)\left(\phi-\frac{F'\psi}{2}\right)' = 2(F\psi')'\left(\frac{4+2\omega}{3+2\omega}-\tfrac{1}{4}F'^2F^{-1}\psi\right). \tag{15}$$

To have an idea about the relation between σ and ρ, we make use of a power-series expansion of F, ψ in terms of ϕ with the stipulation that $\phi\to 0$, $F\to 1$, $g_{11}\to -1$, and $g_{22}=g_{33}\sin^{-2}\theta\to -r^2$ as $r\to 0$ (i.e., the center of symmetry). It then turns out from Eqs. (14), (15), (9), and (5) that

$$\frac{\sigma^2}{\rho^2}=\alpha^2\left[1+\frac{\alpha\phi}{2+\omega}-\alpha^2\frac{126\omega+67}{100(2+\omega)^2}\phi^2+\cdots\right], \tag{16}$$

$$\frac{\sigma^2}{\rho^2}\psi=\frac{4+2\omega}{3+2\omega}\left[1-\alpha^2\phi^2\frac{14\omega+3}{25(2+\omega)^2}+\cdots\right], \tag{17}$$

with

$$\alpha^2\psi_0=\frac{4+2\omega}{3+2\omega}, \tag{18}$$

$$\pi\rho_0\alpha=3\left(\frac{d^2\phi}{dr^2}\right)_0. \tag{19}$$

The subscript zero refers to the values at the origin. Equations (16) and (17) show that in the Brans-Dicke theory neither σ^2/ρ^2 nor $\sigma^2\psi/\rho^2$ is a constant, though in the relativistic limit $\omega\to\infty$ they do lead to the relativity results. In fact with $\rho_0>0$, $\alpha\phi$ is positive near the origin and consequently σ^2/ρ^2 is a minimum at the origin while $\sigma^2\psi/\rho^2$ is a maximum; they have their general relativity values right at the origin.

IV. A CYLINDRICALLY SYMMETRIC DISTRIBUTION WITH RIGID ROTATION

We borrow the picture used by Som and Raychaudhuri[2] of a cylindrically symmetric charged dust distribution in rigid rotation in which the Lorentz force vanishes. As in their case we are led to the line element

$$ds^2=dt^2-e^{2\mu}(dr^2+dz^2)-l\,d\Phi^2+2m\,d\Phi\,dt, \tag{20}$$

with μ, l, and m functions of r alone and

$$F^{31}=-F^{13}=AD^{-1}e^{-2\mu}, \tag{21}$$

$$F^{10}=-F^{01}=mAD^{-1}e^{-2\mu}, \tag{22}$$

where $D^2=l+m^2$ and all other components of $F^{\alpha\beta}$ vanish.

The field equations are now

$$\frac{1}{\sqrt{-g}}\frac{d}{dr}\left(\frac{mm_1}{2D}\right)=\frac{1}{\psi}(4\pi\rho+A^2e^{-2\mu}) -\frac{mm_1}{2D}\frac{1}{\sqrt{-g}}\frac{\psi_1}{\psi}+\frac{1}{2}\frac{\Box\psi}{\psi}, \tag{23}$$

$$\frac{1}{\sqrt{-g}}\frac{d}{dr}\left(\frac{l_1+mm_1}{2D}\right)=-\frac{1}{\psi}(4\pi\rho+A^2e^{-2\mu}) -\frac{l_1+mm_1}{2D}\frac{1}{\sqrt{-g}}\frac{\psi_1}{\psi} +\frac{1}{2}\frac{\Box\psi}{\psi}, \tag{24}$$

$$\frac{1}{\sqrt{-g}}\frac{d}{dr}\left(\frac{ml_1-lm_1}{2D}\right)=-\frac{2}{\psi}m(4\pi\rho+A^2e^{-2\mu}) -\frac{ml_1-lm_1}{2D}\frac{1}{\sqrt{-g}}\frac{\psi_1}{\psi}, \tag{25}$$

$$\frac{d}{dr}\left(\frac{m_1}{2D}\right)=-\frac{m_1}{2D}\frac{\psi_1}{\psi}, \tag{26}$$

$$\frac{1}{\sqrt{-g}}\frac{d}{dr}(D\psi_1)=-\frac{1}{\psi}(4\pi\rho-A^2e^{-2\mu}) -\frac{1}{\sqrt{-g}}D\mu_1\frac{\psi_1}{\psi}+\frac{1}{2}\frac{\Box\psi}{\psi}, \tag{27}$$

$$D\mu_{11} - \frac{m_1^2}{2D} - D_1\mu_1 + D_{11}$$

$$= -\frac{\sqrt{-g}}{\psi}(4\pi\rho + A^2 e^{-2\mu}) - \omega D \frac{\psi_1^2}{\psi^2}$$

$$-D\frac{\psi_{11}}{\psi} + D\mu_1 \frac{\psi_1}{\psi} , \quad (28)$$

$$\Box\psi = \frac{8\pi}{3+2\omega}\rho , \quad (29)$$

$$F^{\mu\nu}{}_{;\nu} = \sigma v^\mu . \quad (30)$$

Equation (26) yields

$$\frac{m_1\psi}{2D} = \text{const} = a \quad (\text{say}) . \quad (31)$$

Using Eq. (30) in (23), we get

$$\frac{1}{\sqrt{-g}}\frac{m_1^2}{2D} = \frac{1}{\psi}(4\pi\rho + A^2 e^{-2\mu}) + \frac{1}{2}\frac{\Box\psi}{\psi} , \quad (32)$$

whereas from Eqs. (23) and (24), we have

$$D\psi = br , \quad (33)$$

where b is an integration constant. Eliminating D and m from Eq. (31) with the help of Eqs. (29) and (33), we get the differential equation for ψ,

$$\psi\psi_{11}r + \psi\psi_1 - \psi_1^2 r = r\frac{A^2\psi - 2a^2}{\omega+2} . \quad (34)$$

From Eqs. (32) and (30) we get

$$4\pi\rho = \frac{2a^2 - A^2\psi}{\psi}\frac{3+2\omega}{4+2\omega}e^{-2\mu} , \quad (35)$$

$$4\pi\sigma = -\frac{2aA}{\psi}e^{-2\mu} , \quad (36)$$

and hence

$$\frac{\sigma}{\rho} = -\frac{2aA}{A^2\psi - 2a^2}\left(\frac{4+2\omega}{3+2\omega}\right) .$$

Thus σ/ρ is not a constant. If the solution is regular at the axis $r=0$, we may write in the neighborhood of the axis

$$\psi = \psi_0 + \sum_1^\infty \alpha_n r^n ,$$

and we get, from Eq. (34),

$$\alpha_1 = 0 ,$$

$$\alpha_2 = -\frac{2a^2 - A^2\psi_0}{\psi_0(4+2\omega)} ,$$

so that, in view of Eq. (35), α_2 is negative, and thus both ψ and $|\sigma/\rho|$ have maxima there.

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[1]U. K. De and A. K. Raychaudhuri, Proc. R. Soc. London A303, 47 (1968).
[2]M. M. Som and A. K. Raychaudhuri, Proc. R. Soc. London A304, 81 (1968).
[3]B. K. Nayak, Aust. J. Phys. 28, 585 (1975).