Solutions of Einstein's field equations for static fluid spheres

Selçuk S. Bayin
Department of Physics, The University of Michigan, Ann Arbor, Michigan 48109
(Received 27 February 1978)

Five new analytic solutions are presented. One of them has the equation of state $P = K_0\rho^{1+1/n} + \sigma_1\rho$, where $\sigma_1 = P_c/\rho_c c^2$ and K_0 and n are constants; both pressure and density diverge at the origin while their ratio remains finite. Since the second term could be negligible at high densities, this solution can be considered as an analog of a relativistic polytrope over a certain range of radius. Each solution has been considered in some detail.

I. INTRODUCTION

Towards the late stages of stellar evolution general-relativistic effects become important and the full field equations have to be solved to get realistic models. Even though this problem is mathematically well defined and straightforward, owing to the nonlinearity of the equations the number of known analytic solutions hardly exceeds ten. In this paper we present various methods of treating field equations and give five new analytic solutions. One of them can be considered as an analog of a relativistic polytrope with the equation of state

$$P = K_0\rho^{1+1/n} + \sigma_1\rho,$$

where $\sigma_1 = P_c/\rho_c c^2$ and K_0 and n are constants.

II. FIELD EQUATIONS AND METHODS OF OBTAINING ANALYTIC SOLUTIONS

For a static and spherically symmetric system the line element can be taken as

$$ds^2 = -B^2dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2 + A^2dt^2, \quad (2.1)$$

where $B = B(r)$, $A = A(r)$ and we set from now on $c = G = 1$. The field equations then take the following form for a perfect fluid[1]:

$$8\pi P = \frac{1}{B^2}\left(\frac{2A'}{Ar} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \quad (2.2)$$

$$8\pi P = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right], \quad (2.3)$$

$$8\pi\rho = \frac{1}{B^2}\left(\frac{2B'}{Br} - \frac{1}{r^2}\right) + \frac{1}{r^2}. \quad (2.4)$$

Here we have a system of three differential equations with four unknowns P, ρ, A, and B as functions of r. The normal way to solve this problem is to use a physically reasonable equation of state to complete the set, but owing to the nonlinear structure of the equations such a physical approach becomes very difficult. Instead one assumes a relation among A, B, and their derivatives such that integrability of the above system is secured and later one eliminates r among $P(r)$ and $\rho(r)$ to obtain an equation of state and checks it for physical reasonableness. We take a solution to be physically reasonable if pressure and density are positive and monotonic decreasing functions of r throughout the star and pressure goes to zero at finite radius.

To find a solution we first equate (2.2) and (2.3) and write the result in the following convenient form:

$$\frac{d}{dr}\left(\frac{1-B^2}{B^2r^2}\right) + \frac{d}{dr}\left(\frac{A'}{B^2Ar}\right) + \frac{1}{B^2A^2}\frac{d}{dr}\left(\frac{A'A}{r}\right) = 0, \quad (2.5)$$

which immediately suggests various possible relations among A and B which will allow us to integrate (2.5). Using this method Tolman[2] has found eight solutions, five of which were new at that time. We will transform this equation into a different form to find other solutions.

Let $A'/Ar = C(r)$ in (2.5) to obtain

$$\frac{d}{dr}\left(\frac{1-B^2}{r^2B^2}\right) + \frac{d}{dr}\left(\frac{C}{B^2}\right) + \frac{1}{B^2}\frac{dC}{dr} + \frac{4r}{2B^2}C^2 = 0, \quad (2.6)$$

which can be written as

$$\left(\frac{1}{r^2B^3} + \frac{C}{B^3}\right)dB - \left(\frac{1}{B^2}\right)dC - \left(\frac{B^2-1}{B^2r^3} + \frac{C^2r}{B^2}\right)dr = 0, \quad (2.7)$$

which is a Pfaffian differential equation in three dimensions, in general given as

$$f_1(B,C,r)dB + f_2(B,C,r)dC + f_3(B,C,r)dr = 0. \quad (2.8)$$

Here we are treating two of the three variables as independent and looking for a surface $F(B,C,r) = 0$ that satisfies the above equation. Equation (2.8) has a necessary and sufficient integrability condition[3] of the form

$$\vec{x}\cdot\mathrm{curl}\,\vec{x}=0, \tag{2.9}$$

where $\vec{x}$ is the vector field defined as

$$\vec{x}=(f_1,f_2,f_3). \tag{2.10}$$

Finding a solution to (2.8) can be geometrically interpreted as finding a surface in B,C,r space whose normal vectors define a vector field normal to $\vec{x}$. For our case Eq. (2.7) is not integrable. Hence, a single surface which would include all the solutions does not exist. The properties of Pfaffian differential equations might deserve further study as far as classification of solutions are concerned in B,C,r space, but that is beyond the purpose of the present paper.

Next we consider $B=B(r)$ and $C=C(r)$ so that (2.7) reduces to a Bernoulli equation[4] for $B=B(r)$ once $C(r)$ is chosen:

$$\frac{dB}{dr}=\frac{1}{r^3(1/r^2+C)}B^3+\frac{-1/r^3+C^2r+dC/dr}{1/r^2+C}B, \tag{2.11}$$

which has the general form

$$\frac{dB}{dr}=g(r)B^3+f(r)B. \tag{2.12}$$

The solution can be immediately given in terms of two quadratures,

$$\frac{1}{B^2}=y_1+y_2, \tag{2.13}$$

where

$$y_1=C_0e^{\phi},\quad y_2=-2e^{\phi}\int e^{-\phi}g(r)dr,$$
$$\phi=-2\int f(r)dr. \tag{2.14}$$

We now describe seven explicit forms of solutions depending on assumed relations between B and C.

Solution I. In order to secure integrability of (2.14) we choose

$$\frac{-1/r^3+C^2r+dC/dr}{1/r^2+C}=-\frac{1}{r}, \tag{2.15}$$

the solution of which is

$$C(r)=\frac{1}{C_0'r+r^2}$$

and $A^2=(a_0+a_1r)^2$, and B^2 is found to be

$$B^2=\frac{1}{C_1r^2-(2/C_0)r+(4r^2/C_0^2)\ln|(C_0+2r)/r|+1}, \tag{2.16}$$

where, C_0 and C_1 are constants. This solution has also been discovered by Kuchowicz,[5] by studying a different differential equation. Various authors have studied other forms of first-order differential equations for $B(r)$, once $A(r)$ is chosen.[6,7,8]

Now we consider the second possibility, that is to solve for $C(r)$ for a given $B(r)$. Equation (2.7) now becomes

$$\frac{dC}{dr}=\left(\frac{1}{r^2B}\frac{dB}{dr}-\frac{B^2-1}{r^3}\right)+\left(\frac{1}{B}\frac{dB}{dr}\right)C-rC^2. \tag{2.17}$$

This is a Riccati equation for $C(r)$ and quite difficult to solve in general.

Solution II. If we assume $B=w_0C(r)$, $w_0=$ constant, one can find the following solution:

$$\ln A=-\tfrac{1}{2}\sin^{-1}\left[\frac{-2r^2+C_0}{(C_0^2+4w_0^2)^{1/2}}\right]+C_1, \tag{2.18}$$

$$B=\frac{w_0}{(-r^4+C_0r^2+w_0^2)^{1/2}}. \tag{2.19}$$

This is a new solution.

Solution III. With $B=w(r)C(r)$ Eq. (2.17) becomes

$$\frac{dC}{dr}=\left(-\frac{w'}{w}-\frac{1}{r}\right)C-\frac{w'}{w}r^2C^2+\left(\frac{w^2}{r}+r^3\right)C^3. \tag{2.20}$$

Assuming $w^2/r+r^3=0$ one gets our second new solution

$$A(r)=C_1\left[r+\left(r^2-\frac{2}{C_0}\right)^{1/2}\right]^{1/\sqrt{C_0}}, \tag{2.21}$$

$$B(r)=\frac{1}{(2/r^2-C_0)^{1/2}}. \tag{2.22}$$

Next we try the substitution

$$\frac{1}{r^2B}\frac{dB}{dr}-\frac{B^2-1}{r^3}+\frac{1}{B}\frac{dB}{dr}C=\theta(r)C \tag{2.23}$$

in (2.17), with $\theta(r)$ to be guessed. The differential equation now becomes

$$\frac{dC}{dr}=\theta(r)C-rC^2, \tag{2.24}$$

which can be reduced to quadratures immediately.

Solution IV. Let $\theta=1/r$; this gives the following solution, which is also new:

$$A=C_2(3C_0+r^3), \tag{2.25}$$

$$B^2=\frac{(3C_0+4r^3)4^{2/3}}{C_1r^2+3C_0-2r^3}. \tag{2.26}$$

Solution V. $\theta=0$ gives $A=a_0+a_1r^2$, which was found by Kuchowicz,[5] Adler,[6] and Adams and Cohen.[7]

Field equations in isotropic coordinates

We will also try to solve the field equations in isotropic coordinates, where the metric is of the form[1]

$$ds^2 = -e^{\mu}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + e^{\nu} dt^2. \quad (2.27)$$

The field equations for a perfect fluid are now given as

$$8\pi P = e^{-\mu}\left(\frac{\mu'^2}{4} + \frac{\mu'\nu'}{2} + \frac{\mu'+\nu'}{r}\right), \quad (2.28)$$

$$8\pi P = e^{-\mu}\left(\frac{\mu''}{2} + \frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\mu'+\nu'}{2r}\right), \quad (2.29)$$

$$8\pi\rho = -e^{-\mu}\left(\mu'' + \frac{\mu'^2}{4} + \frac{2\mu'}{r}\right). \quad (2.30)$$

Buchdahl has also studied the field equations in this form and found a solution whose equation of state resembles that of the polytrope of index 5.[9]

To find a solution we will make the substitution

$$e^{\nu} = A\phi^{-a}, \quad e^{\mu} = B\phi^{b}, \quad (2.31)$$

where $\Phi = \Phi(r)$, into the equation obtained by equating (2.28) and (2.29) which will give us a differential equation to be solved for $\Phi(r)$,

$$\Phi'' - C\frac{\Phi'^2}{\Phi} - \frac{1}{r}\Phi' = 0, \quad (2.32)$$

where

$$C(\text{constant}) = \frac{\frac{1}{2}b^2 - \frac{1}{2}a^2 - ab + b - a}{b - a}. \quad (2.33)$$

Solution VI. For $C = 1$ we obtain

$$\Phi = C_1 \exp(C_0 r^2), \quad (2.34)$$

$$e^{\nu} = AC_1^{-a}\exp(-aC_0 r^2), \quad e^{\mu} = BC_1^{b}\exp(bC_0 r^2). \quad (2.35)$$

Solution VII. For $C \neq 1$,

$$\Phi^{1-C} = C_0 r^2 + C_1, \quad (2.36)$$

$$e^{\nu} = A(C_0 r^2 + C_1)^{-a/(1-C)}, \quad e^{\mu} = B(C_0 r^2 + C_1)^{b/(1-C)}. \quad (2.37)$$

So far we have found seven solutions, where five of them, Solutions II, III, IV, VI, and VII are new.

III. BOUNDARY CONDITIONS

From the Schwarzschild exterior solution we know that, at $r = r_b$,

$$B^2(r_b) = A^{-2}(r_b) = \frac{1}{1 - 2m/r_b}. \quad (3.1)$$

r_b is the boundary of the star which can be defined as the value of the radius where the pressure vanishes,

$$8\pi P(r_b) = 0 = B^{-2}\left(\frac{2A'}{Ar_b} + \frac{1}{r_b^2}\right) - \frac{1}{r_b^2},$$

$$B^{-2}\left(\frac{2A'}{A}r_b + 1\right) - 1 = 0.$$

Also from the requirement that the interior solution has to be connected smoothly to the exterior solution, one easily sees that we must have B, A, and dA/dr continuous at the surface. Hence, $B(r)$ should have one integration constant, while $A(r)$ will have two.

In this respect Wyman,[10] Leibovitz,[11] and recently Whitman[8] generalized some of Tolman's solutions so that they have the proper number of integration constants.

IV. DETAILED STUDY OF SOME OF THE SOLUTIONS

Solution II. In Sec. III we have found

$$\ln A = -\tfrac{1}{2}\sin^{-1}\left(\frac{-2r^2 + C_0}{q}\right) + C_1,$$

$$B = \frac{w_0}{(-r^4 + C_0 r^2 + w_0^2)^{1/2}}, \quad (4.1)$$

$$q = (C_0^2 + 4w_0^2)^{1/2},$$

$$8\pi P(r) = \frac{1}{w_0^2}\left[2(w_0^2 + C_0 r^2 - r^4)^{1/2} + C_0 - r^2\right], \quad (4.2)$$

$$8\pi\rho(r) = \frac{5}{w_0^2}r^2 - \frac{3C_0}{w_0^2}. \quad (4.3)$$

At the origin we have

$$8\pi P_c = \frac{2}{w_0} + \frac{C_0}{w_0^2}, \quad \frac{P_c}{\rho_c} = -\frac{2w_0}{3C_0} - \tfrac{1}{3}, \quad 8\pi\rho_c = -\frac{3C_0}{w_0^2}. \quad (4.4)$$

In order to have positive pressure and density at the origin we must have $C_0 < 0$ and $w_0 > |C_0|/2$. The radius of the star is defined by $P(r) = 0$, which gives

$$2R^2 = -3|C_0| \mp \tfrac{1}{2}(28C_0^2 + 32w_0^2)^{1/2}. \quad (4.5)$$

The pressure and density are positive and finite throughout the star, but, even though pressure is a decreasing function of r, ρ increases from center to surface. Thus this solution is not generally physically reasonable according to our criterion, although it may be a suitable solution for the case where density inversion occurs.[10]

Solution III. We have

$$A(r)=C_1\left[r+\left(r^2-\frac{2}{C_0}\right)^{1/2}\right]^{1/\sqrt{C_0}},$$
$$B(r)=\frac{1}{(2/r^2-C_0)^{1/2}}, \quad (4.6)$$

$$8\pi P(r)=\left(\frac{2}{r^2}-C_0\right)\left(\frac{2}{r(C_0r^2-2)^{1/2}}+\frac{1}{r^2}\right)-\frac{1}{r^2}, \quad (4.7)$$

$$8\pi\rho(r)=+\frac{2}{r^4}+\frac{1}{r^2}(C_0+1), \quad (4.8)$$

$$P_c/\rho_c=+\frac{2}{\sqrt{C_0}}+1. \quad (4.9)$$

The pressure and density both diverge at the origin while their ratio remains constant; they are also monotonic decreasing functions of r. The radius is defined by $P(r)=0$, which gives

$$R=\left(\frac{2}{C_0-1}\right)^{1/2}. \quad (4.10)$$

C_0 and C_1 can be evaluated in terms of M and R from the boundary conditions as follows:

$$\frac{1}{(2/R^2-C_0)^{1/2}}=\frac{1}{(1-2M/R)^{1/2}},$$
$$C_1\left[R+\left(R^2-\frac{2}{C_0}\right)^{1/2}\right]^{1/\sqrt{C_0}}=\left(1-\frac{2M}{R}\right)^{1/2}. \quad (4.11)$$

These give C_0 and C_1 as

$$C_0=\frac{2}{R^2}+\frac{2M}{R}-1,$$
$$C_1=\frac{(1-2M/R)^{1/2}}{[R+(R^2-2/C_0)^{1/2}]^{1/\sqrt{C_0}}}. \quad (4.12)$$

One can also obtain a mass-radius relation as

$$M=R. \quad (4.13)$$

Solution IV. We have also found that

$$A=C_2(3C_0+r^3),\quad B^2=\frac{3C_0+4r^3}{C_1r^2+3C_0-2r^3}, \quad (4.14)$$

which leads to the following pressure and density distributions:

$$8\pi P(r)=\frac{(C_1r^2-2r^3+3C_0)(7r^3+3C_0)}{(3C_0+4r^3)(3C_0+r^3)r^24^{2/3}}-\frac{1}{r^2}, \quad (4.15)$$

$$8\pi\rho(r)=\frac{(C_1r^2-2r^3+3C_0)}{(3C_0+4r^3)4^{2/3}}\left(\frac{4C_1r^3+54C_0r-6C_0C_1}{(C_1r^2-2r^3+3C_0)(3C_0+4r^3)}-\frac{1}{r^2}\right)+\frac{1}{r^2}, \quad (4.16)$$

$$8\pi P_c\to-0.6\frac{1}{r^2},\quad 8\pi\rho_c\to 0.6\frac{1}{r^2},\quad \frac{P_c}{\rho_c}\to-1. \quad (4.17)$$

This solution has a pressure distribution which goes to negative infinity as r approaches zero.

Solutions III and IV, even though they can be used to represent portions of stars, can not provide a physically reasonable model for the entire star, so we will not discuss them any further.

Solution V. In isotropic coordinates we have found the following solution:

$$e^{\nu}=AC_1^{-a}\exp[-(aC_0r^2)],\quad e^{\mu}=BC_1^{b}\exp(bC_0r^2), \quad (4.18)$$

$$8\pi P=(C_1^{-b}/B)\exp[-(C_0br^2)][(b^2C_0^2-2C_0^2ab)r^2+2C_0b-2C_0a], \quad (4.19)$$

$$8\pi\rho=-(C_1^{-b}/B)\exp[-(C_0br^2)](6bC_0+b^2C_0^2r^2), \quad (4.20)$$

$$8\pi\rho_c=-(C_1^{-b}/B)6bC_0, \quad (4.21)$$

$$8\pi P_c=(C_1^{-b}/B)[2C_0(b-a)]. \quad (4.22)$$

The surface is defined by $P(R)=0$, which gives

$$R^2=-\frac{2(b-a)}{bC_0(b-2a)}. \quad (4.23)$$

There are two possibilities which give physically reasonable answers:

(i)

$$b<0,\quad C_0>0,\quad b-a>0,\text{ and } b-2a>0, \quad (4.24)$$

(ii)

$$b>0, \quad C_0<0, \quad b-a<0, \text{ and } b-2a<0. \tag{4.25}$$

Both have positive pressure and density which decrease outward.

Now we connect this solution to the Schwarzschild exterior solution in isotropic coordinates,

$$ds^2 = \frac{(1-M/2R)^2 dt^2}{(1+M/2R)^2} - \left(1+\frac{M}{2R}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2). \tag{4.26}$$

One obtains the following expressions for C_0, C_1, and B in terms of M and R:

$$C_0 = \frac{2(a-b)}{bR^2(b-2a)},$$

$$C_1 = \frac{(1-M/2R)^{-2/a}}{(1+M/2R)^{-2/a}} A^{+1/a} \exp\left(-\frac{2(a-b)}{b(b-2a)}\right), \tag{4.27}$$

$$B = \frac{(1+M/2R)^{4-2b/a}}{(1-M/2R)^{-2b/a}} A^{-b/a}$$

where a, b, and A are arbitrary. We can also find an expression for M in terms of ρ_c, using

$$8\pi\rho_c = -\frac{C_1^{-b}}{B} 6bC_0, \tag{4.28}$$

$$M = \left(-\frac{12(a-b)}{(b-2a)8\pi\rho_c}\right)^{1/4} 2R^{1/2} \exp\left(\frac{a-b}{2(b-2a)}\right) - 2R. \tag{4.29}$$

Solution VI. We have, from Sec. II,

$$e^{\nu} = A(C_0 r^2 + C_1)^{-a/(1-C)}, \quad e^{\mu} = B(C_0 r^2 + C_1)^{b/(1-C)}, \tag{4.30}$$

$$8\pi P = \frac{1}{B}(C_0 r^2 + C_1)^{-b/(1-C)} \left(\frac{C_0^2 b^2 r^2 - 2abC_0^2 r^2 (C_0 r^2 + C_1) + 2C_0(b-a)(1-C)(C_0 r^2 + C_1)}{(1-C)^2 (C_0 r^2 + C_1)^2}\right), \tag{4.31}$$

$$8\pi\rho = -\frac{1}{B}(C_0 r^2 + C_1)^{-b/(1-C)} \left(\frac{(1-C)(6bC_0C_1 + 2bC_0^2 r^2) + b^2 C_0^2 r^2}{(1-C)^2 (C_0 r^2 + C_1)^2}\right), \tag{4.32}$$

$$8\pi\rho_c = -\frac{6bC_0}{B(1-C)} C_1^{(-b-1+C)/(1-C)},$$

$$8\pi P_c = \frac{1}{B}\frac{2C_0(b-a)}{(1-C)} C_1^{(-b-1+C)/(1-C)}, \quad \sigma_c = \frac{P_c}{\rho_c} = \frac{1}{3}\frac{a-b}{b}. \tag{4.33}$$

The radius of the star is given by

$$-2abC_0^2 R^4 + R^2[b^2 - 2abC_1 + 2(b-a)(1-C)]C_0 + 2C_1(b-a)(1-C) = 0. \tag{4.34}$$

This solution has positive pressure and density which are also decreasing functions of r. We determined C_0, C_1, and B from the boundary conditions in terms of M and R, and found that

$$C_1 = \frac{-2HI + F \mp [(2HI-F)^2 - 4(I^2 - E^2)(H^2 - G)]^{1/2}}{2(I^2 - E^2)}, \tag{4.35}$$

$$C_0 = \left(\frac{1-M/2R}{1+M/2R}\right)^{-2(1-C)/a} \frac{A^{(1-C)/a}}{R^2} + \frac{2HI - F \pm [(2HI-F)^2 - 4(I^2-E^2)(H^2-G)]^{1/2}}{2(I^2-E^2)R^2}, \tag{4.36}$$

$$B=\frac{(1+M/2R)^4}{(C_0R^2+C_1)^{b/(1-C)}}. \tag{4.37}$$

Here,

$$H=4ab\left[\left(\frac{1-M/2R}{1+M/2R}\right)^{-2(1-C)/a}A^{(1-C)/a}-\frac{b^2}{4ab}-\frac{2(b-a)(1-C)}{4ab}\right],$$

$$I=-2ab,\quad E^2=4a^2b^2, \tag{4.38}$$

$$F=[2(b-a)(1-C)+b^2](-4ab)+16ab(b-a)(1-C),$$

$$G=[2(b-a)(1-C)+b^2]^2.$$

The mass $M=M(\rho_c)$ can be evaluated from

$$8\pi\rho_c=-\frac{1}{B}C_1^{-b/(1-C)}\frac{6bC_0}{(1-C)C_1}.$$

In Solution VII we have a, b, C_0, C_1, A, B as constants. With the boundary conditions we can determine three of them in terms of M and R, leaving the others arbitrary. So even if we set $C_1=0$ we will still have a sufficient number of constants and also obtain a solution which has an interesting equation of state.

The solution thus takes the form

$$e^{\nu}=A(C_0r^2)^{-a/(1-C)},\quad e^{\mu}=B(C_0r^2)^{b/(1-C)}, \tag{4.39}$$

$$8\pi P(r)=\frac{1}{B}(C_0r^2)^{-b/(1-C)}\left[\frac{b^2+2(b-a)(1-C)}{(1-C)^2}\frac{1}{r^2}-\frac{2abC_0}{(1-C)^2}\right], \tag{4.40}$$

$$8\pi\rho(r)=-\frac{1}{B}(C_0r^2)^{-b/(1-C)}\left[\frac{(1-C)2b+b^2}{(1-C)^2}\left(\frac{1}{r^2}\right)\right]. \tag{4.41}$$

The pressure and density diverge at the origin while their ratio is constant:

$$\sigma_1=\frac{P_c}{\rho_c}=-\frac{2(b-a)(1-C)+b^2}{(1-C)2b+b^2},\quad C=\frac{\frac{1}{2}b^2-\frac{1}{2}a^2-ab+b-a}{b-a}. \tag{4.42}$$

The equation of state can be obtained by eliminating r among $P(r)$ and $\rho(r)$,

$$P=\frac{2abC_0}{(1-C)2b+b^2}\left[\frac{C_0^{-b/(1-C)}}{8\pi B}\left(\frac{(C-1)2b-b^2}{(1-C)^2}\right)\right]^{-(1-C)/(C-1-b)}\rho^{1+(1-C)/(C-1-b)}+\sigma_1\rho. \tag{4.43}$$

This has the form of a polytropic relation except for the second term, which could be negligible at high densities. Also (4.43) reduces to a polytropic relation in the classical limit $\sigma_1\to 0$,[12] so our solution can be considered as an analog of polytropes in general relativity.

If we take C_0, b, A as our constants to be determined by boundary conditions we get

$$C_0=\frac{a+2b}{2bR^2},\quad b=(1-C)\left[\ln\frac{(1+M/2R)^4}{B}\right]\Big/\left(\ln\frac{a+2b}{2b}\right),\quad A=\frac{(1-M/2R)^2}{(1+M/2R)^2}\left(\frac{a+2b}{2b}\right)^{a/(1-C)}. \tag{4.44}$$

V. SUMMARY AND CONCLUSIONS

We saw that once the two equations (2.2) and (2.3) are equated we get a coupled differential equation for the metric coefficients, which can be solved when a relation between them is assumed. The most common path taken in the literature is to assume either $A(r)$ or $B(r)$ and then solve the remaining differential equation for the other metric coefficient.

The differential equation to be solved for $B(r)$ is of first order and can be reduced to quadratures immediately. On the other hand, the equation for $A(r)$ is of second order and cannot be solved in general.

In this paper we have studied the second alternative and managed to reduce the problem to the solution of two first-order differential equations, both of which can be written in terms of quadratures. The equation to be solved for $C=A'/Ar$ is

$$\frac{dC}{dr}=\left(\frac{1}{r^2B}\frac{dB}{dr}-\frac{B^2-1}{r^3}\right)+\frac{1}{B}\frac{dB}{dr}C-rC^2. \tag{5.1}$$

With the substitution

$$\frac{1}{r^2B}\frac{dB}{dr}-\frac{B^2-1}{r^3}+\frac{1}{B}\frac{dB}{dr}C=\theta(r)C \tag{5.2}$$

the equations to be solved become

$$\frac{dC}{dr}=\theta(r)C-rC^2 \text{ for } C, \tag{5.3}$$

and

$$\frac{dB}{dr}=\frac{\theta(r)r^3C-1}{(1+r^2C)r}B+\frac{1}{(1+r^2C)r}B^3 \text{ for } B. \tag{5.4}$$

Both are Bernoulli equations[4] and can be solved immediately once $\theta(r)$ is chosen. Note that once $C(r)$ is found one need not solve another integral to get $A(r)$, since all the physical consequences follow from $P(r)$ and $\rho(r)$ which involve only $C(r)$; see Eqs. (2.2) and (2.4).

We have also substituted $B=w(r)C$ into (5.1), where in this case $w(r)$ is the function to be guessed. By trying various functional forms for $w(r)$ and $\theta(r)$ we have found three new solutions; solutions *II*, *III*, and *IV*. Writing the field equations in isotropic coordinates also leads us to two new solutions, where one of them has an equation of state that might be used to approximate analytically numerical solutions for polytropes at high densities, as discussed by Tooper.[13]

It is a pleasure to thank R. L. Sears for his constant guidance and help throughout the preparation of this work. I would also like to thank D. Hegyi for a critical reading of the manuscript.

---

[1]R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford Univ. Press, New York, 1934), p. 244.
[2]R. C. Tolman, Phys. Rev. 55, 367 (1939).
[3]I. N. Sneddon, *Elements of Partial Differential Equations* (McGraw-Hill, New York, 1957), p. 21.
[4]G. M. Murphy, *Ordinary Differential Equations and Their Solutions* (Van Nostrand, Princeton, 1960), p. 26.
[5]B. Kuchowicz, Acta Pol. 33, 541 (1968).
[6]R. J. Adler, J. Math. Phys. 15, 727 (1974).
[7]R. C. Adams and J. M. Cohen, Astrophys. J. 198, 507 (1975).
[8]P. G. Whitman, J. Math. Phys. 18, 869 (1977).
[9]H. A. Buchdahl, Astrophys. J. 160, 1512 (1966).
[10]M. Wyman, Phys. Rev. 75, 1930 (1949); Wyman studies a similar solution, where the density distribution is given by $\rho=ar^n$, where a and n are constants.
[11]C. Leibovitz, Phys. Rev. 185, 1664 (1969).
[12]S. A. Bludman, Astrophys. J. 183, 637 (1973).
[13]R. F. Tooper, Astrophys. J. 140, 434 (1964).