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Lagrangian theory of the motion of spinning particles in torsion gravitational theories

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The (second-order differential) equations of motion of spinning test particles (tops) are derived from a variational principle in a given gravitational background defined by a Riemannian metric and a torsion tensor. The mass and (magnitude of) the spin of the top are conserved. There exists a Regge trajectory linking the mass and the spin of the top. Constants of the motion associated with Killing vectors of the metric along which the Lie derivative of the torsion tensor vanish are found.

I. INTRODUCTION

The increasing interest in torsion theories and the fact that spin interacts with torsion makes it necessary to study in detail the motion of particles and fields with spin in the presence of torsion.

The purpose of this work is to present a Lagrangian theory of the motion of spinning particles (tops) in torsion theories. The method used here was developed in Refs. 1 and 2 for the special and general relativistic cases, respectively. A preliminary report of this work was presented in Ref. 3.

In spite of the fact that Hehl has already derived equations of motion for tops in the Einstein-Cartan theory,[4] it seems necessary to display the new results obtained here and comment on the difference between some of the results of Ref. 4 and the ones we obtain. In fact, the method used in Ref. 4 is the one devised by Papapetrou[5] and therefore (in spite of the fact that the equations in Ref. 4 were not spelled out in full detail) the equations of motion are third-order differential equations for the position of the top or the angles defining its orientation. Our differential equations of motion are second order.

In what follows we will describe new results that do not appear in Ref. 4 or anywhere else in the literature to our knowledge.

First of all, we obtain the equations of motion for the spin. In addition, we show that if there are Killing vectors of the metric such that the Lie derivatives of the torsion along them vanish, then there are constants of the motion associated with those Killing vectors and we exhibit them explicitly. Finally, in the same way as has been done previously[1,2] we choose a set of constraints for the spin to ensure the correct nonrelativistic limit of the motion of the top. A Regge trajectory linking the mass and the magnitude of the spin top is obtained. Both the mass and magnitude of the spin of the top are conserved.

The Lagrangian method also has at least two more advantages over the energy-momentum-tensor method, namely, constants of the motion can be easily obtained when space-time symmetries exist by means of Noether's theorem (as is done in Sec. III) and the transition to a quantum theory can be made following well-known prescriptions.

In Sec. II we introduce a way of describing the top and its Lagrangian and equations of motion. In Sec. III, we obtain the constants of motion associated with Killing vectors of the metric (along which the Lie derivatives of the torsion vanish).

The conclusions are contained in Sec. IV.

II. LAGRANGIAN AND EQUATIONS OF MOTION

We will treat the top as a test particle, i.e., we will ignore the contribution of its energy-momentum tensor and spin to the equations of motion of the gravitational and torsion fields which we will consider as a given background.

We describe the top by its position $x^\mu(\tau)$ and a tetrad $e^\mu_{(\alpha)}(\tau)$ to denote its orientation, where τ parametrizes the world line of the top. The tetrad consists of four orthonormal vectors where the index in parentheses labels the vector while the other one is a space-time index.

The orthonormality relation is

$$g_{\mu\nu}e^\mu_{(\alpha)}e^\nu_{(\beta)}=\eta_{(\alpha\beta)}=\eta^{(\alpha\beta)}=\mathrm{diag}\,(1,-1,-1,-1)\,, \tag{1}$$

where $\eta_{(\alpha\beta)}$ is a symmetric scalar matrix and $g_{\mu\nu}(x)$ is the metric of space-time. The completeness relation

$$\eta^{(\alpha\beta)}e^\mu_{(\alpha)}e^\nu_{(\beta)}=g^{\mu\nu} \tag{2}$$

follows from (1) and $g^{\mu\nu}(x)$ is the matrix inverse of $g_{\mu\nu}(x)$.

We consider a curved space-time with torsion and vanishing nonmetricity, i.e., the connection $\Gamma_{\alpha\beta}{}^\gamma$ is defined by[6]

$$\Gamma_{\alpha\beta}{}^\gamma=\left\{{}^{\;\gamma}_{\alpha\beta}\right\}-K_{\alpha\beta}{}^\gamma\,, \tag{3}$$

where

$$\left\{ {\gamma \atop \alpha\beta} \right\} = \tfrac{1}{2} g^{\gamma\delta}(-g_{\alpha\beta,\delta} + g_{\beta\delta,\alpha} + g_{\delta\alpha,\beta})$$

is the usual Christoffel symbol of the metric $g_{\mu\nu}$. The contortion tensor $K_{\alpha\beta}{}^{\gamma}$ and the torsion tensor $S_{\alpha\beta}{}^{\gamma}$ are defined by

$$S_{\alpha\beta}{}^{\gamma} = \tfrac{1}{2}(\Gamma_{\alpha\beta}{}^{\gamma} - \Gamma_{\beta\alpha}{}^{\gamma}) = \Gamma_{[\alpha\beta]}{}^{\gamma} = -S_{\beta\alpha}{}^{\gamma} \tag{4}$$

and

$$K_{\alpha\beta}{}^{\gamma} = -S_{\alpha\beta}{}^{\gamma} + S_{\beta}{}^{\gamma}{}_{\alpha} - S^{\gamma}{}_{\alpha\beta} = -K_{\alpha}{}^{\gamma}{}_{\beta}, \tag{5}$$

which means that the covariant derivative of the metric g (taken with respect to the connection Γ) vanishes, i.e., the nonmetricity tensor Q is equal to zero:

$$Q_{\gamma\alpha\beta} = -g_{\alpha\beta;\gamma} = 0. \tag{6}$$

We denote by a semicolon the covariant derivative constructed with the connection Γ, while we use a vertical bar for the one constructed with the connection $\{\ \}$. Let us define the velocity vector u^{μ} and the angular velocity tensor $\sigma^{\mu\nu}$ by

$$u^{\mu} = \dot{x}^{\mu} = \frac{dx^{\mu}}{d\tau}, \tag{7}$$

$$\sigma^{\mu\nu} = \eta^{(\alpha\beta)} e^{\mu}_{(\alpha)} \frac{De^{\nu}_{(\beta)}}{D\tau} = \eta^{(\alpha\beta)} e^{\mu}_{(\alpha)} \left(\frac{de^{\nu}_{(\beta)}}{d\tau} + \Gamma_{\lambda\rho}{}^{\nu} e^{\rho}_{(\beta)} u^{\lambda} \right) = -\sigma^{\nu\mu}. \tag{8}$$

The antisymmetry of $\sigma^{\mu\nu}$ is due to the completeness relation (2) and the fact that nonmetricity vanishes. Because of this antisymmetry, the angular velocity tensor $\sigma^{\mu\nu}$ has only six independent components. One has

$$\frac{De^{\mu}_{(\alpha)}}{D\tau} = -\sigma^{\mu\nu} e_{(\alpha)\nu}.$$

The orthonormality of the tetrad (1) leaves only six degrees of freedom in the four vectors.

For variational purposes we define $\delta\theta^{\mu\nu}$,

$$\delta\theta^{\mu\nu} = \eta^{(\alpha\beta)} e^{\mu}_{(\alpha)} De^{\nu}_{(\beta)} = \eta^{(\alpha\beta)} e^{\mu}_{(\alpha)} (\delta e^{\nu}_{(\beta)} + \Gamma_{\lambda\rho}{}^{\nu} e^{\rho}_{(\beta)} \delta x^{\lambda}) = -\delta\theta^{\nu\mu}, \tag{9}$$

which has only six independent components, and using the relation (1) one gets

$$De^{\mu}_{(\alpha)} = -\delta\theta^{\mu\nu} e_{(\alpha)\nu}.$$

One can then prove that

$$D\sigma^{\mu\nu} \equiv \delta\sigma^{\mu\nu} + \Gamma_{\lambda\rho}{}^{\mu}\sigma^{\rho\nu}\delta x^{\lambda} + \Gamma_{\lambda\rho}{}^{\nu}\sigma^{\mu\rho}\delta x^{\lambda} = \frac{D}{D\tau}(\delta\theta^{\mu\nu}) + \sigma^{\mu\lambda}\delta\theta_{\lambda}{}^{\nu} - \delta\theta^{\mu\lambda}\sigma_{\lambda}{}^{\nu} - g^{\mu\rho} R_{\tau\lambda\rho}{}^{\nu} u^{\lambda} \delta x^{\tau}, \tag{10}$$

where the Riemann tensor R is defined by

$$R_{\alpha\beta\gamma}{}^{\delta} = -\Gamma_{\beta\gamma}{}^{\delta}{}_{,\alpha} + \Gamma_{\alpha\gamma}{}^{\delta}{}_{,\beta} - \Gamma_{\alpha\mu}{}^{\delta}\Gamma_{\beta\gamma}{}^{\mu} + \Gamma_{\beta\mu}{}^{\delta}\Gamma_{\alpha\delta}{}^{\mu}, \tag{11}$$

using Schouten's conventions[7] (up to a sign).

In the special relativistic case the theory of the free top is constructed by defining a Lagrangian as a function of all the scalars that can be formed from the velocities u and σ only, namely

$$\begin{aligned} a_1^0 &= u_{\mu}u^{\mu}, \\ a_2^0 &= \sigma_{\mu\nu}\sigma^{\nu\mu}, \\ a_3^0 &= u_{\mu}\sigma^{\mu\nu}\sigma_{\nu\rho}u^{\rho}, \\ a_4^0 &= \sigma_{\alpha\beta}\sigma^{\beta\gamma}\sigma_{\gamma\delta}\sigma^{\delta\alpha}. \end{aligned} \tag{12}$$

Using the equivalence principle as a guide, we define the Lagrangian as a function of a_1, a_2, a_3, and a_4 only,

$$\begin{aligned} a_1 &= g_{\mu\nu}u^{\mu}u^{\nu}, \\ a_2 &= g_{\alpha\beta}g_{\gamma\delta}\sigma^{\delta\alpha}\sigma^{\beta\gamma}, \\ a_3 &= g_{\alpha\beta}g_{\gamma\delta}g_{\mu\nu}u^{\alpha}\sigma^{\beta\gamma}\sigma^{\delta\mu}u^{\nu}, \\ a_4 &= g_{\alpha\beta}g_{\gamma\delta}g_{\mu\nu}g_{\lambda\rho}\sigma^{\rho\alpha}\sigma^{\beta\gamma}\sigma^{\delta\mu}\sigma^{\nu\lambda}, \end{aligned} \tag{13}$$

where u and σ are defined by (7) and (8), respectively. These definitions are in strict accordance with the equivalence principle which states that the flat space-time metric and partial derivatives must be replaced by the curved space-time metric and covariant derivatives with respect to the connection Γ, respectively. The Lagrangian L for the top is then

$$L = L(a_1, a_2, a_3, a_4), \tag{14}$$

but because τ is arbitrary and therefore unobservable, L must be a homogeneous function of degree one in the velocities so that the action will be τ-reparametrization invariant, i.e., L may be written as

$$L = (a_1)^{1/2} \mathcal{L}\left(\frac{a_2}{a_1}, \frac{a_3}{a_1^2}, \frac{a_4}{a_1^2}\right). \tag{15}$$

We may now define the momentum vector P^{μ} and the spin tensor $S^{\mu\nu}$,

$$P^{\mu} = -\frac{\partial L}{\partial u_{\mu}} = -2u^{\mu}\frac{\partial L}{\partial a_1} - 2\sigma^{\mu\nu}\sigma_{\nu\lambda}u^{\lambda}\frac{\partial L}{\partial a_3}, \tag{16}$$

$$S^{\mu\nu} = -\frac{\partial L}{\partial \sigma_{\mu\nu}} = -4\sigma^{\nu\mu}\frac{\partial L}{\partial a_2} - 4u^{[\mu}\sigma^{\nu]\lambda}u_{\lambda}\frac{\partial L}{\partial a_3} - 8\sigma^{\nu\lambda}\sigma_{\lambda\tau}\sigma^{\tau\mu}\frac{\partial L}{\partial a_4}, \tag{17}$$

and we may prove that

$$P^{\mu}u^{\nu} - P^{\nu}u^{\mu} = S^{\mu\lambda}\sigma_{\lambda}{}^{\nu} - \sigma^{\mu\lambda}S_{\lambda}{}^{\nu}. \tag{18}$$

The Euler-Langrange equations obtained from the action

$$S=\int L\,d\tau \tag{19}$$

for arbitrary variations δx^μ and $\delta\theta^{\mu\nu}$ are

$$\frac{DP^\mu}{D\tau}=-\tfrac{1}{2}R^\mu{}_{\nu\alpha\beta}u^\nu S^{\alpha\beta}+2S^\mu{}_{\alpha\beta}P^\beta u^\alpha\,, \tag{20}$$

$$\frac{DS^{\mu\nu}}{D\tau}=S^{\mu\lambda}\sigma_\lambda{}^\nu-\sigma^{\mu\lambda}S_\lambda{}^\nu \tag{21a}$$

$$=P^\mu u^\nu-P^\nu u^\mu\,. \tag{21b}$$

These equations exhibit clearly the spin-torsion interaction.

From the first line of Eq. (21) one gets that the magnitude of the spin $J^2=\frac{1}{2}S_{\mu\nu}S^{\mu\nu}$ is a constant of the motion. In the same way as in Refs. 1 and 2 we choose the constraint

$$S^{\mu\nu}P_\nu=0 \tag{22}$$

in order to obtain the proper nonrelativistic limit (for details see Ref. 2).

This constraint, the identity

$$u^\mu\equiv\frac{(u_\nu P^\nu)}{M^2}P^\mu-\frac{(P^\mu u^\nu-P^\nu u^\mu)P_\nu}{M^2}$$

(where $M^2=P_\mu P^\mu$), the second line of Eq. (21), and the equation of motion (20) for P^μ allow us to show that the mass M^2 is a constant of the motion.

The τ-parametrization invariance of the action implies that there exists a function f much that

$$M^2=f(J^2)\,, \tag{23}$$

i.e., a Regge trajectory links the mass and the spin. At this stage, the function f as well as the Lagrangian are arbitrary functions. As a matter of fact, one may prove that for each L one can find an f and for each f there is at least one L of the form defined by (14) and (15) such that the constraint (22) can be obtained as a consequence of the definitions (16) and (17) for P^μ and $S^{\mu\nu}$, respectively. For details the reader is referred to Ref. 1.

To simplify the solution of the equations of motion one may choose the gauges and invariant relations associated with the constraints (22) and (23) which were proposed in Ref. 1:

$$e^\mu_{(0)}-\frac{P^\mu}{M}=0\,, \tag{24}$$

$$x^0-\tau=0\,. \tag{25}$$

It is worthwhile noting that the equations of motion presented here are second-order differential equations as opposed to those of Refs. 4 and 5 which contain *third* derivatives with respect to τ.

Both sets of equations look the same, though, when written in terms of momentum and spin, but they do not coincide when written in terms of x^μ and $e^\mu_{(\alpha)}$ because of the different definitions of momentum in terms of the position and the tetrad vectors.

III. KILLING VECTORS AND CONSTANTS OF THE MOTION

Let us consider the case where both the metric and the torsion possess the same symmetry, that is, when there exists at least one Killing vector ξ^μ of the metric such that the Lie derivative of the torsion along it vanishes. These facts can be precisely stated as follows: ξ^μ is such that

$$\xi_{\mu|\nu}+\xi_{\nu|\mu}=0 \tag{26}$$

or

$$\xi_{\mu,\nu}+\xi_{\nu,\mu}-2\left\{{\lambda \atop \mu\nu}\right\}\xi_\lambda=0 \tag{27}$$

and

$$\mathcal{L}_\xi K_{\alpha\beta}{}^\gamma=0=K_{\alpha\beta}{}^\gamma{}_{,\rho}\xi^\rho+K_{\mu\beta}{}^\gamma\xi^\mu{}_{,\rho}+K_{\alpha\mu}{}^\gamma\xi^\mu{}_{,\rho}-K_{\alpha\beta}{}^\mu\xi^\rho{}_{,\mu} \tag{28}$$

are true simultaneously.

These statements are equivalent to saying that the metric and contorsion tensors are invariant under the transformation

$$x'^\mu=x^\mu+\xi^\mu(x) \tag{29}$$

with

$$g'_{\mu\nu}(x')=\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(x)\,, \tag{30}$$

$$K'^{\rho}_{\mu\nu}(x')=\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x'^\rho}{\partial x^\gamma}K_{\alpha\beta}{}^\gamma(x)\,, \tag{31}$$

i.e., that $g'_{\mu\nu}(x)=g_{\mu\nu}(x)$ and $K'_{\alpha\beta}{}^\gamma(x)=K_{\alpha\beta}{}^\gamma(x)$ are satisfied simultaneously.

For spaces with vanishing torsion one can relate these invariances to constants of the motion.[2] The same can be done here and we will use Noether's theorem to find the constant of the motion C_ξ associated with the invariance of the action under the transformation (29).

To prove that the Lagrangian (14) is invariant under (29) it is useful to prove that

$$\delta u^\mu=\xi^\mu{}_{,\lambda}u^\lambda\,, \tag{32}$$

$$\delta\sigma^{\mu\nu}=\xi^\mu{}_{,\lambda}\sigma^{\lambda\nu}+\xi^\nu{}_{,\lambda}\sigma^{\mu\lambda}\,, \tag{33}$$

and then one can readily verify that a_1, a_2, a_3, and a_4 (and therefore the Lagrangian) are invariant under the transformation considered.

The constant of the motion associated with the Killing vector is therefore (using Noether's theorem)

$$C_\xi = P^\mu \xi_\mu - \tfrac{1}{2} S^{\mu\nu}(\xi_{\mu|\nu} + K_{\alpha\mu\nu}\xi^\alpha). \quad (34)$$

This constant also exhibits clearly the spin-torsion interaction.

In a forthcoming paper[8] the constants of the motion associated with Killing vectors of solutions of the graviton-tlaplon system will be constructed to test the theory using tops as probes.

IV. CONCLUSIONS

We have constructed a Lagrangian theory of the motion of tops in given gravitational background specified by any Riemannian metric and torsion.

The equations of motion that we obtain for the momentum and spin are second-order differential equations for the position and the angles (related to the orientation) of the top. Both the mass and the magnitude of the spin are conserved and there exists a Regge trajectory relating them. The constants of the motion related to Killing vectors of the metric that leave the torsion invariant are presented.

The theory presented here may be useful in testing torsion theories using tops as probes. Such an application will be presented elsewhere.[8]

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