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Inverse scattering problem for quarkonium systems. I. One-dimensonal formalism and methodology

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The inverse scattering formalism for reflectionless potentials is applied to the reconstruction of confining potentials from bound-state properties. An explicit algebraic technique is presented and tested on several one-dimensional examples. The connection with a classical problem of interacting solitons is exhibited.

I. INTRODUCTION

The families of extremely massive hadrons discovered during the past three years are widely believed to indicate the existence of new flavors of heavy quarks. The psions[1] now are firmly established as bound states of a charmed quark and antiquark, and it is popular to assume that $\Upsilon(9.4)$, $\Upsilon'(10.0), \ldots$[2] are bound states of one or more species of new quarks. The general agreement[3] between the spectroscopy of psions and the predictions of simple nonrelativistic potential models encourages the belief that heavy quarkonium systems may be meaningfully discussed in the context of the Schrödinger equation for a central potential,

$$\left(-\frac{\nabla^2}{2\mu}+V(r)\right)\Psi(\vec{\mathbf{r}},t)=i\frac{\partial\Psi}{\partial t}(\vec{\mathbf{r}},t)\,. \tag{1.1}$$

Such an approximation should be even more reliable for the quarks which make up Υ than for the ψ.[4] This opens the possibility of studying the interaction of quarks in a situation which is greatly simplified in comparison with ordinary light hadrons.

Within the framework of nonrelativistic potential models, the question of how quarks interact becomes sharply defined. It is the inverse scattering problem of the Schrödinger equation: How and to what extent does the spectrum of a quarkonium system measure the interquark potential? The mathematical problem this poses has been studied for many years in other contexts, and a rich formalism has grown up around it.[5-11] In this article and the sequel, we shall explore some of these techniques and study the possibility of deriving the interquark potential directly from spectroscopic data. This first paper deals primarily with formalism and methodology. Specific applications to heavy-quark systems are presented in the following paper.

The literature on the inverse scattering problem is voluminous and we will not attempt to review it thoroughly. The procedure derived and discussed in Sec. II is based upon the techniques of Gel'fand and Levitan[8] and of Kay and Moses.[12] In order to bring out the various aspects of the problem in proper sequence, we shall restrict our attention for the present to the one-dimensional problem

$$\left(-\frac{\partial^2}{\partial x^2}+V(x)\right)\phi(x,k)=k^2\phi(x,k)\,. \tag{1.2}$$

The extension to the radial equation in three spatial dimensions is mentioned in Sec. V and discussed at length in the following paper.

Broadly speaking, the Gel'fand-Levitan method may be viewed as a dispersion theory for the Schrödinger wave function. From solutions to (1.2) with k^2 replaced by a complex eigenvalue ζ^2, one can construct an analytic function $\Phi(x,\zeta)$ which approaches unity as $|\zeta|\to\infty$. Thus, $\Phi(x\ \zeta)$ is completely determined by its singularity structure which consists of a cut along the real ζ axis and some number N of bound-state poles on the positive imaginary axis. The spectral weight of the cut is essentially a scattering-state wave function multiplied by the reflection coefficient, both evaluated at real k. Similarly, the pole residues are essentially constants times bound-state wave functions. Upon Fourier transformation the dispersion relation for Φ becomes the Gel'fand-Levitan integral equation which determines the wave functions. The kernel of this equation is given entirely in terms of the reflection coefficient and $2N$ bound-state parameters.

If the reflection coefficient vanishes for all real values of k, the Gel'fand-Levitan equation can be solved exactly by algebraic techniques. The only singularities of $\Phi(x,\zeta)$ are the bound-state poles, and the integral equation reduces to a system of N linear algebraic equations for the bound-state wave functions. Cramer's rule gives an explicit formula for the bound-state wave functions in terms of $2N$

parameters. A $2N$-parameter expression for the potential $V(x)$ is also obtained. Half of the parameters are determined by the bound-state energies. The remaining N parameters may be fixed either by some explicit piece of information about the bound-state wave functions (e.g., their values at $x=0$) or by imposing the requirement $V(x)=V(-x)$.[13] The latter procedure is of more interest to us since it forms the most convenient bridge to the radial Schrödinger equation, for which we must concern ourselves with boundary conditions at $x=0$. The general choice of parameters which yields a symmetric potential is derived in Sec. III.

Thus, when the physical situation is such that the continuum part of the spectral function can be ignored, the inverse problem is completely and explicitly solved. The result is a symmetric, reflectionless potential which binds N states at arbitrarily adjustable energies. For a strictly confining potential there is perforce no continuum, but it would be impractical to reconstruct a potential from an infinite number of bound states. We therefore must ask how well the inverse scattering formulas work with only partial information, namely the energies of a few low-lying bound states. The behavior of the sequence of approximations $V_N(x)$ to $V(x)$ obtained as more bound states are included poses a well-defined mathematical question of convergence, which we shall not address here. Instead we study the successive approximations obtained for three simple examples: The linear, harmonic-oscillator, and infinite-square-well potentials. The results, described in Sec. IV, clearly suggest that an arbitrarily accurate local approximation to any reasonably smooth confining potential is provided by a reflectionless potential as more bound states are included. For any finite number of bound states, the potential constructed in this fashion is evidently not unique. It is possible to imagine a great variety of other parameterizations which may be adjusted to fit the observed bound-state spectrum. The method we propose is attractive because the parameters which arise are directly related to the bound-state energies, because the reconstructed potential is given algebraically in terms of those parameters, and because successive approximations to the potential are easily generated. Moreover, in the region of x for which the relevant bound-state wave functions are not negligible, the approximations are quite good even for $N=3$ or 4. An interquark potential constructed from information about $\psi(3095)$ and $\psi'(3684)$ corresponds to the case if $N=4$ in one dimension. Consequently, we expect to obtain from the accessible data a fairly accurate impression of the interquark interaction.

In the next section, we derive the Gel'fand-Levitan equations in the form most convenient for the quarkonium problem. The connection between the inverse scattering problem of the Schrödinger equation and the nonlinear Korteweg–de Vries equation[14] is explained in Sec. III. We compare in Sec. IV approximate potentials and wave functions with exact results for confining potentials. Discussion of these results and the application to quarkonium systems occupies Sec. V.

II. THE GEL'FAND-LEVITAN EQUATION[8]

Consider the eigenvalue problem of the Schrödinger equation for a potential $V(x)$ in one space dimension,

$$\left(-\frac{\partial^2}{\partial x^2}+V(x)\right)\phi(x,k)=k^2\phi(x,k)\,. \tag{2.1}$$

In order to formulate the direct and inverse scattering theory for (2.1), we shall assume that $V(x)$ approaches at infinity a constant which we take to be zero,

$$V(x)\xrightarrow[|x|\to\infty]{}0\,. \tag{2.2}$$

From an operational standpoint, the restriction (2.2) will be removed in Sec. IV where we show that confining potentials $[V(\pm\infty)=\infty]$ can be locally reconstructed by the same analysis. The scattering data for the potential $V(x)$ can be defined in terms of particular solutions to (2.1) which obey prescribed boundary conditions at infinity. Let ϕ_1 and ϕ_2 be solutions to (2.1) with asymptotic behavior

$$\phi_1(x,k)\sim e^{ikx}\text{ as }x\to+\infty\,, \tag{2.3}$$

$$\phi_2(x,k)\sim e^{-ikx}\text{ as }x\to-\infty\,. \tag{2.4}$$

For real values of k, the function $\phi_1^*(x,k)=\phi_1(x,-k)$ is also a solution to (2.1) which is linearly independent of $\phi_1(x,k)$. Hence ϕ_2 can be written as a linear combination of ϕ_1 and ϕ_1^* as

$$\phi_2(x,k)=a(k)\phi_1^*(x,k)+b(k)\phi_1(x,k)\,. \tag{2.5}$$

The usual reflection and transmission coefficients are related to the coefficients in (2.5) by

$$R(k)=b(k)/a(k)\,, \tag{2.6a}$$

$$T(k)=1/a(k)\,. \tag{2.6b}$$

The Wronskian of any two solutions $\Psi_1(x,k)$ and $\Psi_2(x,k)$ of Eq. (2.1) is independent of x,

$$\partial_x(\Psi_1\overleftrightarrow{\partial}_x\Psi_2)=\Psi_1\Psi_2'-\Psi_1'\Psi_2=0\,. \tag{2.7}$$

Thus we may compute the Wronskian of $\phi_1(x,k)$ or $\phi_1^*(x,k)$ with both sides of (2.5), evaluating the right-hand side in the asymptotic regime $x\to+\infty$. By this device we arrive at expressions for the scattering data directly in terms of the wave functions:

$$a(k)=\frac{i}{2k}(\phi_1\overleftrightarrow{\partial}_x\phi_2)\,, \tag{2.8}$$

$$b(k)=\frac{-i}{2k}(\phi_1^*\overleftrightarrow{\partial}_x\phi_2)\,. \tag{2.9}$$

The functions $\phi_1(x,\zeta)$ and $\phi_2(x,\zeta)$ can be analytically continued into the upper-half ζ plane. (We shall use k and ζ when referring to real and complex values, respectively.) Analyticity for $\mathrm{Im}\zeta>0$ follows[9] from the Volterra integral equations which incorporate (2.1) and the boundary conditions (2.3) and (2.4):

$$\phi_1(x,\zeta)=e^{i\zeta x}+\frac{1}{\zeta}\int_x^\infty dy\,\sin[\zeta(y-x)]V(y)\phi_1(y,\zeta)\,, \tag{2.10a}$$

$$\phi_2(x,\zeta)=e^{-i\zeta x}+\frac{1}{\zeta}\int_{-\infty}^x dy\,\sin[\zeta(x-y)]V(y)\phi_2(y,\zeta)\,. \tag{2.10b}$$

Equation (2.8) also allows us to regard a(k) as the boundary value of an analytic function $a(\zeta)$. Because of the reality of the potential $V(x)$, it follows from (2.1) that

$$\phi_1^*(x,\zeta)=\phi_1(x,-\zeta^*)\,, \tag{2.11}$$

$$\phi_2^*(x,\zeta)=\phi_2(x,-\zeta^*)\,, \tag{2.12}$$

and hence, using (2.8),

$$a^*(\zeta)=a(-\zeta^*)\,. \tag{2.13}$$

From this result it is easily shown that the zeros of $a(\zeta)$ in the upper half-plane must lie along the imaginary axis at $\zeta=\zeta_n\equiv i\kappa_n$, $n=1,2,\ldots,N$, with κ_n real. As shown by (2.6), these zeros correspond to bound-state poles in the S matrix.

In essence, the Gel'fand-Levitan equation is a dispersion relation for an analytic function $\Phi(x,\zeta)$ which is defined as[15]

$$\Phi(x,\zeta)=\begin{cases}a^{-1}(\zeta)\phi_2(x,\zeta)e^{i\zeta x}, & \mathrm{Im}\zeta>0 \qquad (2.14a)\\ \phi_1^*(x,\zeta^*)e^{i\zeta x}, & \mathrm{Im}\zeta<0\,. \qquad (2.14b)\end{cases}$$

The choice of (2.14a) is suggested by the fact that $a^{-1}(k)\phi_2(x,k)$ is a wave function which obeys scattering boundary conditions. It consists of an incoming wave (from the right) with coefficient unity and transmitted and reflected waves multiplied by the S-matrix elements $T(k)$ and $R(k)$, respectively. In addition to the poles at $\zeta=i\kappa_n$, $\Phi(x,\zeta)$ will in general have a cut along the real axis with discontinuity

$$\Phi(x,k+i\epsilon)-\Phi(x,k-i\epsilon)\equiv\rho(x,k)$$

$$=\frac{b(k)}{a(k)}e^{ikx}\phi_1(x,k)\,. \tag{2.15}$$

Thus, the choice of (2.14b) in the lower half-plane provides a function with discontinuity proportional to the reflection coefficient. From the integral equations (2.10) it is easily shown that $\Phi(x,\zeta)$ approaches unity as $|\zeta|\to\infty$. Thus the function $\Phi(x,\zeta)$ can be reconstructed from its singularities as

$$\Phi(x,\zeta)=1+\sum_{n=1}^N\frac{e^{-\kappa_n x}}{\zeta-i\kappa_n}\frac{\phi_2(x,i\kappa_n)}{a'(i\kappa_n)}$$

$$+\frac{1}{2\pi i}\int_{-\infty}^\infty\frac{dk'\rho(x,k')}{k'-\zeta}\,. \tag{2.16}$$

The Gel'fand-Levitan equation is obtained from (2.16) by letting $\zeta\to k-i\epsilon$ and Fourier transforming with respect to k. For our purposes, however, the dispersive form will be more useful. For the reasons outlined in Sec. I, we shall be particularly concerned with reflectionless potentials, for which $R(k)$ vanishes for all real values of k. Thus we set $\rho(x,k)=0$ in (2.16), whereupon

$$\Phi(x,\zeta)=1+\sum_{n=1}^N\frac{e^{-\kappa_n x}}{\zeta-i\kappa_n}\frac{\phi_2(x,i\kappa_n)}{a'(i\kappa_n)}\,. \tag{2.17}$$

From (2.5) it is seen that the bound-state wave functions $\phi_1(x,i\kappa_n)$ and $\phi_2(x,i\kappa_n)$ are proportional,

$$\phi_2(x,i\kappa_n)=b(i\kappa_n)\phi_1(x,i\kappa_n)\,. \tag{2.18}$$

Thus if (2.17) is evaluated in the lower half-plane at the N points $\zeta=-i\kappa_m$, $m=1,2,\ldots,N$, it yields a system of linear algebraic equations for the N bound-state wave functions,

$$\phi_1(x,i\kappa_m)e^{\kappa_m x}=1-\sum_{n=1}^N\frac{c_n^{\,2}e^{-\kappa_n x}}{\kappa_m+\kappa_n}\phi_1(x,i\kappa_n)\,, \tag{2.19}$$

where

$$c_n^{\,2}=-ib(ik_n)/a'(i\kappa_n) \tag{2.20}$$

and m runs from 1 to N.

A more symmetric form results from the definition

$$\psi_n(x)=c_n\phi_1(x,i\kappa_n)\,, \tag{2.21}$$

where $\psi_n(x)$ is a normalized bound-state wave function, which satisfies

$$\int_{-\infty}^\infty dx[\psi_n(x)]^2=1\,. \tag{2.22}$$

We rewrite (2.19) as

$$\sum_{n=1}^N A_{mn}\psi_n(x)=\lambda_m(x)\,, \tag{2.23}$$

where

$$\lambda_n(x)\equiv c_n e^{-\kappa_n x} \tag{2.24}$$

and the symmetric matrix A is defined by

$$A_{mn}=\delta_{mn}+\frac{\lambda_m\lambda_n}{\kappa_m+\kappa_n}\,. \tag{2.25}$$

Finally, we define a matrix $A^{(n)}$ which is obtained from A by replacing the nth column by its deriva-

tive. Noting that

$$\partial_x\left(\frac{\lambda_m\lambda_n}{\kappa_m+\kappa_n}\right)=-\lambda_m\lambda_n\,, \tag{2.26}$$

we write the bound-state wave functions explicitly as

$$\psi_n(x)=-\frac{1}{\lambda_n}\frac{\mathrm{Det}\,A^{(n)}}{\mathrm{Det}\,A}\,. \tag{2.27}$$

Equation (2.27) gives the bound-state wave functions in a reflectionless potential in terms of the $2N$ parameters κ_n and c_n. The potential itself may be expressed in terms of the same parameters by considering the function

$$\chi(x,\zeta)=\phi_1(x,\zeta)e^{-i\zeta x}\,. \tag{2.28}$$

From the Schrödinger equation (2.1), we have

$$\chi''+2i\zeta\chi'=V\chi\,, \tag{2.29}$$

where prime denotes a derivative with respect to x. The function $\chi(x,\zeta)$ may be expanded in inverse powers of ζ as

$$\chi(x,\zeta)\sim 1-\frac{1}{2i\zeta}\int_x^\infty dx'V(x')+O\left(\frac{1}{\zeta^2}\right). \tag{2.30}$$

This is to be compared with the $|\zeta|\to\infty$, $\mathrm{Im}\zeta<0$ limit of (2.17)

$$\phi_1^*(x,\zeta^*)e^{i\zeta x}\sim 1-\frac{1}{i\zeta}\sum_{n=1}^{N}c_n^{\,2}e^{-\kappa_n x}\phi_1(x,i\kappa_n)+O\left(\frac{1}{\zeta^2}\right), \tag{2.31}$$

or

$$\chi(x,\zeta)\sim 1-\frac{i}{\zeta}\sum_{n=1}^{N}c_n^{\,2}e^{-\kappa_n x}\phi_1(x,i\kappa_n)+O\left(\frac{1}{\zeta^2}\right). \tag{2.32}$$

By comparing (2.30) and (2.32) we obtain the integrated potential in terms of the bound-state wave functions,

$$\int_x^\infty dx'V(x')=-2\sum_{n=1}^{N}\lambda_n(x)\psi_n(x)\,. \tag{2.33}$$

A concise expression for the potential follows from the substitution of (2.27) into (2.33) and the observation that

$$\frac{d}{dx}\ln(\mathrm{Det}\,A)=\sum_{n=1}^{N}\frac{\mathrm{Det}\,A^{(n)}}{\mathrm{Det}\,A}\,, \tag{2.34}$$

which leads to

$$\int_x^\infty dx'V(x')=2\frac{d}{dx}\ln(\mathrm{Det}\,A)\,. \tag{2.35}$$

We therefore find a $2N$-parameter formula for a reflectionless potential with N bound states,[16]

$$V(x)=-2\frac{d^2}{dx^2}\ln[\mathrm{Det}\,A(x)]\,. \tag{2.36}$$

Equations (2.36) and (2.25) form the basis of our subsequent analysis of specific potentials in terms of their spectral properties.

To illustrate the use of these formulas, let us consider the case of a reflectionless potential with a single bound state, $N=1$. Equation (2.25) becomes

$$A=1+\frac{c^2}{2\kappa}e^{-2\kappa x}\,, \tag{2.37}$$

so the potential obtained from (2.36) is

$$V(x)=-2\frac{d^2}{dx^2}\ln(A)=-2\kappa^2\mathrm{sech}^2[\kappa(x-x_0)]\,, \tag{2.38}$$

where

$$x_0=\frac{1}{2\kappa}\ln\left(\frac{c^2}{2\kappa}\right). \tag{2.39}$$

This result clarifies the significance of the parameters c and κ in this simplest case. The binding energy κ^2 fixes the depth of the potential, and the parameter c is related through (2.39) to the position of its center. We compute the bound-state wave function using (2.27):

$$\psi(x)=\frac{-1}{\lambda A}\frac{dA}{dx}=\left(\frac{\kappa}{2}\right)^{1/2}\mathrm{sech}[\kappa(x-x_0)]\,. \tag{2.40}$$

The Schrödinger equation

$$\left(-\frac{\partial^2}{\partial x^2}+V(x)\right)\psi(x)=-\kappa^2\psi(x) \tag{2.41}$$

may be verified directly from (2.38) and (2.40).

III. REFLECTIONLESS POTENTIALS, SOLITONS, AND THE KORTEWEG-DE VRIES EQUATION

Some very useful intuition about the formalism constructed in the preceding section can be gained by reviewing the famous connection[17,18] between the inverse scattering problem of the Schrödinger equation and the nonlinear Korteweg–de Vries (KdV) equation,[14]

$$v_t-6vv_x+v_{xxx}=0\,. \tag{3.1}$$

Here $v(x,t)$ is a function of x which depends on a time parameter t, and the subscripts denote partial derivatives. The KdV equation first arose in the study of shallow water waves and is relevant to a variety of systems which exhibit a balance between nonlinear and dispersive effects [the second and third terms in (3.1), respectively]. A most conspicuous result of this balance is the existence of soliton[19] solutions to (3.1). In this section we shall review these well-known results and bring out the direct connection between reflectionless potentials of the Schrödinger equation and N-soliton systems

of the KdV equation.

The reflectionless potential formula (2.36) is precisely a pure N-soliton configuration at a fixed time t_0, the potential $V(x)$ being identified with the KdV "field" $v(x,t_0)$. The parameters c_n and κ_n are directly related to the positions and sizes of the individual solitons. The speed of a KdV soliton is not independent, but is determined by its size. In addition to providing us with a better perspective on the nature of reflectionless potentials, the KdV connection allows us rather easily to construct potentials with certain desired properties such as reflection symmetry by manipulation of solitons.

Our treatment follows the approach of Lax.[20] We define a differential operator

$$B \equiv -4i\frac{\partial^3}{\partial x^3} + 3i\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial x}v\right), \tag{3.2}$$

where $v(x,t)$ is a solution to the KdV equation (3.1). At a given time t, the function $v(x,t)$ may be interpreted as a scattering potential, with the corresponding Schrödinger operator

$$L = -\frac{\partial^2}{\partial x^2} + v(x,t). \tag{3.3}$$

The commutator of the operators defined by (3.2) and (3.3) is found, after some calculation, to be

$$[B,L] = -iv_{xxx} + 6ivv_x = i\frac{\partial L}{\partial t}, \tag{3.4}$$

the last equality following from the KdV equation (3.1) for v. Thus B may be interpreted as the generator of time evolution which propagates the potential v (and its Schrödinger operator) according to the KdV equation. (The propagation of v generated by B should not be confused with the usual time evolution of the Schrödinger equation generated by L, which propagates particles through a fixed potential.)

Since the operator B is Hermitian, the time evolution of the potential is effected by a unitary transformation

$$U(t) = e^{-iBt}. \tag{3.5}$$

The Schrödinger operator $L(t)$ is expressed in terms of its form at $t=0$ by

$$L(t) = U(t)L(0)U^{\dagger}(t). \tag{3.6}$$

Moreover, the potential associated with $L(t)$ supports N bound states with wave functions given by

$$\psi_n(x,t) = U(t)\psi_n(x). \tag{3.7}$$

That is to say, if the wave functions $\psi_n(x)$ satisfy the Schrödinger equation at $t=0$ [with $v(x,0) \equiv V(x)$],

$$L(0)\psi_n(x) = -\kappa_n{}^2\psi_n(x), \tag{3.8}$$

then the functions defined by (3.7) satisfy the corresponding equation at time t, *with the same eigenvalues*,

$$L(t)\psi_n(x,t) = -\kappa_n{}^2\psi_n(x,t). \tag{3.9}$$

The unitarity of U enables us to verify the normalization condition (2.22) at $t=-\infty$, when all solitons are widely separated. Thus, as a reflectionless potential (an N-soliton system) evolves in time according to the KdV equation, its bound-state energy spectrum remains unchanged.[21]

Equation (3.7) can be rewritten in differential form as

$$i\frac{\partial\psi_n}{\partial t}(x,t) = B\psi_n(x,t). \tag{3.10}$$

The idea of the inverse scattering approach to KdV solitons is to locate an individual soliton in the system by studying the asymptotic behavior of the corresponding bound-state wave function. It follows from Eqs. (2.3) and (2.21) that

$$\psi_n(x,t) \sim c_n(t)e^{-\kappa_n x} \text{ as } x \to +\infty). \tag{3.11}$$

In the asymptotic regions $x \to \pm\infty$, the operator B assumes the simple form

$$B \to -4i\frac{\partial^3}{\partial x^3} \quad (x \to \pm\infty). \tag{3.12}$$

Thus (3.10) simplifies to an equation for the time dependence of the coefficients $c_n(t)$,

$$\frac{\partial c_n}{\partial t}(t) = 4\kappa_n{}^3 c_n(t), \tag{3.13}$$

with the solution

$$c_n(t) = c_n(0)e^{4\kappa_n{}^3 t}. \tag{3.14}$$

To recapitulate, if a reflectionless potential is written in terms of the parameters c_n and κ_n by Eq. (2.36), and if the coefficients $c_n(t)$ are permitted to vary according to (3.14), the resulting time-dependent function is an N-soliton solution to the KdV equation.

Since our ultimate aim is to consider the radial Schrödinger equation for a system in three spatial dimensions, we will be particularly interested in the reconstruction of potentials which are symmetric about $x=0$,

$$V(x) = V(-x). \tag{3.15}$$

For the remainder of this section we will use the intuition derived from the foregoing discussion of solitons to find a general ansatz for the parameters c_n which will guarantee symmetry of the potential. If there are no degeneracies in the spectrum of $V(x) = v(x,0)$, which we hereafter assume, (3.15) can be satisfied only if each of the N solitons described by $v(x,t)$ is located at the origin at $t=0$. Let us order the eigenvalues as

$$\kappa_1 > \kappa_2 > \cdots > \kappa_N . \tag{3.16}$$

By transforming to a reference frame that is co-moving with the nth soliton

$$\xi = x - 4\kappa_n^2 t , \tag{3.17}$$

it can be shown[22] that

$$\lim_{\substack{t\to\infty,\\ \xi \text{ fixed}}} v(x,t) = -2\kappa_n^2 \operatorname{sech}^2[\kappa_n(x - 4\kappa_n^2 t - \xi_n)] . \tag{3.18}$$

This is precisely the expression for an isolated KdV soliton, all others having moved off to $x = \pm\infty$ where they may be neglected. The nth soliton is displaced from the origin of the comoving frame by an amount

$$\xi_n = \frac{1}{\kappa_n}\left[\tfrac{1}{2}\ln\left(\frac{c_n(0)^2}{2\kappa_n}\right) - \sum_{m=1}^{n-1} \ln\left(\frac{\kappa_m + \kappa_n}{\kappa_m - \kappa_n}\right)\right] . \tag{3.19}$$

Similarly it is found that, at large negative times, the N-soliton solution reduces in the frame (3.17) to

$$\lim_{\substack{t\to-\infty\\ \xi \text{ fixed}}} v(x,t) = -2\kappa_n^2 \operatorname{sech}^2[\kappa_n(x - 4\kappa_n^2 t - \bar{\xi}_n)] , \tag{3.20}$$

where

$$\bar{\xi}_n = \frac{1}{\kappa_n}\left[\tfrac{1}{2}\ln\left(\frac{c_n(0)^2}{2\kappa_n}\right) - \sum_{m=n+1}^{N} \ln\left(\frac{\kappa_m + \kappa_n}{\kappa_m - \kappa_n}\right)\right] . \tag{3.21}$$

Thus the total shift of the soliton's trajectory is

$$\xi_n - \bar{\xi}_n = \frac{1}{\kappa_n}\left[\sum_{m=1}^{N} \ln\left(\frac{\kappa_n + \kappa_m}{\kappa_n - \kappa_m}\right) - \sum_{m=1}^{n-1} \ln\left(\frac{\kappa_m + \kappa_n}{\kappa_m - \kappa_n}\right)\right] . \tag{3.22}$$

It is clear that a symmetric potential is obtained at $t = 0$ if we choose the parameters $c_n(0)$ to ensure that

$$\xi_n = -\bar{\xi}_n , \tag{3.23}$$

which requires

$$\frac{c_n(0)^2}{2\kappa_n} = \prod_{m\neq n} \left|\frac{\kappa_m + \kappa_n}{\kappa_m - \kappa_n}\right| . \tag{3.24}$$

The construction of a symmetric potential at $t = 0$ by the coalescence of several solitons is depicted graphically in Fig. 1. For large negative values of t, the N potentials (each of which supports a single bound state) are isolated and do not interact. At $t = 0$ the (nonlinear) superposition is the symmetric potential which supports N bound states with the

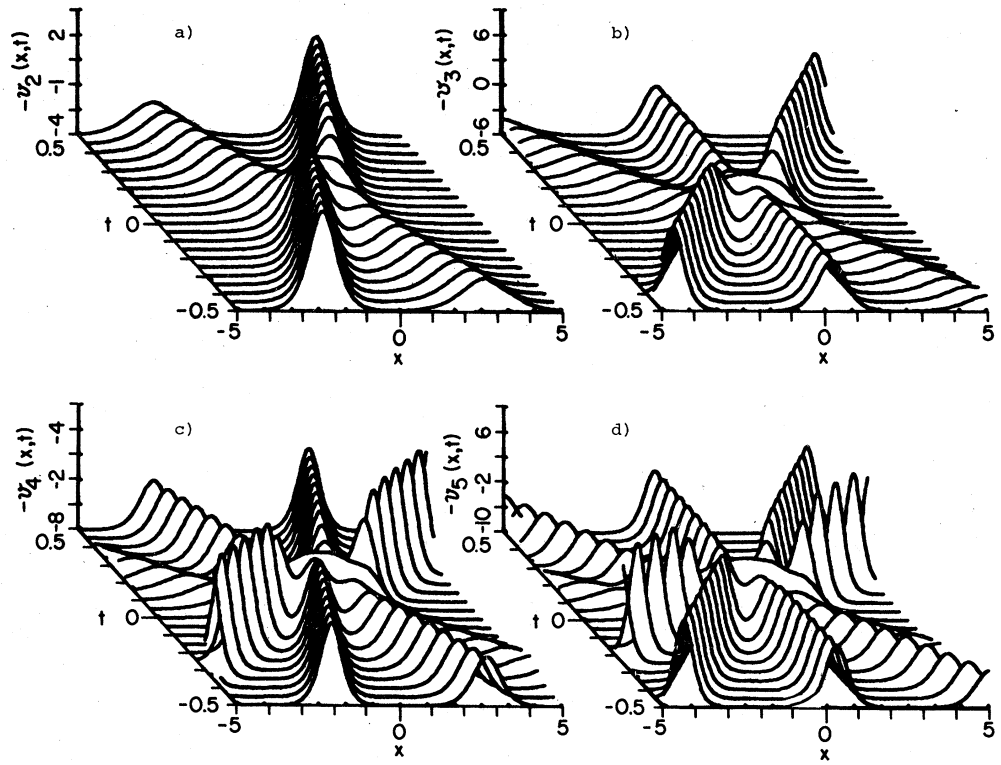

FIG. 1. The scattering of solitons appropriate to the reconstruction of a symmetric potential. The negative of the KdV field $v_N(x,t)$ is plotted for a better visual effect. The isolated disturbances for large values of $|t|$ correspond to potential wells, each supporting a single bound state with energy eigenvalues of the harmonic-oscillator system. The symmetric disturbance at $t = 0$ corresponds to the approximate potential reconstructed from N bound states: (a) two-bound-states case, (b) three-bound-states case, (c) four-bound-states case, (d) five-bound-states case.

prescribed energies. The individual solitons recover their initial shapes as $t \to +\infty$. The phase shift (3.22) is immediately apparent.

A simple semiclassical argument serves to illustrate the importance of the symmetry condition (3.15). Let $V(x) = V(-x)$ be monotonically increasing for $x > 0$. The WKB quantization condition may be written as

$$2\int_0^{x_0} dx[E_n - V(x)]^{1/2} = (n+\tfrac{1}{2})\pi\,, \tag{3.25}$$

where the classical turning point x_0 is defined by $V(x_0) = E_n$. If (3.25) is differentiated with respect to n, and the variable of integration is changed to V, one finds

$$\int_{V(0)}^{E} \frac{dV}{(E-V)^{1/2}} \frac{dx}{dV} = \pi\left(\frac{dE}{dn}\right)^{-1}. \tag{3.26}$$

We may invert (3.26) by multiplying by $(V-E)^{-1/2}$ and integrating with respect to E. The result is that[23]

$$x(V) = \int_{V(0)}^{V} \frac{dE}{(V-E)^{1/2}} \left(\frac{dE}{dn}^{-1}\right). \tag{3.27}$$

Hence, knowledge of the bound-state spectrum suffices to determine a unique, symmetric, monotonic potential, in the semiclassical approximation. In this limit, moreover, information about levels below $E = E_0$ determines the potential for all values of $x \leq x(E_0)$. We shall find very similar behavior in the examples to be discussed in the following section.

IV. RECONSTRUCTION OF SIMPLE POTENTIALS

Using formulas (2.36) and (3.24) we are able to explicitly construct a one-dimensional potential which is symmetric in x and has any desired bound-state spectrum. The result is unique, provided the potential is required to be reflectionless. That it is possible in principle to derive a potential directly from experimental data is evident. However, it is not immediately apparent how well the method can be expected to work if, for example, we are trying to reconstruct a confining potential knowing only the energies of a few low-lying bound states. The purpose of this section is to test the method on some simple potentials. The results encourage the application to quarkonium systems which is reported in a companion paper.[24]

A. Harmonic oscillator

We first discuss the harmonic-oscillator potential

$$V(x) = x^2\,, \tag{4.1}$$

which supports bound states at energies

$$E_{n+1} = 2n+1, \quad n = 0, 1, 2, \ldots\,. \tag{4.2}$$

The normalized bound-state wave functions are

$$\psi_{n+1}(x) = \frac{\pi^{-1/4}}{(2^n n!)^{1/2}} H_n(x) e^{-x^2/2}\,, \tag{4.3}$$

where H_n is a Hermite polynomial

$$H_n(x) = (-1)^n e^{x^2}\left(\frac{d^n}{dx^n}\right) e^{-x^2}\,. \tag{4.4}$$

We now wish to use the bound-state spectrum (4.2) as input for the inverse formalism and investigate how well (4.1) and (4.3) are approximated by (2.36) and (2.27).

At this point we encounter an ambiguity which is characteristic of confining potentials. Because the true potential does not approach a constant for large values of $|x|$, as was assumed in (2.2), we must select a criterion by which the zero of energy is set. In other words, in order to construct a reflectionless potential which supports N bound states at the same energies as the first N harmonic-oscillator levels (4.2), it is necessary to choose a parameter E_0 to define the binding energies

$$\kappa_n^2 = E_0 - E_n, \quad n = 1, 2, \ldots, N \tag{4.5}$$

where E_n is the energy of the nth level. We view this "E_0 ambiguity" in the following light. If E_0 is chosen equal to E_N, then the N-soliton formula (2.36) is identical to the $(N-1)$-soliton formula which ignores the Nth bound state. This is obvious from the fact that the depth of the Nth soliton is proportional to $\kappa_N^2 = E_0 - E_N$. It seems reasonable to conclude that if N levels are to be included in the approximation, the value of E_0 should be restricted to the range

$$E_N < E_0 < E_{N+1}\,. \tag{4.6}$$

In specific examples we find that the best approximation is obtained for $E_0 \approx \frac{1}{2}(E_N + E_{N+1})$. The variation of $V_4(x)$, the $N = 4$ approximation to the harmonic-oscillator potential, as E_0 is varied over the range (4.6) is shown in Fig. 2. Hereafter we shall consistently make the choice

$$E_0 = \tfrac{1}{2}(E_N + E_{N+1}) \tag{4.7}$$

for the N-bound-state approximation.

In Figs. 3(a)–3(e) we compare the first five approximations to the potential with the exact result (4.1). The agreement is excellent in the region of x relevant to the specified energy levels. Successive approximations to the wave functions are plotted in Figs. 3(f)–3(j). It is seen that they are converging rapidly to the exact solutions shown in Fig. 3(k).

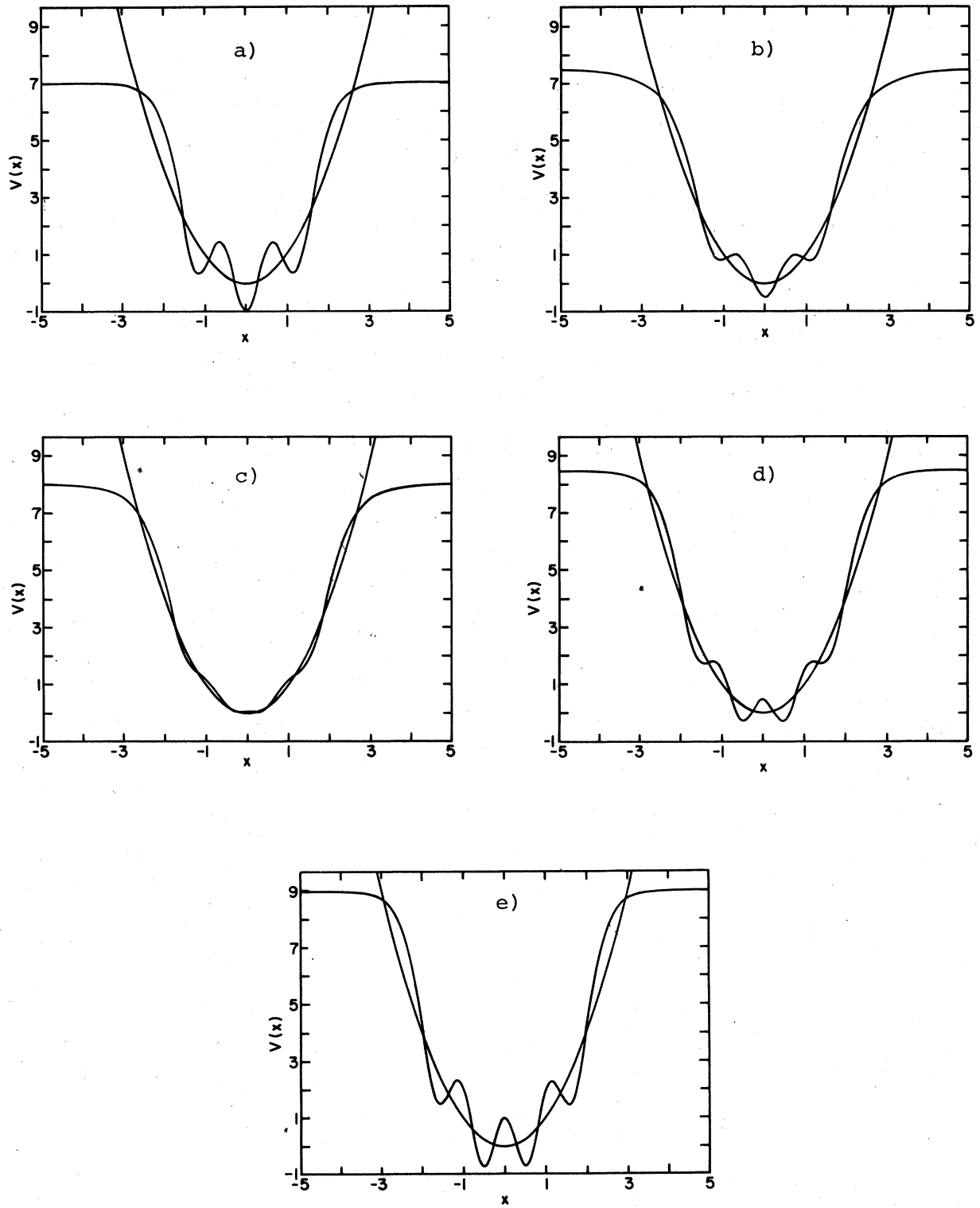

FIG. 2. Effect of the choice of the parameter E_0 upon the $N=4$ approximation to the harmonic-oscillator potential. (a) $V_4(x;E_0=7)=V_3(x;E_0=7)$, (b) $V_4(x;E_0=7.5)$, (c) $V_4(x;E_0=8)$, (d) $V_4(x;E_0=8.5)$, (e) $V_4(x;E_0=9)=V_5(x;E_0=9)$. The exact potential $V(x)=x^2$ is shown for comparison.

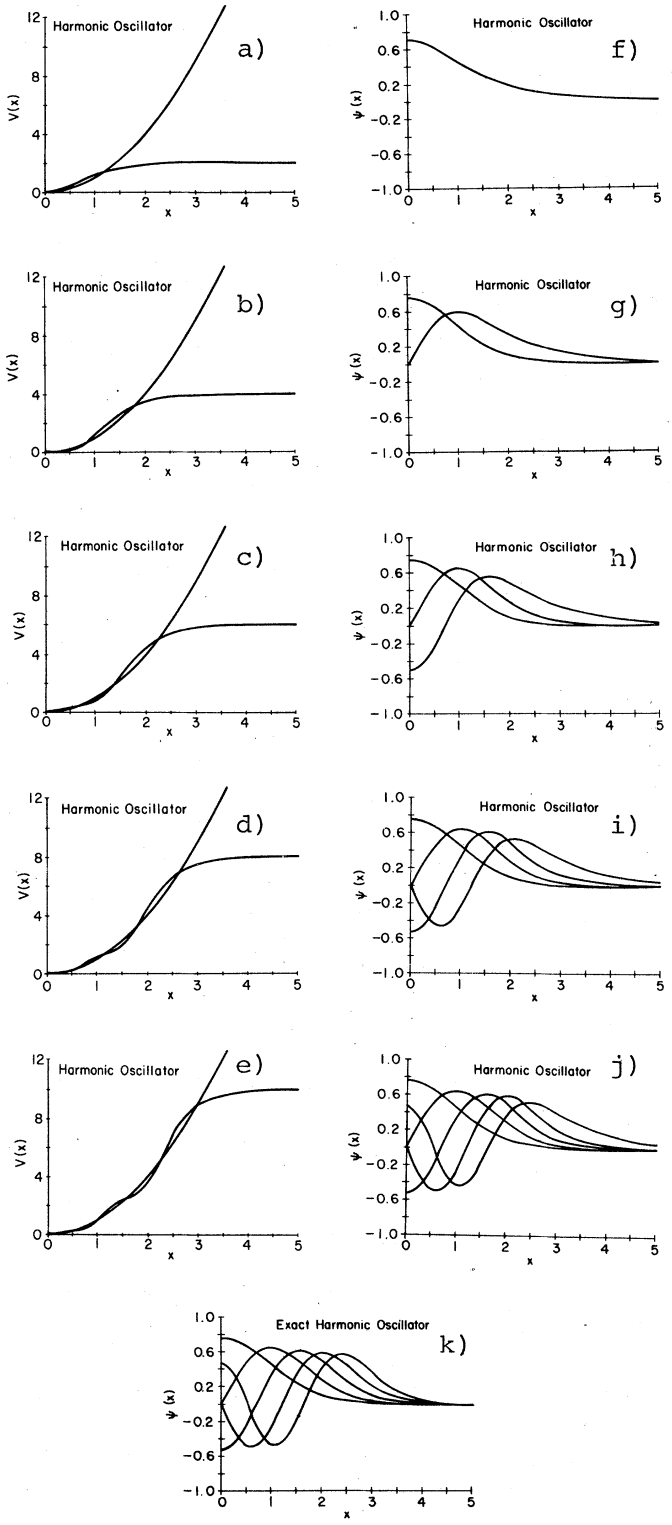

FIG. 3. Approximate reconstruction of the harmonic-oscillator potential. (a)-(e): $N = 1, 2, 3, 4, 5$ approximations to the potential. The true potential is shown for comparison. (f)–(j): wave functions obtained in the $N = 1, 2, 3, 4, 5$ approximations; (k): Exact wave functions.

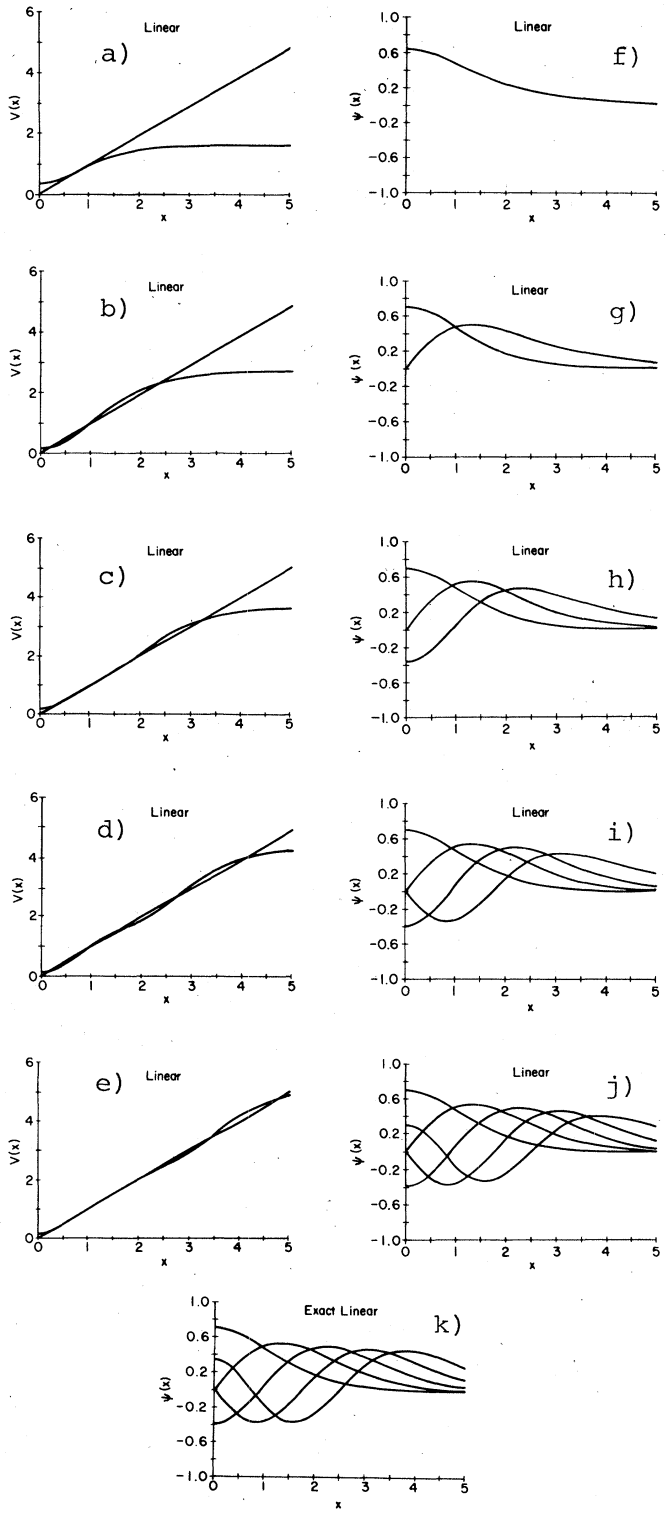

FIG. 4. Approximate reconstruction of the linear potential. See the caption to Fig. 3.

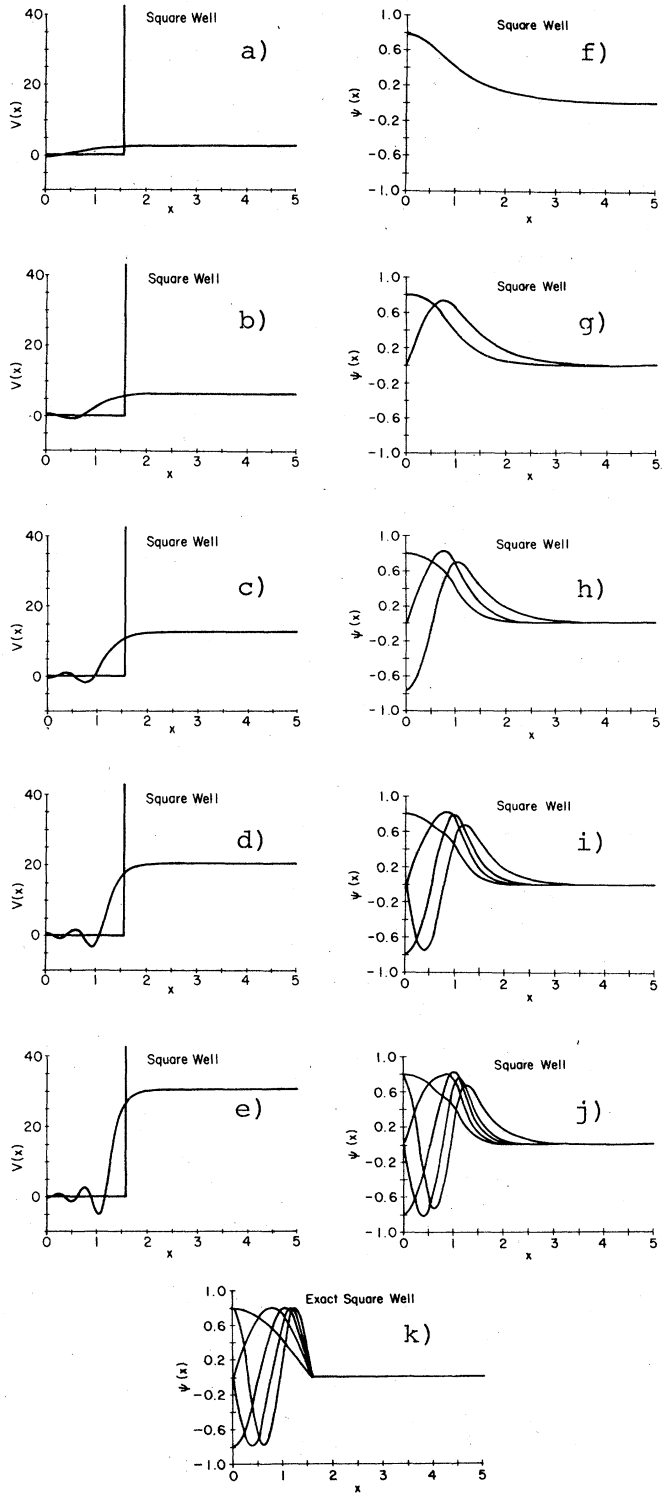

FIG. 5. Approximate reconstruction of the infinite square-well potential. See the caption to Fig. 3.

B. Linear potential

As a second example we consider the linear potential

$$V(x) = |x| , \tag{4.8}$$

for which the bound-state energies are given by the zeros of Airy functions[25]

$$\begin{aligned} &\mathrm{Ai}'(-E_n) = 0, \quad n = 1,3,5,\ldots \\ &\mathrm{Ai}(-E_n) = 0, \quad n = 2,4,6,\ldots . \end{aligned} \tag{4.9}$$

The bound-state wave functions are the Airy functions

$$\psi_n(x) = \begin{cases} \dfrac{1}{\sqrt{\mathfrak{N}_n}} \mathrm{Ai}(x - E_n), & x > 0 \\ \dfrac{(-1)^{n-1}}{\sqrt{\mathfrak{N}_n}} \mathrm{Ai}(-x - E_n), & x < 0 \end{cases} \tag{4.10}$$

where the normalization integral $\mathfrak{N}_n$ is given by

$$\begin{aligned} \mathfrak{N}_n &= 2\int_0^\infty dx[\mathrm{Ai}(x - E_n)]^2 \\ &= \begin{cases} 2E_n[\mathrm{Ai}(-E_n)]^2, & n = 1,3,5\ldots \\ 2[\mathrm{Ai}'(-E_n)]^2, & n = 2,4,6\ldots . \end{cases} \end{aligned} \tag{4.11}$$

With the energy spectrum given by (4.9) we obtain the approximate potentials and wave functions displayed in Fig. 4. The results are again extremely impressive.

C. Infinite square well

Finally we examine the pathological case of an infinitely deep square-well potential,

$$V(x) = \begin{cases} 0, & |x| < \pi/2 \\ \infty, & |x| > \pi/2 \end{cases} \tag{4.12}$$

which has bound states at

$$E_n = n^2, \quad n = 1,2,\ldots . \tag{4.13}$$

The bound-state wave functions are

$$\psi_n(x) = \begin{cases} \pi^{-1/2}\cos nx, n \text{ odd}, & |x| < \pi/2 \\ \pi^{-1/2}\sin nx, n \text{ even}, & |x| < \pi/2 \\ 0, & |x| > \pi/2 . \end{cases} \tag{4.14}$$

The approximate results obtained from (2.36) and (2.27) are shown in Fig. 5. The manner in which the wave functions are increasingly confined to the allowed region of space is noteworthy.

V. DISCUSSION

We have presented a method for systematically and explicitly calculating the shape of a symmetric, one-dimensional, confining potential from its bound-state energy levels. Approximations to the bound-state wave functions for all the levels included in the calculation are also obtained. As additional levels are included, the approximation to $V(x)$ is improved locally and is extended to larger values of $|x|$. The extension of the method to the s-wave radial equation in three dimensions is straightforward and will be described in the following paper. In this case two pieces of information are required for each bound state, namely the energy and the magnitude of the wave function at $r = 0$. The latter is measured by the leptonic decay width of the state.

It is particularly encouraging that in the specific examples of Sec. IV, the approximations are already excellent for $N = 4$, which corresponds to four bound states in one dimension or two bound states in three dimensions. In the following paper we shall calculate the charmonium potential using the ψ and ψ' masses and leptonic widths. There we explore in detail the ambiguities of the quarkonium problem and the additional experimental data needed to resolve them.

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