Self-consistent torsion potentials

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The absence of an autonomous dynamics for the torsion field in the usual version of the Einstein-Cartan theory can be avoided through introduction of torsion potentials which yield the contortion tensor upon covariant differentiation. The ensuing self-consistency problem is solved in closed form for the special case of totally antisymmetric contortion tensor. The gauge-invariant feature of the torsion potentials, in the weak-field approximation, is the propagation of a longitudinal massless tordion mode whose parity is that of an axial vector.

I. INTRODUCTION

More than half a century ago Cartan[1] pointed out that the lines of reasoning which led Einstein to his equations for the gravitational field can be followed through even if one aims at conceiving gravitation as a geometrical feature in an affine space which admits the presence of torsions through a nonsymmetrical connection

$$\Gamma^a{}_{bc} = \left\{ {a \atop bc} \right\} - K^a{}_{bc} \tag{1.1}$$

with a contortion tensor $K_{abc} = -K_{bac}$ of 24 components. The ensuing theory has been rediscovered repeatedly by Weyl[2] and others,[3,4] and has recently been the object of a thorough review.[5] It is derivable from an action principle with the Lagrangian density

$$\mathfrak{L} = \sqrt{|g|}\, L\,, \quad L = k_0 R_0 + k_1 R_1 + k_M L_M\,. \tag{1.2}$$

$R_0 = R(\{\ \})$ is the Riemann scalar serving as the action function of the gravitational field in Einstein's theory, R_1 is the bilinear combination

$$R_1 = K^a{}_{bc} K^{cb}{}_a - K^a{}_{ba} K^{cb}{}_c \tag{1.3}$$

representing the contribution to the curvature scalar made by the contortion, L_M is the action function of the material sources, and k_0, k_1, k_M are three constants. Most authors take it for granted that k_0 and k_1 are identical, but this is not necessary.[6]

The Einstein-Cartan theory does not fit into the mold of conventional field theories, because it does not provide, in its usual form, an autonomous dynamics for the torsion field. Indeed, if one treats the components of the contortion tensor as independent field variables, and invokes the principle of minimal coupling to fix the form of L_M, e.g. as in the case of a spinor source field

$$L_M = (\bar{\psi}_{,a}\gamma^a\psi - \bar{\psi}\gamma^a\psi_{,a}) + \tfrac{1}{2}K^{[abc]}(\bar{\psi}\gamma_{[a}\gamma_b\gamma_{c]}\psi)\,, \tag{1.4}$$

then the field equations obtained upon variation with respect to K_{abc} are the purely algebraic relations

$$C_{abc} = \kappa\sigma_{abc}\,, \quad \kappa = k_M/k_1 \tag{1.5}$$

which connect a certain linear combination of the K_{abc}, the "Cartan tensor" C_{abc}, with the spin tensor σ_{abc} of the material source. Thus, the contortion can be eliminated, enabling one to look upon R_1 as a modification of the source describing contact interactions[7] only.

The purpose of this paper is to demonstrate how this troublesome feature[8] of the Einstein-Cartan theory can be avoided through introduction of a 6-component potential tensor $\phi_{ab} = -\phi_{ba}$ by writing the components of the contortion tensor as covariant derivatives

$$K_{abc} = \phi_{ab:c}\,. \tag{1.6}$$

There is, of course, no fundamental reason why the contortion should be "curl-free." Just as one studies in hydrodynamics the case of curl-free flow prior to getting entangled in more general velocity fields, the arrangement (1.6) aims at providing a simple specific proposal that will yield a dynamic equation for the torsion, without sacrificing the tensor character of the contortion. Alternative options such as

$$K_{ab}{}^c = \phi^c{}_{a:b} - \phi^c{}_{b:a} \tag{1.6$'$}$$

or

$$K_{abc} = \phi_{ab:c} + \epsilon_{abij}(\eta^{ik}\xi^j{}_k)_{:c} \tag{1.6$''$}$$

which may merit further study are eschewed for the purpose of this paper because they lead to more complicated field equations.

The work reported in this paper also aims at retaining as many features of the Einstein-Cartan theory as possible. In particular, the coupling of torsion to sources and to itself is envisaged to be governed by the principle of minimal coupling, which is implemented by employing *everywhere* covariant derivatives with the nonsymmetrical connection (1.1). Also, the Lagrangian density (1.2) of the Einstein-Cartan theory will be retained. This has the advantage of avoiding introduction of additional arbitrary parameters required, for example, if one were to consider the most general quadratic expression in K_{abc}. The introduction of terms involving the derivatives of K_{abc} into the Lagrangian, which has been suggested[8] as a possible means of acquiring dynamical equations for the torsion, is also excluded by that premise.

After completion of the work reported here there appeared a paper by S. Hojman, M. Rosenbaum, M. P. Ryan, and L. C. Shepley [Phys. Rev. D 17, 3141 (1978)] in which a *scalar* potential is used to generate torsion. Although these authors also aim at retaining the principle of minimal coupling without compromise, and obtain a dynamic equation for their torsion potential, their specific proposal [Eq. (21)] is contingent upon a modification of the usual form of gauge invariance and differs substantially from the proposal (1.6) put forward here.

Since the covariant derivatives in turn contain the contortion through the connections (1.1), the proposal (1.6) raises a self-consistency problem. For the special case of the totally antisymmetric contortion tensor, which governs the coupling to a spinor field as in (1.4), this problem is solved in closed form (Sec. II).

The resulting field equations for the torsion potentials are highly nonlinear. In the weak-field approximation they are invariant under the gauge transformation

$$\phi_{ab} \to \phi'_{ab} = \phi_{ab} + C_{a,b} - C_{b,a}\,, \tag{1.7}$$

where C_a is an arbitrary vector field. The gauge-invariant feature of the torsion potentials is the propagation of a massless longitudinal phenomenon whose parity is that of an axial vector. Its particle aspect (the "tordion") may be obtained, in the radiation gauge, by application of the Gupta-Bleuler method (Sec. III).

Notations and conventions used throughout this paper are summarized in an appendix.

II. THE SELF-CONSISTENCY PROBLEM

By definition of the covariant derivatives, Eqs. (1.6) read explicitly

$$K_{abc} = \phi_{ab;c} + \phi_b{}^n K_{nac} - \phi_a{}^n K_{nbc}\,. \tag{2.1}$$

In general, for given ϕ_{ab}, these are 24 equations for the 24 components K_{abc}. In the special case of the totally antisymmetric contortion tensor $K_{abc} = K_{[abc]}$ they reduce to four equations

$$a^i{}_k K^k = \phi^i \tag{2.2}$$

for the four components of the dual vector

$$K^d = \epsilon^{dabc} K_{abc}\,, \text{ i.e., } K_{abc} = (1/3!)\epsilon_{abcd} K^d \tag{2.3}$$

where, using the symmetry of the Christoffel symbols,

$$\phi^d = \epsilon^{dabc}\phi_{ab;c} = \epsilon^{dabc}\phi_{ab,c} \tag{2.4}$$

and

$$a^i{}_k = \delta^i{}_k + \tfrac{2}{3}\phi^i{}_k\,. \tag{2.5}$$

The solution of the equations (2.2) is

$$K^k = A^{-1} A^k{}_i \phi^i\,, \tag{2.6}$$

where

$$A^k{}_i = \delta^k{}_i - \tfrac{2}{3}\phi^k{}_i - (8b/27)\hat{\phi}^k{}_i\,, \tag{2.7}$$

$$A = 1 - \tfrac{4}{9}a - (4b/9)^2\,, \tag{2.8}$$

and $\hat{\phi}$ is the dual tensor,

$$\hat{\phi}_{cd} = \tfrac{1}{2}\epsilon_{cdab}\phi^{ab}\,, \text{ i.e., } \phi^{ab} = -\tfrac{1}{2}\epsilon^{abcd}\hat{\phi}_{cd}\,, \tag{2.9}$$

which satisfies the well-known relations

$$\phi^i{}_j \phi^j{}_k - \hat{\phi}^i{}_j \hat{\phi}^j{}_k = a\delta^i{}_k\,, \tag{2.10}$$

$$\phi^i{}_j \hat{\phi}^j{}_k = \hat{\phi}^i{}_j \phi^j{}_k = b\delta^i{}_k \tag{2.11}$$

with the two invariants

$$a = \tfrac{1}{2}\phi^{ab}\phi_{ba}\,, \tag{2.12}$$

$$b = -\tfrac{1}{2}\epsilon_{abcd}\phi^{ab}\phi^{cd}\,. \tag{2.13}$$

Since traces $K^a{}_{ba}$ vanish in this special case, the action scalar (1.3) reduces to

$$R_1 = K^a{}_{bc} K^{cb}{}_a = (1/3!) K_k K^k \tag{2.14}$$

and upon substitution of the solution (2.6) one obtains by a simple calculation, aided by the relations (2.10) and (2.11), the scalar action function

$$L_1 = (\kappa_1/4)\,\frac{\phi_k\phi^k}{1 - \frac{4}{9}a - (4b/9)^2}\,, \quad \kappa_1 = (4/3!)\,k_1 \tag{2.15}$$

which describes the dynamics of the self-consistent torsion potentials ϕ_{ab}.

III. THE WEAK-FIELD APPROXIMATION

The field equations arising out of the action function (2.15) are obviously highly nonlinear. To gain some insight into the propagation properties of the torsion field neglect in L_1 all terms of quartic and higher order in the contortion and its derivatives. Also, neglect in this "weak-field approximation" the effect of the torsion on the metric by setting $g_{ab} = \delta_{ab}$. In the language of the elastomechanical analogy[9] this amounts to conceiving space-time as a Minkowski space with dislocations.

The resulting action function for the "free" torsion field

$$L_0=(\kappa_1/4)\phi_k\phi^k \tag{3.1}$$

is invariant under the gauge transformations

$$\phi_{ab}\to\phi'_{ab}=\phi_{ab}+C_{a,b}-C_{b,a}, \tag{3.2}$$

where C_a is an arbitrary vector field. The field equations

$$\phi^{ab,c}{}_{,c}+\phi^{ca,b}{}_{,c}+\phi^{bc,a}{}_{,c}=0 \text{ or } \epsilon^{abcd}\phi_{c,d}=0 \tag{3.3}$$

in conjunction with the identity

$$\phi^a{}_{,a}=0 \tag{3.4}$$

show that the torsion potentials are derivable from a scalar massless field satisfying $\Box\eta=0$ so that $\phi_a=\eta_{,a}$. Among the six possible polarization modes of the torsion potentials ϕ_{ab} only one has a gauge-invariant meaning. This feature is most conveniently exhibited by working in the "radiation gauge"

$$\phi^{ab}{}_{,b}=0\,. \tag{3.5}$$

The canonical formalism requires addition of the "Fermi term"

$$L_F=(\kappa_1/2)(\phi^{ab}{}_{,a}\phi_{cb}{}^{,c}+\phi^{ab}{}_{,b}\phi_{ac}{}^{,c}) \tag{3.6}$$

to L_0, and the field equations

$$\phi^{ab,c}{}_{,c}=0 \tag{3.7}$$

are solved by the expansion in plane waves

$$\phi_{ab}(x)=(1/\sqrt{V})\sum_{\vec{\kappa}}\sum_{P=1}^{6}(2\omega)^{-1/2}\epsilon_{ab}(P)\times[b(\vec{\kappa},P)e^{-i\kappa x}+b^\dagger(\vec{\kappa},P)e^{i\kappa x}]\,. \tag{3.8}$$

The radiation gauge (3.5) affects only the polarization modes $P\neq 3$,

$$b(1)+b(5)=0\,,\quad b(2)+b(4)=0\,,\quad b(6)=0\,. \tag{3.9}$$

Accordingly, only the mode $P=3$ ($j=1$, $m=0$) is gauge invariant. It describes a longitudinally propagating massless phenomenon whose parity is that of an axial vector. Its particle aspect (the "tordion") makes its appearance upon transition to quantum field theory, which proceeds without difficulty upon application of the Gupta-Bleuler method.

APPENDIX

Throughout this paper partial and covariant derivatives are denoted, respectively,

$$\phi_{ab,c}=\partial\phi_{ab}/\partial x^c\,, \tag{A1}$$

$$\phi_{ab;c}=\phi_{ab,c}-\left\{\begin{matrix}n\\ac\end{matrix}\right\}\phi_{nb}-\left\{\begin{matrix}n\\bc\end{matrix}\right\}\phi_{an}\,, \tag{A2}$$

$$\phi_{ab:c}=\phi_{ab,c}-\Gamma^n{}_{ac}\phi_{nb}-\Gamma^n{}_{bc}\phi_{an}\,. \tag{A3}$$

The Minkowski metric is taken to be $\delta_{ab}=\mathrm{diag}(-1,-1,-1,+1)$, and the Levi-Civita density ϵ_{abcd} is normalized in Minkowski space to $\epsilon_{1234}=+1$. For a given propagation vector there are four independent polarization vectors $\epsilon(S)$ ($S=1,2,3,4$) which satisfy the orthogonality and completeness relations

$$\epsilon^a(S)\epsilon_a(S')=\delta_{SS'}=\mathrm{diag}(-1,-1,-1,+1)\,, \tag{A4}$$

$$\sum_S\sum_{S'}\delta_{SS'}\epsilon_a(S)\epsilon_b(S')=\delta_{ab}\,. \tag{A5}$$

Their components in an angular momentum representation associate $S=1,2,3,4$ by convention with $(j,m)=(1,1),(1,-1),(1,0),(0,0)$. From these vectors the six polarization tensors $\epsilon_{ab}(P)=-\epsilon_{ba}(P)$ are obtained by a composition which associates $P=1,2,3,4,5,6$ with $(j,m)=(1,-1)$, $(1,1)$, $(1,0)$, $(1,1)$, $(1,-1)$, $(1,0)$. They satisfy the orthogonality and completeness relations

$$\epsilon^{ab}(P)\epsilon_{ab}(P')=\delta_{PP'}=\mathrm{diag}(1,1,1,-1,-1,-1), \tag{A6}$$

$$\sum_P\sum_{P'}\delta_{PP'}\epsilon_{ab}(P)\epsilon_{cd}(P')=\tfrac{1}{2}(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})\,. \tag{A7}$$

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