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Spectrum-generating SU(4) in particle physics: SU(n) and particle assignments*

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The conventions and representations of SU(n) as used in the spectrum-generating-group approach to particle physics are described. The particle assignments are given in terms of Gelfand-Zetlin basis vectors, and the most useful Clebsch-Gordan coefficients are listed. The electric charge operator and the electromagnetic current operator are defined and the matrix elements of the electromagnetic current operator are discussed.

INTRODUCTION

In this series of papers, we will develop the spectrum-generating-group approach to particle physics. This paper contains a description of the conventions and representations of SU(4) which are useful for particle physics, and includes tables of Clebsch-Gordan coefficients in a readily usable form. The definition of the electromagnetic current operator in SU(4) is also discussed. In a later paper, we will apply these ideas to the radiative decays of vector mesons, beginning with a derivation of the decay-rate formula from the spectrum-generating-group assumption. Subsequent papers will describe the application of these methods to other areas of particle physics.

In view of the profusion of different conventions and notations used for SU(4) in the literature, it is worthwhile to review the subject and establish a consistent notation. This is especially important for calculations in which the relative phases are crucial (such as the electromagnetic-decay calculations to be discussed in the next paper of this series). In this paper we will present the representations of SU(n) in the Gelfand-Zetlin pattern notation,[1] without giving any proofs.[2] The representation of basis vectors in SU(3) by Gelfand-Zetlin patterns is already a substantial simplification over other conventions, and for SU(4) this simplification is even more appreciable. Moreover, for a systematic study of the application of the chain of unitary groups,

$$\mathrm{SU}(2)\subset\mathrm{SU}(3)\subset\mathrm{SU}(4)\subset\cdots\subset\mathrm{SU}(n)\subset\cdots,$$

of which the apparent discovery of charm[3] may have just revealed one further step, the use of Gelfand-Zetlin patterns becomes essential.

In Sec. I, we discuss those aspects of the Gelfand-Zetlin pattern which are necessary for particle physics. Most of this material has been taken from papers by Louck, Biedenharn, and collaborators.[1,2] In Sec. II we discuss the assignment of particle multiplets to SU(4) irreducible representations,[4] and define the "charges" in terms of the group generators. We define the electric charge operator in Sec. III, and extend this to the electromagnetic current operator. Finally, in Sec. IV we discuss the assignment of particles to SU(4) states.

I. REVIEW OF THE PATTERN NOTATION

A Gelfand-Zetlin pattern for U(n) is a triangular scheme

$$(m)=\begin{pmatrix} m_{1,n} & & m_{2,n} & & m_{3,n} & \cdots & & m_{n,n} \\ & m_{1,n-1} & & m_{2,n-1} & & \cdots & m_{n-1,n-1} & \\ & & \ddots & & & \cdot^{\cdot^{\cdot}} & & \\ & & & m_{1,2} & & m_{2,2} & & \\ & & & & m_{1,1} & & & \end{pmatrix} \qquad (1)$$

of non-negative integers $m_{i,n}$ which satisfy the following inequalities:

$$m_{1,n}\geq m_{2,n}\geq\cdots\geq m_{n,n}\geq 0\,, \qquad (2)$$

$$m_{i,j+1}\geq m_{i,j}\geq m_{i+1,j+1}\,. \qquad (3)$$

The first row in the pattern (1) characterizes an irreducible representation (irrep) of U(n) and is

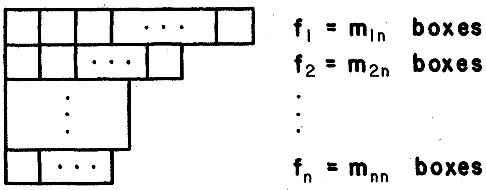

FIG. 1. Young diagram corresponding to the irrep $(m_{1,n}, m_{2,n}, \cdots, m_{n,n})$.

TABLE I. Reduction of the 15-dimensional irrep (2, 1, 1, 0) of U(4) or SU(4) into irreps of U(3). The eigenvalues of $h_3^{SU(4)}$ are also given.

$m_{1,3}, m_{2,3}, m_{3,3}$	$D(\lambda_1, \lambda_2)$	$\underline{N}$	$h_3^{SU(4)}$
2, 1, 1	$D(1,0)$	$\underline{3}$	$(\frac{3}{2})^{1/2}\,\frac{1}{3}(+1)$
2, 1, 0	$D(1,1)$	$\underline{8}$	0
1, 1, 1	$D(0,0)$	$\underline{1}$	0
1, 1, 0	$D(0,1)$	$\underline{3}^*$	$(\frac{3}{2})^{1/2}\,\frac{1}{3}(-1)$

connected with the characterization of the irrep by a Young diagram through the lengths of its rows

$$(f_1, f_2, \ldots, f_n) = (m_{1,n}, m_{2,n}, \ldots, m_{n,n}),$$

as illustrated in Fig. 1. The second row in pattern (1) characterizes the irrep of the $U(n-1)$ subgroup. The inequalities (3) for $j+1=n$ then constitute the Weyl branching law: A given irrep $(m_{1,n}, m_{2,n}, \ldots, m_{n,n})$ of $U(n)$ restricted to the subgroup $U(n-1)$ reduces into the direct sum of all posible irreps $(m_{1,n-1}, m_{2,n-1}, \ldots, m_{n-1,n-1})$ for which the $m_{i,n+1}$ fulfill (3), i.e., such that the integer $m_{i,n-1}$ lies between the pair of integers directly above it in the pattern.

As an example, consider the 15-dimensional representation of U(4) [the adjoint representation of SU(4)] characterized by (2, 1, 1, 0). The four possible sets of numbers fulfilling (3) which may appear in the second row of the pattern

$$\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{pmatrix}$$

are shown in the first column of Table I. The irrep (2, 1, 1, 0) of U(4) therefore reduces with respect to U(3) into the direct sum of the four irreps characterized by the numbers in the first column of Table I.

The irreps of $SU(n)$ can also be characterized by the first row of a Gelfand-Zetlin pattern, and each irrep of $U(n)$ is also an irrep of $SU(n)$. However, $(m_{1,n}, m_{2,n}, \ldots, m_{n,n})$ and

$$(m_{1,n}+e, m_{2,n}+e, \ldots, m_{n,n}+e),$$

where e is any integer, characterize equivalent representations of $SU(n)$, so long as (2) holds. Therefore one may characterize an irrep of $SU(n)$ by just the $n-1$ numbers

$$(f_1 = m_{1,n} - m_{n,n}, f_2 = m_{2,n} - m_{n,n}, \ldots,$$
$$f_{n-1} = m_{n-1,n} - m_{n,n}, 0).$$

In this convention, the last entry in the first row of a Gelfand-Zetlin pattern for $SU(n)$ will always be zero:

$$\begin{pmatrix} f_1 & & f_2 & \cdots & f_{n-1} & & 0 \\ & m_{1,n-1} & & \cdots & & m_{n-1,n-1} & \\ & & m_{1,n-2} & \cdots & & & \\ & & & \cdots & & & \end{pmatrix}. \tag{4}$$

The connection between the characterization of an irrep by the first row $(f_1 \cdots f_{n-1}, 0)$ of a pattern (or a Young tableau) and by the highest weight $[\lambda_1, \lambda_2, \ldots, \lambda_{n-1}]$ is

$$\begin{aligned} \lambda_1 &= f_1 - f_2, \\ \lambda_2 &= f_2 - f_3, \\ &\vdots \\ \lambda_{n-1} &= f_{n-1}. \end{aligned} \tag{5}$$

The highest-weight characterizations of the irreps of SU(3) contained in the adjoint irrep (2, 1, 1, 0) $\simeq [\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 1]$ of SU(4) are also shown in Table I.

The dimension of the irrep $D([\lambda_1, \lambda_2, \ldots, \lambda_{n-1}])$ of $SU(n)$ is given by

$$N_{[\lambda_1,\ldots,\lambda_{n-1}]} = (1+\lambda_1)(1+\lambda_2)\cdots(1+\lambda_{n-1})\left(1+\frac{\lambda_1+\lambda_2}{2}\right)\left(1+\frac{\lambda_2+\lambda_3}{2}\right)\cdots\left(1+\frac{\lambda_{n-2}+\lambda_{n-1}}{2}\right)$$
$$\times\left(1+\frac{\lambda_1+\lambda_2+\lambda_3}{3}\right)\cdots\left(1+\frac{\lambda_{n-3}+\lambda_{n-2}+\lambda_{n-1}}{3}\right)\times\cdots\times\left(1+\frac{\lambda_1+\lambda_2+\cdots+\lambda_{n-1}}{n-1}\right). \tag{6}$$

In the example above, the dimensions of the irreps of SU(3) are also shown in Table I.

In the same way that the second row of pattern (4) gives the reduction of U(n) with respect to U($n-1$), the third row gives the reduction of the irrep $(m_{1,n-1}, \ldots, m_{n-1,n-1})$ of U($n-1$) with respect to U($n-2$). For example, the irrep (2, 1, 1) of SU(3) in the pattern

$$\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & 2 & & 1 & & 1 & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{pmatrix}$$

reduces to a direct sum of the two SU(2) irreps shown in Table II. The usual labeling of the SU(2) irreps by the isospin I

$$I=\frac{m_{1,2}-m_{2,2}}{2}$$

is also shown in the table.

Proceeding in this manner from one row to the next in the pattern, we obtain the complete reduction of our irrep of SU(n) with respect to the chain of subgroups

$$\mathrm{SU}(n) \supset \mathrm{U}(n-1) \supset \mathrm{U}(n-2) \supset \cdots \supset \mathrm{U}(1) \tag{7}$$

with

$$\mathrm{U}(i) \simeq \mathrm{U}(1) \otimes \mathrm{SU}(i)/Z_i\,, \tag{8}$$

where Z_i is the cyclic group of i objects. The irreps of U(1) are one-dimensional and are characterized by the integer $m_{1,1}$. For example, in the irrep (2, 1) of U(2), $m_{1,1}$ can have the two possible values 2 and 1 (corresponding to the two I_3 values for $I=\frac{1}{2}$).

A fully specified pattern such as

$$\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & 2 & & 1 & & 1 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{pmatrix}$$

characterizes a one-dimensional subspace of the irrep space of SU(n). Therefore, up to a phase factor, it also characterizes a basis vector of this irrep space $\mathfrak{R}(m_{1,n}, m_{2,n}, \ldots, m_{n,n})$. We denote this basis vector by

$$|m\rangle = \left| \begin{matrix} m_{1,n} & \cdots & & m_{n,n} \\ & m_{1,n-1} & \cdots & \\ & & \vdots & \\ & & m_{11} & \end{matrix} \right\rangle . \tag{9}$$

TABLE II. Reduction of the irrep (2, 1, 1) of SU(3) into irreps of SU(2). Isospin I, eigenvalues of $h_2^{\mathrm{SU}(4)}$ and $h_1^{\mathrm{SU}(4)}$ are shown.

$m_{1,2}, m_{2,2}$	I	$h_2^{\mathrm{SU}(4)}$	$h_1^{\mathrm{SU}(4)}$
2, 1	$\frac{1}{2}$	$\frac{\sqrt{3}}{4}(+\frac{1}{3})$	$\frac{1}{2}(\frac{1}{2}), \frac{1}{2}(-\frac{1}{2})$
1, 1	0	$\frac{\sqrt{3}}{4}(-\frac{2}{3})$	0

The Gelfand-Zetlin basis vectors (9) are eigenvectors of $(n-1)$ invariants I_i^n $(i=1, \ldots, n-1)$ of SU(n), of $(n-2)$ invariants I_i^{n-1} of SU($n-1$), etc., of the generator H_{n-1} of the U(1) in

$$\mathrm{U}(n-1) \simeq \mathrm{U}(1) \otimes \mathrm{SU}(n-1)/Z_{n-1}\,, \tag{10}$$

of the generator H_{n-2} of the U(1) in

$$\mathrm{U}(n-2) \simeq \mathrm{U}(1) \otimes \mathrm{SU}(n-2)/Z_{n-2}\,, \tag{11}$$

and so on.

The generators $H_1, H_2, \ldots, H_{n-1}$ of the Abelian Cartan subalgebra (representing the "charges" I_3, Y, charm, etc.) have the following eigenvalues on the Gelfand-Zetlin basis vectors:

$$H_i\,|(m)\rangle = \left[\frac{i(i+1)}{2n}\right]^{1/2} \times \frac{1}{i}\left(\sum_{j=1}^{i} m_{j,i} - \frac{i}{i+1}\sum_{j=1}^{i+1} m_{j,i+1}\right)|(m)\rangle . \tag{12}$$

From this we see that the eigenvalue h_i of H_i depends only upon the ith and the $(i+1)$th rows in the Gelfand-Zetlin pattern, except that its normalization depends upon n. For example, h_3 for SU(4) is given by

$$h_3 = (\tfrac{3}{2})^{1/2}\tfrac{1}{3}[(m_{1,3}+m_{2,3}+m_{3,3}) - \tfrac{3}{4}(m_{1,4}+m_{2,4}+m_{3,4}+m_{4,4})] . \tag{13}$$

The values of h_3 for the patterns

$$\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & m_{1,3} & & m_{2,3} & & m_{3,3} & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{pmatrix}$$

of the adjoint irrep of SU(4) are given in the fourth column of Table I.

Likewise, the eigenvalue h_2 in SU(4) is given by

$$h_2 = \frac{\sqrt{3}}{4}\left[m_{1,2}+m_{2,2} - \tfrac{2}{3}(m_{1,3}+m_{2,3}+m_{3,3})\right] , \tag{14}$$

and the values for the patterns

$$\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & 2 & & 1 & & 1 & \\ & & m_{1,2} & & m_{2,2} & & \\ & & & \cdot & & & \end{pmatrix}$$

are given in the third column of Table II. Finally, the values of

$$h_1 = \tfrac{1}{2}[m_{1,1} - \tfrac{1}{2}(m_{1,2} + m_{2,2})] \tag{15}$$

for the pattern

$$\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & 2 & & 1 & & 1 & \\ & & m_{1,2} & & m_{2,2} & & \\ & & & m_{1,1} & & & \end{pmatrix}$$

are also shown in Table II.

II. MULTIPLET ASSIGNMENTS IN SU(4)

The (pseudoscalar) mesons should be assigned to the adjoint representation; so for SU(4) we need to consider the representation (2,1,1,0), which has been used above as an example. The old pseudoscalar mesons are assigned to the octet, and to retain the usual values of isospin I_3 for these states we connect I_3 to $h_1^{\mathrm{SU}(4)}$ by

$$I_3 = 2H_1 . \tag{16}$$

Likewise, the usual values of hypercharge Y will be retained if we set

$$Y = \frac{4}{\sqrt{3}} H_2 . \tag{17}$$

If we now keep the convention (17) for states with $h_3 \neq 0$ (charmed particles), then the values of hypercharge for the $\underline{3}$ representation characterized by the patterns

$$\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & 2 & & 1 & & 1 & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{pmatrix}$$

are $\frac{1}{3}$ for $I = \frac{1}{2}$ and $-\frac{2}{3}$ for $I = 0$ (see Table II).

Similarly, for the $\underline{3}^*$ representation described by the patterns

$$\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & 1 & & 1 & & 0 & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{pmatrix},$$

TABLE III. Reduction of the 20-dimensional irrep (2,1,0,0) of SU(4) into irreps of U(3). Eigenvalues of $h_3^{\mathrm{SU}(4)}$ and charm are shown.

$m_{1,3}, m_{2,3}, m_{3,3}$	$\underline{N}$	$h_3^{\mathrm{SU}(4)}$	C
2, 1, 0	$\underline{8}$	$\frac{3}{4\sqrt{6}}$	0
2, 0, 0	$\underline{6}$	$-\frac{1}{4\sqrt{6}}$	1
1, 1, 0	$\underline{3}^*$	$-\frac{1}{4\sqrt{6}}$	1
1, 0, 0	$\underline{3}$	$-\frac{5}{4\sqrt{6}}$	2

one finds from (17) and (14) that $Y = -\frac{1}{3}$ for $I = \frac{1}{2}$ and $Y = +\frac{2}{3}$ for $I = 0$.

It has already become conventional to assign the charm +1 to the mesons in the $\underline{3}^*$ representation (called D^0, D^+, F^+). In order to retain this convention, the connection between the charm (for mesons) or supercharge χ and the value $h_3^{\mathrm{SU}(4)}$ given by (13) must be

$$\chi = -\sqrt{6} H_3 . \tag{18}$$

The $\frac{1}{2}^+$ baryons are assigned in SU(4) to the 20-dimensional representation (2,1,0,0), which we also denote $\underline{20}^1$.[4] The possible values of $m_{1,3}, m_{2,3}, m_{3,3}$ in the pattern

$$\begin{pmatrix} 2 & & 1 & & 0 & & 0 \\ & m_{1,3} & & m_{2,3} & & m_{3,3} & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{pmatrix}$$

are listed in Table III together with the corresponding dimensional notation for the irrep of SU(3) and the value of $h_3^{\mathrm{SU}(4)}$ calculated from (13). I_3 and Y are still given by (16) and (17).

If the assignment of charm is to be the same as in the usual convention[4] (see Table III) then the connection between the charm C and supercharge χ (= charm for mesons) is given by

$$C = \tfrac{3}{4}B - \sqrt{6} h_3 = \tfrac{3}{4}B + \chi , \tag{19}$$

where B represents the baryon number.

At the present time, there is no experimental evidence for assigning the $\frac{1}{2}^+$ baryons to $\underline{20}^1$. For example, no disagreement with experimental facts results from choosing another 20-dimensional irrep, (2,2,0,0), instead of (2,1,0,0), even though (2,2,0,0) is not contained in the reduction of the product $\underline{4} \times \underline{4} \times \underline{4}$. This irrep is self-conjugate, has an $\underline{\text{octet}}$ with $h_3 = 0$ and a $\underline{6}$ and $\underline{6}^*$ with $h_3 = -1/\sqrt{6}$ and $+1/\sqrt{6}$, respectively.

The $\frac{3}{2}^+$ baryons are assigned to the 20-dimensional representation (3, 0, 0, 0), denoted $\underline{20}$. The possible values of $m_{1,3}$, $m_{2,3}$, and $m_{3,3}$ in the pattern

$$\begin{pmatrix} 3 & & 0 & & 0 & & 0 \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{pmatrix}$$

are shown in Table IV, along with the values of $h_3^{SU(4)}$ and charm, and the corresponding dimensional notation for the SU(3) irreps. Relations (16), (17), (18), and (19) are still valid for $\underline{20}$. We note that instead of a 3^* with $C=1$ (as in $\underline{20}^{1)}$), there is a singlet with $C=3$.

III. CHARGE OPERATOR AND ELECTROMAGNETIC CURRENT

The connection between the electric charge Q and the eigenvalues of the operators H_1, H_2, H_3 in SU(4) is specified by the requirement that

(1) the Gell-Mann–Nishijima formula, $\frac{1}{2}Q = h_1 + h_2/\sqrt{3}$, is to be generalized for values of $H_3 \neq 0$,

(2) there exists a new particle doublet with charge 0 (D^0) and charge +1 (D^+), and

(3) the isodoublet with $\chi = -1$ is the charge conjugate of the isodoublet with $\chi = +1$ [or that the conjugation operator for SU(4) is the charge-conjugation operator for mesons].

Then the charge (operator) for mesons is given by

$$\tfrac{1}{2}Q = H_1 + \frac{1}{\sqrt{3}}H_2 - (\tfrac{2}{3})^{1/2}H_3 . \tag{20}$$

If this is taken to be the charge operator for all $B=0$ particles, and if the baryon octet with the usual charges is assigned to the irrep space characterized by

$$\begin{pmatrix} 2 & & 1 & & 0 & & 0 \\ & 2 & & 1 & & 0 & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{pmatrix},$$

then the charge operator must have an additional term:

$$\tfrac{1}{2}Q = H_1 + \frac{1}{\sqrt{3}}H_2 - (\tfrac{2}{3})^{1/2}H_3 + \tfrac{1}{4}B , \tag{21}$$

or

$$Q = I_3 + \tfrac{1}{2}Y + \tfrac{2}{3}C . \tag{21$'$}$$

These are the same expressions one gets by quark charge counting, where the charmed quark charge is $Q_C = \frac{2}{3}$.

TABLE IV. Reduction of the 20-dimensional irrep (3, 0, 0, 0) of SU(4) into irreps of U(3). Eigenvalues of $h_3^{SU(4)}$ and charm are shown.

$m_{1,3}, m_{2,3}, m_{3,3}$	$\underline{N}$	$h_3^{SU(4)}$	C
3, 0, 0	$\underline{10}$	$\frac{3}{4\sqrt{6}}$	0
2, 0, 0	$\underline{6}$	$-\frac{1}{4\sqrt{6}}$	1
1, 0, 0	$\underline{3}$	$-\frac{5}{4\sqrt{6}}$	2
0, 0, 0	$\underline{1}$	$-\frac{9}{4\sqrt{6}}$	3

As we have seen, the experimental discovery of the isodoublet with D^0 and D^+ is sufficient to establish (21) if one assigns the $\frac{1}{2}^+$ baryons to the 20-dimensional representation (2, 1, 0, 0). If one would instead assign them to the representation (2, 2, 0, 0), as discussed above, then the charge operator would be

$$\tfrac{1}{2}Q = H_1 + \frac{1}{\sqrt{3}}H_2 - (\tfrac{2}{3})^{1/2}H_3 \tag{22}$$

in order to give the Gell-Mann–Nishijima formula for the octet. The distinction between (21) and (22), which we do not want to elaborate upon here, is that in the case (22) the group is $U_B(1) \times SU(4)/Z_4$. In case (21), however, the group is U(4), where U(4) is generated by B and the SU(4) generators.

If the charge is given by (21) one would choose for the electromagnetic "current"[5]

$$V_\mu^{el} = V_\mu^{\pi^0} + \frac{1}{\sqrt{3}}V_\mu^{\eta} - (\tfrac{2}{3})^{1/2}V_\mu^{\chi} + V_\mu^{\sigma} \tag{23}$$

[in the case of (22) one would choose $V_\mu^\sigma = 0$, but below we will see that such a term is essential for the radiative decays], where $V_\mu^{\pi^0}, V_\mu^{\eta}, V_\mu^{\chi}$ are the components of a regular tensor operator of SU(4) that transform like H_1, H_2, H_3, respectively, and where V_μ^σ is an SU(4)-scalar operator. The phases are specified such that the F-type Clebsch-Gordan coefficients of the matrix elements of $V_\mu^{\pi^0}, V_\mu^{\eta}, V_\mu^{\chi}, V_\mu^{\sigma}$ are proportional to the charges h_1, h_2, h_3, B.

From the physical interpretation of the charges it follows that, under charge conjugation U_C, V_μ^{el} and each of its terms should transform like

$$U_C^\dagger V_\mu^{el} U_C = -V_\mu^{el} . \tag{24}$$

In particular,

$$U_C^\dagger V_\mu^{\sigma} U_C - V_\mu^{\sigma} . \tag{25}$$

Taking (25) between meson state vectors $|M\rangle$ gives

$$\langle M|V^\sigma|M\rangle = -\bar{\alpha}_M\alpha_M\langle\bar{M}|V^\sigma|\bar{M}\rangle\,, \tag{26}$$

where

$$U_C|M\rangle = \alpha_M|\bar{M}\rangle\,, \tag{27}$$

with $\bar{M}$ denoting the antiparticle of M and α_M being a phase factor (with $\alpha_M=+1$ or -1 as a consequence of $U_C^2=1$).

Applying (23) for zero-charge mesons, $\bar{M}^0=M^0$ (such as π^0, η, ρ^0, etc.), it follows that

$$\langle M^0|V^\sigma|M^0\rangle = 0\,. \tag{28}$$

Since all the other quantum numbers in addition to the SU(4) quantum numbers are the same for the whole SU(4) multiplet, it follows that

$$\langle M|V^\sigma|M\rangle = 0 \tag{29}$$

for the whole multiplet.

By the same argument it follows that

$$\langle M_1|V^\sigma_\mu|M_2\rangle = 0$$

for any two members of two different SU(4) multiplets with $\alpha_{M_1}=\alpha_{M_2}$. However, if (the neutral numbers of) the SU(4) multiplets M_1 and M_2 have opposite charge parity, $\alpha_{M_1}\neq\alpha_{M_2}$, then $\langle M_1|V^\sigma_\mu|M_2\rangle \neq 0$. Thus, if M_1 are the pseudoscalar mesons P and M_2 are the vector mesons V, then in general

$$\langle P|V^\sigma_\mu|V\rangle \neq 0\,. \tag{30}$$

As we mentioned after Eq. (23), Eq. (30) will become important for radiative decays $V\to P\gamma$.

We shall now show that the F reduced matrix elements between the vector-meson 15-plet and the pseudoscalar-meson 15-plet must be zero as a consequence of the transformation property of $V^{\rm el}_\mu$ under charge conjugation. Let us for the sake of definiteness consider the matrix elements

$$\begin{aligned}\langle\pi^+|V^{\rm el}_\mu|\rho^+\rangle = {}& C(F,\rho^+,{\rm el},\pi^+)\langle 15P\|V^F_\mu\|15V\rangle\\ &+C(D,\rho^+,{\rm el},\pi^+)\langle 15P\|V^D_\mu\|15V\rangle\\ &+\langle 15P|V^\sigma_\mu|15V\rangle\,.\end{aligned} \tag{31}$$

According to (23),

$$\begin{aligned}C(D,\rho^+,{\rm el},\pi^+) = {}& C(D,\rho^+,\pi^0,\pi^+)\\ &+\frac{1}{\sqrt{3}}C(D,\rho^+,\eta,\pi^+)\\ &-(\tfrac{2}{3})^{1/2}C(D,\rho^+,\chi,\pi^+)\end{aligned} \tag{32}$$

and similarly for the $C(F,\dots)$, where ρ^+, π^0, π^+, etc., stand for the SU(4) quantum numbers of the particles ρ^+,π^0,π^+, etc., and $C(D,\rho^+,\pi^0,\pi^+)$, etc., [$C(D,\dots)$] are the symmetric [antisymmetric] SU(4) Clebsch-Gordan coefficients. Just as for the SU(3) Clebsch-Gordan coefficients the SU(4) Clebsch-Gordan coefficients also fulfill the well-known relation

$$\begin{aligned}C(F,\rho^+,\dots,\pi^+) &= -C(F,\rho^-,\dots,\pi^-)\,,\\ C(D,\rho^+,\dots,\pi^+) &= +C(D,\rho^-,\dots,\pi^-)\,.\end{aligned} \tag{33}$$

From (24) it follows that

$$\begin{aligned}\langle\pi^+|V^{\rm el}_\mu|\rho^+\rangle &= -\langle\pi^+|U_C^{-1}V^{\rm el}_\mu U_C|\rho^+\rangle\\ &= -\alpha_V\alpha_P\langle\pi^-|V^{\rm el}_\mu|\rho^-\rangle\,,\end{aligned} \tag{34}$$

where $\alpha_P=+1$ and $\alpha_V=-1$ are the charge parities of the pseudoscalar and vector mesons, respectively.

Writing $\langle\pi^-|V^{\rm el}_\mu|\rho^-\rangle$ in the form (31) and using (34) we obtain

$$\begin{aligned}\langle\pi^+|V^{\rm el}_\mu|\rho^+\rangle = {}& C(F,\rho^-,{\rm el},\pi^-)\langle 15P\|V^F_\mu\|15V\rangle\\ &+C(D,\rho^-,{\rm el},\pi^-)\langle 15P\|V^D_\mu\|15V\rangle\\ &+\langle 15P|V^\sigma_\mu|15V\rangle\,.\end{aligned} \tag{35}$$

Using (33) and comparing it with (31) we obtain

$$\langle 15P\|V^F_\mu\|15V\rangle = 0\,. \tag{36}$$

IV. STATES AND PARTICLE ASSIGNMENTS

It is well known that the vector-meson states are not described by the basis vectors of the 15-dimensional representation adapted to the chain of subgroups

$$\mathrm{SU}(4)\supset U_X(1)\times\mathrm{SU}(3)\supset U_X(1)\times U_Y(1)\times\mathrm{SU}_I(2) \tag{37}$$

[we call this basis the SU(4) basis], where $U_X(1)$ and $U_Y(1)$ are U(1) groups of charm and hypercharge, but instead that a much better approximation is given by the basis vectors which are connected to the subgroup chain

$$\begin{aligned}\mathrm{SU}(8)&\supset\mathrm{SU}(6)\times\mathrm{SU}_{S_X}(2)\\ &\supset\mathrm{SU}_W(4)\times\mathrm{SU}_{S_Y}(2)\times\mathrm{SU}_{S_X}(2)\\ &\supset\mathrm{SU}_{S_N}(2)\times\mathrm{SU}_I(2)\times\mathrm{SU}_{S_Y}(2)\times\mathrm{SU}_{S_X}(2)\end{aligned} \tag{38}$$

[we call this basis the SU(8) basis].

The basis vectors which one would use in the case (38) are the vectors (9) labeled by the Gelfand-Zetlin pattern and, in the rest system, by the spin S, spin component S_3, and the mass, which in the spectrum-generating-group approach is not an independent label but a function of the discrete quantum numbers.

The basis vectors for the case (38) are not the basis vectors adapted to the subgroup chain (38) because then the spin would not be diagonal, but rather the basis in which the charmed spin S_X, the hypercharged spin S_Y, and the charged spin S_N are combined according to the rules of addition of angular momenta. In this basis the operators

$$\vec{S}^2, S_3, Y, \vec{S}_Y{}^2, \chi, \vec{S}_\chi{}^2, I, I_3, \tilde{C}_2^4, \tilde{C}_2^6 \tag{39}$$

are diagonal. Here $\tilde{C}_2^6$ is (a set of) the second-order Casimir operator of the Gürsey-Radicati SU(6) and $\tilde{C}_2^4$ is (a set of) the Casimir operator of the Wigner $SU_W(4)$, and is physically distinct from the Casimir operator C_2^4 of the SU(4) of the charges given in (13), (14), and (15). For the few low-lying irreps of SU(6) and $SU_W(4)$ that we need, the second-order Casimir operator suffices for the characterization. In general, however, higher-order Casimir operators are also needed. The second-order Casimir operator for SU(n) of the irrep $[\lambda_1, \ldots, \lambda_{n-1}]$ is given by[6]

$$\tilde{C}_2^n([\lambda_1, \ldots, \lambda_{n-1}]) = \frac{1}{n}\Big[\sum_{i=1}^{n-1} i(n-i)(\lambda_i{}^2 + n\lambda_i) + 2\sum_{1\le i<j\le n-1} i(n-j)\lambda_i\lambda_j\Big]. \tag{40}$$

By analogy to SU(6) (Ref. 7) the pseudoscalar and vector mesons are assigned to the adjoint (63-dimensional) representation of SU(8).[8] We then make the assumption that each meson state is represented by a basis vector that belongs to the system of commuting observables (39). This is an approximate description, but from the analogy with the Gürsey-Radicati SU(6) we expect this to be a much better description than the description of particle states by the Gelfand-Zetlin basis vectors. In this approximate description the vector mesons belong to an ideally mixed 16-plet of SU(4) and the pseudoscalar mesons belong to a pure 15-plet of SU(4). There is no mixing of the 1- and 63-dimensional irreps of SU(8), and no η-η' mixing in this approximation. The η', assigned to the 1-dimensional irrep of SU(8), and therewith a 1-dimensional irrep of any of its subgroups, e.g., also SU(4), is not considered here. The predictions obtained with this assumption will, therefore, be only approximately correct, since η-η' mixing, deviation from ideal mixing, and isospin mixing (ρ^0-ω mixing or η-π^0 mixing) will alter these results. However, we hope that for those processes which we shall consider, these "symmetry"-breaking corrections are not very important. We shall discuss this point later and describe the possible effects of these corrections. An inclusion of any of these effects in our general considerations would introduce too many parameters and would lead to a description which would not enable us to make a prediction.

We denote the basis vectors belonging to the system of commuting observables (39) by the particle symbol of the meson which the basis vector represents (in the above-described approximation). The correspondence is shown in Table V. It should be noted that the fact that S_χ or S_Y has a definite value for a particle (e.g., D^*) does not in general mean that this particle state transforms according to the corresponding irrep of $SU(2)_{S_\chi}$ or $SU(2)_{S_Y}$.

The basis system adapted to the chain of subgroups (37) will be denoted by the particle symbol of the pseudoscalar mesons, and a subscript V for $S^P = 1^-$ and P for $S^P = 0^-$. For example,

$$|\pi_V^0\rangle = |I_3=0, I=1, Y=0, \chi=0, \underline{8}, \underline{15}, S^P=1^-, S_3\rangle = \left|\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & 2 & & 1 & & 0 & \\ & & 2 & & 0 & & \\ & & & 1 & & & \end{pmatrix}; 1^-, S_3\right\rangle,$$

$$|\eta_V\rangle = -|I_3=0, I=0, Y=0, \chi=0, \underline{8}, \underline{15}, S^P=1^-, S_3\rangle = -\left|\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & 2 & & 1 & & 0 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{pmatrix}; 1^-, S_3\right\rangle,$$

$$|\chi_V\rangle = |I_3=0, I=0, Y=0, \chi=0, \underline{1}, \underline{15}, S^P=1^-, S_3\rangle = -\left|\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & 1 & & 1 & & 1 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{pmatrix}; 1^-, S_3\right\rangle, \tag{41}$$

$$|D_P^+\rangle = |I_3=\tfrac{1}{2}, I=\tfrac{1}{2}, Y=0, \chi=1, \underline{\bar{3}}, \underline{15}, 0^-\rangle = \left|\begin{pmatrix} 2 & & 1 & & 1 & & 0 \\ & 1 & & 1 & & 0 & \\ & & 1 & & 0 & & \\ & & & 1 & & & \end{pmatrix}; 0^-\right\rangle.$$

The phase factors are conventions but have to be made in conjunction with the phase conventions for the generators or regular tensor operators and Clebsch-Gordan coefficients. For the calculation of many presently available experimental data the relative phases are not important; however, in the calculation of the electromagnetic decays these phases are essential. For this reason, we list in Table VI the coefficients of the re-

TABLE V. Meson assignments in the 63-dimensional irrep of SU(8), with the eigenvalues of the complete system of commuting observables (39).

Particle	Eigenvalues of $\tilde{C}_2^6$	$\tilde{C}_2^4$	S	S_X	χ	S_Y	Y	I
$\pi^{\pm 0}$	$C_{35}=12$	$C_{15}=8$	0	0	0	0	0	1
$\rho^{\pm 0}$	C_{35}	C_{15}	1	0	0	0	0	1
η	C_{35}	$C_1=0$	0	0	0	0	0	0
ω	C_{35}	C_{15}	1	0	0	0	0	0
ϕ	C_{35}	C_1	1	0	0	1	0	0
$K^{+,0}$	C_{35}	$C_4=\frac{15}{4}$	0	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
$K^-\bar{K}^0$	C_{35}	$C_{\bar{4}}=\frac{15}{4}$	0	0	0	$\frac{1}{2}$	-1	$\frac{1}{2}$
$K^{*+,0}$	C_{35}	C_4	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
$\bar{K}^{*-}\bar{K}^{*0}$	C_{35}	$C_{\bar{4}}$	1	0	0	$\frac{1}{2}$	-1	$\frac{1}{2}$
D^+D^0	$C_{\bar{6}}=\frac{35}{6}$	$C_{\bar{4}}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$
$D^-\bar{D}^0$	$C_6=\frac{35}{6}$	C_4	0	$\frac{1}{2}$	-1	0	0	$\frac{1}{2}$
$D^{*+}D^{*0}$	$C_{\bar{6}}$	$C_{\bar{4}}$	1	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$
$\bar{D}^{*-}\bar{D}^{*0}$	C_6	C_4	1	$\frac{1}{2}$	-1	0	0	$\frac{1}{2}$
F^+	$C_{\bar{6}}$	$C_1=0$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	0
F^-	C_6	C_1	0	$\frac{1}{2}$	-1	$\frac{1}{2}$	-1	0
F^{+*}	$C_{\bar{6}}$	C_1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	0
F^{-*}	C_6	C_1	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	-1	0
$\psi(3.1)$	$C_1=0$	C_1	1	1	0	0	0	0
$\chi(2.75)$	C_1	C_1	0	0	0	0	0	0

TABLE VI. Coefficients of the D, F, and singlet reduced matrix elements in the matrix element of the electromagnetic operator given by Eq. (23) between vector- and pseudoscalar-meson states.

Initial state	Final state	$\langle \underline{15} \Vert \underline{15} \Vert \underline{15} \rangle_D$	$\langle \underline{15} \Vert \underline{15} \Vert \underline{15} \rangle_F$	$\langle \underline{15} \Vert \underline{1} \Vert \underline{15} \rangle$
$F^{\pm *}$	$F^{\pm}$	0	$\pm\frac{1}{2}$	1
$D^{\pm *}$	$D^{\pm}$	0	$\pm\frac{1}{2}$	1
$D^{0*}(\bar{D}^{0*})$	$D^0(\bar{D}^0)$	$-1/\sqrt{3}$	0	1
$K^{\pm *}$	$K^{\pm}$	0	$\pm\frac{1}{2}$	1
$K^{0*}(\bar{K}^{0*})$	$K^0(\bar{K}^0)$	$1/\sqrt{3}$	0	1
$\pi_V^{\pm}$	$\pi^{\pm}$	0	$\pm\frac{1}{2}$	1
π_V^0	π^0	0	0	1
η_V	η	$\frac{2}{3\sqrt{3}}$	0	1
η_{C_V}	η_C	$-\frac{2}{3\sqrt{3}}$	0	1
η_V	π^0	$-\frac{1}{3}$	0	0
η_{C_V}	π^0	$1/\sqrt{18}$	0	0
η_{C_V}	η	$\frac{1}{3\sqrt{6}}$	0	0

duced matrix elements which occur in the matrix elements of V^{el} between vector- and pseudoscalar-meson states. These coefficients have been taken from the literature and adapted to the phase convention used for the form (23) of the electromagnetic operator.

If the mesons are assigned to the basis vectors of the SU(8) basis system [corresponding to (39)] as given in Table V, then the pseudoscalar mesons are also represented by the SU(4) basis vectors (41) corresponding to (37). [It is for this reason that we can use the same pseudoscalar-meson symbol to denote the basis system corresponding to (37).] The vector mesons with $I \neq 0$ are also represented by the SU(4) basis vectors (41). However, the $I=0$ vector mesons are expressed in terms of the SU(4) basis vectors by

TABLE VII. Baryon assignments in the 120-dimensional irrep of SU(8), with the eigenvalues of the complete system of commuting observables (39). Names of the $\frac{1}{2}^+$ baryons are the same as in Ref. 4. The corresponding $\frac{3}{2}^+$ states are denoted by asterisks. The doubly charged $\frac{3}{2}^+$ SU(4) singlet is arbitrarily denoted B^{++}.

Particle	Eigenvalues of $\tilde{C}_2^6$	$\tilde{C}_2^6$	S	S_X	χ	S_Y	Y	I
p, n	$C_{56}=\frac{45}{2}$	$C_{20}=\frac{63}{4}$	$\frac{1}{2}$	0	0	0	1	$\frac{1}{2}$
$\Delta^{++,+,0,-}$	C_{56}	C_{20}	$\frac{3}{2}$	0	0	0	1	$\frac{3}{2}$
$\Sigma^{\pm,0}$	C_{56}	$C_{10}=9$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	1
Λ^0	C_{56}	C_{10}	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
$\Sigma^{*\pm 0}$	C_{56}	C_{10}	$\frac{3}{2}$	0	0	$\frac{1}{2}$	0	1
$\Xi^{-,0}$	C_{56}	$C_4=\frac{15}{4}$	$\frac{1}{2}$	0	0	1	-1	$\frac{1}{2}$
$\Xi^{*-,0}$	C_{56}	C_4	$\frac{3}{2}$	0	0	1	-1	$\frac{1}{2}$
Ω^-	C_{56}	$C_1=0$	$\frac{3}{2}$	0	0	$\frac{3}{2}$	-2	0
$C_1^{++,+,0}$	$C_{21}=\frac{40}{3}$	C_{10}	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{2}{3}$	1
C_0^+	C_{21}	C_{10}	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{2}{3}$	0
$C_1^{*++,+,0}$	C_{21}	C_{10}	$\frac{3}{2}$	$\frac{1}{2}$	1	0	$\frac{2}{3}$	1
$A^{0,+}$	C_{21}	C_4	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{2}$
$S^{0,+}$	C_{21}	C_4	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{2}$
$S^{*0,+}$	C_{21}	C_4	$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{2}$
T^0	C_{21}	C_1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$-\frac{1}{3}$	0
T^{*0}	C_{21}	C_1	$\frac{3}{2}$	$\frac{1}{2}$	1	1	$-\frac{4}{3}$	0
$X^{++,+}$	$C_6=\frac{35}{6}$	C_4	$\frac{1}{2}$	1	2	0	$\frac{1}{3}$	$\frac{1}{2}$
$X^{*++,+}$	C_6	C_4	$\frac{3}{2}$	1	2	0	$\frac{1}{3}$	$\frac{1}{2}$
X_S^+	C_6	C_1	$\frac{1}{2}$	1	2	$\frac{1}{2}$	$-\frac{2}{3}$	0
X_S^{*+}	C_6	C_1	$\frac{3}{2}$	1	2	$\frac{1}{2}$	$-\frac{2}{3}$	0
B^{++}	$C_1=0$	C_1	$\frac{3}{2}$	$\frac{3}{2}$	3	0	0	0

TABLE VIII. The SU(4) Clebsch-Gordan coefficients, taken from Ref. 9 and adapted to our phase convention.

		$(F^+)_D$	$(F^+)_F$
F^+	π^0	0	0
F^+	η	$\frac{1}{3}$	$(\frac{1}{12})^{1/2}$
F^+	χ	$(\frac{1}{18})^{1/2}$	$-(\frac{1}{6})^{1/2}$
D^+	K^0	$-(\frac{1}{6})^{1/2}$	$-(\frac{1}{8})^{1/2}$
D_0	K^+	$-(\frac{1}{6})^{1/2}$	$-(\frac{1}{8})^{1/2}$
		$(D^+)_D$	$(D^+)_F$
F^+	$\bar{K}^0$	$-(\frac{1}{6})^{1/2}$	$-(\frac{1}{8})^{1/2}$
D^+	π^0	$(\frac{1}{12})^{1/2}$	$\frac{1}{4}$
D^+	η	$-\frac{1}{6}$	$-(\frac{1}{48})^{1/2}$
D^+	χ	$(\frac{1}{18})^{1/2}$	$-(\frac{1}{6})^{1/2}$
D^0	π^+	$-(\frac{1}{6})^{1/2}$	$-(\frac{1}{8})^{1/2}$
		$(D^0)_D$	$(D^0)_F$
F^+	K^-	$-(\frac{1}{6})^{1/2}$	$-(\frac{1}{8})^{1/2}$
D^+	π^-	$-(\frac{1}{6})^{1/2}$	$-(\frac{1}{8})^{1/2}$
D^0	π^0	$-(\frac{1}{12})^{1/2}$	$-\frac{1}{4}$
D^0	η	$-\frac{1}{6}$	$-(\frac{1}{48})^{1/2}$
D^0	χ	$(\frac{1}{18})^{1/2}$	$-(\frac{1}{6})^{1/2}$
		$(K^+)_D$	$(K^+)_F$
F^+	$\bar{D}^0$	$-(\frac{1}{6})^{1/2}$	$(\frac{1}{8})^{1/2}$
K^+	π^0	$-(\frac{1}{12})^{1/2}$	$\frac{1}{4}$
K^+	η	$\frac{1}{6}$	$(\frac{3}{16})^{1/2}$
K^+	χ	$-(\frac{1}{18})^{1/2}$	0
K^0	π^+	$-(\frac{1}{6})^{1/2}$	$(\frac{1}{8})^{1/2}$
		$(K^0)_D$	$(K^0)_F$
F^+	D^-	$-(\frac{1}{6})^{1/2}$	$(\frac{1}{8})^{1/2}$
K^+	π^-	$-(\frac{1}{6})^{1/2}$	$(\frac{1}{8})^{1/2}$
K^0	π^0	$(\frac{1}{12})^{1/2}$	$-\frac{1}{4}$
K^0	η	$\frac{1}{6}$	$(\frac{3}{16})^{1/2}$
K^0	χ	$-(\frac{1}{18})^{1/2}$	0
		$(\pi^+)_D$	$(\pi^+)_F$
D^+	$\bar{D}^0$	$-(\frac{1}{6})^{1/2}$	$(\frac{1}{8})^{1/2}$
K^+	$\bar{K}^0$	$-(\frac{1}{6})^{1/2}$	$-(\frac{1}{8})^{1/2}$
π^+	π^0	0	$\frac{1}{2}$
π^+	η	$-\frac{1}{3}$	0
π^+	χ	$-(\frac{1}{18})^{1/2}$	0

TABLE VIII. (*Continued*)

		$(\pi^0)_D$	$(\eta)_D$	$(\chi)_D$	$(\pi^0)_F$	$(\eta)_F$	$(\chi)_F$	(η')
F^+	F^-	0	$\frac{1}{3}$	$(\frac{1}{18})^{1/2}$	0	$-(\frac{1}{12})^{1/2}$	$(\frac{1}{6})^{1/2}$	$-(\frac{1}{15})^{1/2}$
D^+	D^-	$(\frac{1}{12})^{1/2}$	$-\frac{1}{6}$	$(\frac{1}{18})^{1/2}$	$-\frac{1}{4}$	$(\frac{1}{48})^{1/2}$	$(\frac{1}{6})^{1/2}$	$-(\frac{1}{15})^{1/2}$
D^0	$\bar{D}^0$	$-(\frac{1}{12})^{1/2}$	$-\frac{1}{6}$	$(\frac{1}{18})^{1/2}$	$\frac{1}{4}$	$(\frac{1}{48})^{1/2}$	$(\frac{1}{6})^{1/2}$	$-(\frac{1}{15})^{1/2}$
K^+	K^-	$-(\frac{1}{12})^{1/2}$	$\frac{1}{6}$	$-(\frac{1}{18})^{1/2}$	$-\frac{1}{4}$	$-(\frac{3}{16})^{1/2}$	0	$-(\frac{1}{15})^{1/2}$
K^0	$\bar{K}^0$	$(\frac{1}{12})^{1/2}$	$\frac{1}{6}$	$-(\frac{1}{18})^{1/2}$	$\frac{1}{4}$	$-(\frac{3}{16})^{1/2}$	0	$-(\frac{1}{15})^{1/2}$
π^+	π^-	0	$-\frac{1}{3}$	$-(\frac{1}{18})^{1/2}$	$-\frac{1}{2}$	0	0	$-(\frac{1}{15})^{1/2}$
π^0	π^0	0	$-\frac{1}{3}$	$-(\frac{1}{18})^{1/2}$	0	0	0	$-(\frac{1}{15})^{1/2}$
π^0	η	$-\frac{1}{3}$	0	0	0	0	0	0
π^0	χ	$-(\frac{1}{18})^{1/2}$	0	0	0	0	0	0
η	η	0	$\frac{1}{3}$	$-(\frac{1}{18})^{1/2}$	0	0	0	$-(\frac{1}{15})^{1/2}$
η	χ	0	$-(\frac{1}{18})^{1/2}$	0	0	0	0	0
χ	χ	0	0	$(\frac{2}{9})^{1/2}$	0	0	0	$-(\frac{1}{15})^{1/2}$

$$
\begin{aligned}
|\phi\rangle &= (\tfrac{2}{3})^{1/2}|\eta_V\rangle + (12)^{-1/2}|\chi_V\rangle - \tfrac{1}{2}|\sigma_V\rangle ,\\
|\omega\rangle &= -3^{-1/2}|\eta_V\rangle + 6^{-1/2}|\chi_V\rangle - 2^{-1/2}|\sigma_V\rangle ,\\
|\psi\rangle &= (\tfrac{3}{4})^{1/2}|\chi_V\rangle + \tfrac{1}{2}|\sigma_V\rangle .
\end{aligned}
\qquad (42)
$$

The phases depend upon the phase conventions and are important for our calculations.

In an analogous way, we can classify the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ baryons according to the subgroup chain (38) of SU(8). The results are shown in Table VII.

For convenience, we list in Table VIII the Clebsch-Gordan coefficients for the meson 15-plet. The F- and D-type coefficients $\langle\alpha\|\beta\|\alpha'\rangle_{F,D}$ for the physical matrix element $\langle\alpha'|V^\beta|\alpha\rangle$ appear in the table as follows:

		$(\alpha')_D$	$(\alpha')_F$
α	β	$\cdots$	$\cdots$

Coefficients not appearing in the tables may be formed by using the properties

$$\langle\alpha'\|\alpha\|\beta\rangle_{F,D} = \langle\alpha'\|\beta\|\alpha\rangle_{F,D} \times \begin{cases} -1 & \text{for } F \\ +1 & \text{for } D \end{cases} .$$

These coefficients have been adapted to our own phase convention and were extracted from the Tables by Miyata *et al.*[9]

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[1]L. C. Biedenharn, J. Math. Phys. 4, 436 (1963); G. E. Baird and L. C. Biedenharn, *ibid.* 4, 1449 (1963).
[2]J. D. Louck and L. C. Biedenharn, J. Math. Phys. 11, 2368 (1970).
[3]G. Goldhaber *et al.*, Phys. Rev. Lett. 37, 255 (1976).
[4]M. K. Gaillard, B. W. Lee, and J. L. Rosner, Rev. Mod. Phys. 47, 277 (1975).
[5]An explanation of the exact meaning of the "current" operators V_μ^α is given in A. Bohm, Phys. Rev. D 13, 2110 (1976).
[6]A. M. Perelomov and V. S. Popov, Yad. Fiz. 3, 924 (1966) [Sov. J. Nucl. Phys. 3, 676 (1966)].
[7]F. Gürsey and L. A. Radicati, Phys. Rev. Lett. 13, 173 (1964).
[8]S. Iwao, Ann. Phys. (N.Y.) 35, 1 (1965); J. W. Moffat, Phys. Rev. 140, B1681 (1965); S. Okubo, Phys. Rev. D 11, 3261 (1975); S. Eliezer and B. R. Holstein, *ibid.* 11, 3344 (1975); D. H. Boal, R. H. Graham, J. W. Moffat, and P. J. O'Donnell, Phys. Rev. Lett. 34, 541 (1975); J. W. Moffat, Phys. Rev. D 12, 286 (1975); 12, 288 (1975); J. Franklin, *ibid.* 12, 2077 (1975); A. W. Hendry and D. B. Lichtenberg, *ibid.* 12, 2756 (1975); D. B. Lichtenberg, Lett. Nuovo Cimento 13, 346 (1975); C. A. Nelson, Phys. Rev. D 13, 2139 (1976).
[9]Y. Miyata, S. Iwai, and K. Kudoh, Tokyo Inst. of Tech. Report No. TIT/HEP-21, 1975 (unpublished).