Analyticity bounds involving pion polarizabilities and $\pi\pi\to\gamma\gamma$ threshold amplitudes

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Some simple analyticity bounds involving the pion electromagnetic polarizabilities and threshold values of the (properly normalized) $I=0,\ 2$ $\pi\pi\to\gamma\gamma$ d-wave amplitudes are derived and discussed.

The possibility of investigating two-pion production in electron-positron colliding-beam experiments renewed interest in the theoretical study of the $\pi\pi\to\gamma\gamma$ process.[1] There is now also some interest in the electromagnetic polarizabilities of the hadrons (in particular of the pions).[2] As the pion polarizabilities are intimately related to the pion Compton scattering, we find it useful to report here some simple analyticity bounds involving the pion polarizabilities and the threshold values of the (properly normalized) $I=0$ and $I=2$ d partial waves of the annihilation process $\pi\pi\to\gamma\gamma$. In deriving such bounds we have been guided by methods previously employed in connection with the purely hadronic process $\pi\pi\to\pi\pi$.[3]

Our considerations will be restricted to the lowest order in electromagnetism. The processes $\gamma\pi\to\gamma\pi$ and $\pi\pi\to\gamma\gamma$ have been simultaneously studied within the Mandelstam representation framework in Ref. 4. As we are working here with other invariant amplitudes and with somewhat different partial-wave projections, we begin by displaying in some detail the relevant kinematics.

The S matrix for pion Compton scattering is

$$S=\delta_{i,f}+i(2\pi)^{-2}(16k_0k_0'p_0p_0')^{-1/2}\epsilon_\mu^*(k')T_{\mu\nu}\epsilon_\nu(k)\,, \qquad (1)$$

with

$$\begin{aligned}T_{\mu\nu}=&A(s,t,u)[(k\cdot k')g_{\mu\nu}-k_\mu k_\nu']\\&-B(s,t,u)[(k\cdot k')P_\mu P_\nu-(P\cdot K)(P_\mu k_\nu'+P_\nu k_\mu)\\&\qquad+(P\cdot K)^2 g_{\mu\nu}]\,,\end{aligned} \qquad (2)$$

where k (k') and p (p') are the initial (final) photon and pion four-momenta, ϵ,ϵ^* are the polarization vectors of the photons and

$$P_\mu=\tfrac{1}{2}(p_\mu+p_\mu'),\quad K_\mu=\tfrac{1}{2}(k_\mu+k_\mu')$$
$$\Delta=(p+k)^2,\quad t=(k-k')^2,\quad s+t+u=2\mu^2$$
$$\mu=\text{pion mass}\,.$$

The invariant amplitudes A and B (both even under s-u crossing) are known to be free of kinematical singularities and zeros[5] and in the case of γ scattering on charged pions have the following Born pole structure:

$$\begin{aligned}A^{\text{Born}}(s,t,u)&=\frac{-e^2t}{(s-\mu^2)(u-\mu^2)}\,,\\B^{\text{Born}}(s,t,u)&=\frac{8e^2}{(s-\mu^2)(u-\mu^2)}\,,\quad\frac{e^2}{4\pi}\simeq\frac{1}{137}\,.\end{aligned} \qquad (3)$$

(For $\gamma\pi^0$ scattering, of course, there are no similar contributions.) In the t channel the same amplitudes describe the annihilation process $\pi\pi\to\gamma\gamma$ which occurs in the isotopic states $I=0$ or $I=2$. The relationship between the amplitudes with given isospin in the t channel and those referring to γ scattering on charged (ch) and neutral (N) pions in the direct channel is given by

$$\begin{aligned}T^{(I=0)}&=T^{(\text{ch})}+\tfrac{1}{2}T^{(\text{N})}\,,\\T^{(I=2)}&=T^{(\text{ch})}-T^{(\text{N})}\,.\end{aligned} \qquad (4)$$

The needed kinematics in the barycentric systems of the s and t channels is established by the following relations:

s channel

$$t=-2p^2(1-\cos\theta),\quad p=\frac{s-\mu^2}{2s^{1/2}}\,,\quad \cos\theta=1+\frac{2st}{(s-\mu^2)^2}\,. \qquad (5)$$

t channel

$$\begin{aligned}&s=-(q^2+\omega^2+2q\omega\cos\varphi),\quad \omega^2=q^2+\mu^2\\&u=-(q^2+\omega^2-2q\omega\cos\varphi)\,,\quad \cos\varphi=\frac{u-s}{[t(t-4\mu^2)]^{1/2}}\\&q=\tfrac{1}{2}(t-4\mu^2)^{1/2},\quad k=\tfrac{1}{2}(t)^{1/2}\,.\end{aligned} \qquad (6)$$

The connection between the invariant amplitudes A,B and the independent helicity amplitudes $f_{0,1;0,\pm1}$ (of the s channel) and $f_{0,0;1,\pm1}$ (of the t channel) is

$$\begin{aligned}A&=-\frac{t-4\mu^2}{2[st+(s-\mu^2)^2]}f_{0,1;0,1}-\frac{2}{t}f_{0,1;0,-1}\,,\\B&=\frac{4}{st+(s-\mu^2)^2}f_{0,1;0,1}\\f_{0,0;1,1}&=\frac{t}{2}A+\frac{t(t-4\mu^2)}{16}B\,,\\f_{0,0;1,-1}&=-\frac{(s-\mu^2)^2+st}{4}B\,.\end{aligned} \qquad (7)$$

The development of $f_{0,0;1,-1}$ in partial waves,

$$f_{0,0;1,-1}=\sum_{J=\text{even}}(2J+1)\,g_-^{(J)}(t)(kq)^J d_{2,0}^J(\cos\varphi)\,, \tag{8}$$

allows us, using a fixed-t unsubtracted dispersion relation for the B amplitude,

$$B(s,t,u)=B^{\text{Born}}(s,t,u)+\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds'\,\text{Im}B(s',t)\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right), \tag{9}$$

to express the threshold value of the partial wave $g_-^{(J=2)}(t)$ in terms of the s-channel absorptive part of the B amplitude at $t=4\mu^2$. Indeed, using Eqs. (7) and (8) one has

$$g_-^{(J=2)}(t)=\frac{\sqrt{3}}{4\sqrt{2}}\int_{-1}^{+1}\sin^4\varphi\,B(s(t,\cos\varphi),t)d(\cos\varphi)\,, \tag{10}$$

which, in turn, through Eq. (9) formally gives

$$\gamma\equiv g_-^{(J=2)}(t=4\mu^2)-g_-^{(J=2)\text{Born}}(t=4\mu^2)=\rho\int_{4\mu^2}^{\infty}\frac{ds'}{s'+\mu^2}\text{Im}B(s',t=4\mu^2)\,, \tag{11}$$

where the second term on the right-hand side in the first line is explicitly known by Eqs. (10) and (3), and ρ is a numerical factor,

$$\rho\equiv\frac{8\sqrt{3}}{15\pi\sqrt{2}}\,. \tag{12}$$

The optical theorem in the s channel reads as follows:

$$\text{Im}f_{0,1;0,1}(s',t=0)=\frac{(s'-\mu^2)^2}{4}\,\text{Im}B(s',t=0)=(s'-\mu^2)\sigma^T(s')\,, \tag{13}$$

where $\sigma^T(s')$ is the total cross section for photoabsorption on pion. The generalized electric (α) and magnetic (β) polarizabilities of the pion [which appear in the low-energy expansion of the pion Compton-scattering differential cross section as coefficients specifying the angular distribution in the terms containing the second power of the photon frequency (Rayleigh scattering)] can be easily identified as

$$\alpha+\beta=\frac{\mu}{8\pi}B^{\text{cont}}\Big|_{s=\mu^2,\,t=0}\,,\qquad \alpha-\beta=-\frac{1}{8\mu\pi}(2A^{\text{cont}}-\mu^2B^{\text{cont}})\Big|_{s=\mu^2,\,t=0}\,, \tag{14}$$

where the superscript "cont" indicates that part of the amplitude which remains after the s and u channel Born poles have been taken off. Using Eq. (9) at $t=0$ and the optical theorem Eq. (13) one has the known relation

$$\alpha+\beta=\frac{\mu}{\pi^2}\int_{4\mu^2}^{\infty}ds'\frac{\sigma^T(s')}{(s'-\mu^2)^2}\,. \tag{15}$$

Now, using the partial-wave projection of the s channel helicity amplitude $f_{0,1;0,1}$,

$$f_{0,1;0,1}(s,t)=\sum_{J=1}^{\infty}(2J+1)f_{0,1;0,1}^{(J)}(s)d_{1,1}^J(\cos\theta)\,, \tag{16}$$

noting the positivity of $\text{Im}f_{0,1;0,1}^{(J)}(s)$, Eqs. (5), (11), (7), (13), (14), and the simple property of the $d_{1,1}^J$ functions

$$d_{1,1}^J(x)\geq d_{1,1}^1(x)=\tfrac{1}{2}(1+x) \tag{17}$$

for $x\geq 1$ and for any integer J, one immediately arrives at the desired bound through the following chain of inequalities:

$$\frac{\gamma}{4\rho}=\int_{4\mu^2}^{\infty}\frac{ds}{(s+\mu^2)^3}\sum_{J=1}^{\infty}(2J+1)\,\text{Im}f_{0,1;0,1}^{(J)}(s)d_{1,1}^J\left[1+\frac{8\mu^2s}{(s-\mu^2)^2}\right]$$
$$\geq\int_{4\mu^2}^{\infty}\frac{ds}{(s-\mu^2)^3}\frac{(s-\mu^2)}{(s+\mu^2)}\sum_{J=1}^{\infty}(2J+1)\,\text{Im}f_{0,1;0,1}^{(J)}(s)=\int_{4\mu^2}^{\infty}ds\frac{\sigma^T(s)}{(s-\mu^2)^2}\frac{(s-\mu^2)}{(s+\mu^2)}\geq\frac{3}{5}\frac{\pi^2}{\mu}(\alpha+\beta)\,. \tag{18}$$

Recalling the isospin specifications from Eqs. (4) we can finally write the results of this paper as follows:

$$\frac{2}{3}[g_-^{(J=2)(I=0)}(t=4\mu^2)-g_-^{(J=2)(I=2)}(t=4\mu^2)]\geq 4\rho\int_{4\mu^2}^{\infty}\frac{\sigma_{\gamma\pi^0}^T(s)ds}{s^2-\mu^4}\geq\frac{3}{5}4\rho\frac{\pi^2}{\mu}(\alpha+\beta)^{(\pi^0)}>0\,, \tag{19}$$

$$\frac{2}{3}\left[g_-^{(J=2)(I=0)}(t=4\mu^2)+\frac{1}{2}g_-^{(J=2)(I=2)}(t=4\mu^2)\right]-\rho\frac{e^2\pi}{\mu^4}\geq 4\rho\int_{4\mu^2}^{\infty}\frac{\sigma_{\gamma\pi^\pm}^T(s)}{s^2-\mu^4}\geq\frac{3}{5}4\rho\frac{\pi^2}{\mu}(\alpha+\beta)^{(\pi^\pm)}>0 \tag{20}$$

and

$$g_-^{(J=2)(I=0)}(t=4\mu^2) - \rho\frac{e^2\pi}{\mu^4} \geq 4\rho\int_{4\mu^2}^{\infty} \frac{\sigma^T_{\gamma\pi^\pm}(s)+\frac{1}{2}\sigma^T_{\gamma\pi^0}(s)}{s^2-\mu^4} \geq \frac{3}{5}4\rho\frac{\pi^2}{\mu}[(\alpha+\beta)^{(\pi^\pm)}+\frac{1}{2}(\alpha+\beta)^{(\pi^0)}]>0\,, \tag{21}$$

which follows directly from Eqs. (19) and (20).

For completeness we note the connection between the partial waves $g_-^{(J)}$ used by us and the corresponding amplitudes G^J used in Ref. 1:

$$(2J+1)(kq)^J g_-^{(J)} = [(J-1)J(J+1)(J+2)]^{1/2}G^J\,.$$

Note added in proof. We also note that from kinematical considerations alone one can further relate $g_-^{(J=2)\,(\mathrm{ch})}(t=4\mu^2)$ from the left-hand side of the inequality (20) to the threshold behavior of the center-of-mass total cross section for $\pi^+\pi^-$ annihilation into two photons with opposite helicities:

$$\lim_{q\to 0} q^{-3}\sigma_{\pi^+\pi^-\to\gamma(+1),\gamma(-1)} = \frac{5\mu^3}{32\pi}[g_-^{(J=2)\,(\mathrm{ch})}(t=4\mu^2)]^2 \tag{22}$$

Indeed, if one integrates over the solid angle the center-of-mass differential cross section for $\pi^+\pi^-$ annihilating into two photons with helicities +1 and −1,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\pi^+\pi^-\to\gamma(+1),\gamma(-1)} = \frac{1}{32\pi^2 t}\frac{k}{q}|f^{(\mathrm{ch})}_{0,0;1,-1}|^2\,, \tag{23}$$

with the aid of the partial-wave projection of $f_{0,0;1,-1}$ displayed in Eq. (8), one arrives immediately at Eq. (22) when taking the specified threshold limit.

The analyticity assumptions required to get the bounds (19)–(21) are quite clear from their derivation and refer mainly to the existence of the quantities we have been operating with. The inequalities obtained above tell us that the d-wave $\pi\pi\to\gamma\gamma$ annihilation somehow controls the ability of the hadronic cloud surrounding the pion to become polarized in the presence of external electric and magnetic fields. Equation (21), for instance, shows that the threshold value of $g_-^{(J=2)(I=0)}$ should not be less than its Born value and that the deviation between the actual threshold value and the corresponding Born value imposes an upper limit on the sum $\alpha+\beta$ of the electric and magnetic polarizabilities of the pions. The sum $\alpha+\beta$ for pions is expected to be very small, at least in comparison with the standard unit of 10^{-3} fm^3, owing to the small $\rho\pi\gamma$ coupling (current-algebra[6] and chiral-Lagrangian[7] calculations also support this indication), and so the stringency of these bounds will depend essentially on how close the threshold d waves are to their Born parts. Although probably (as it usually happens with model-independent results of this kind) the inequalities written down in this paper will turn out to be only of academic interest, we think it would be worthwhile to know more (at least as much as we do for $\pi\pi$ scattering lengths) about the threshold values of $\pi\pi\to\gamma\gamma$ amplitudes and/or α, β which are essentially the threshold values of the continuum parts of the invariant $\gamma\pi\to\gamma\pi$ amplitudes.

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