

Internal structure of multicomponent fluids

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The equations of hydrostatic equilibrium for the constituents of a spherical multicomponent fluid derived earlier are interpreted along thermodynamic lines. An example of a 3-component neutron star of ideal gases is worked out, with the internal charge density and electric field computed.

I. INTRODUCTION

In a previous paper¹ the general-relativistic equations of hydrostatic equilibrium for a fluid composed of different species of charged or neutral particles were derived for the special case of spherical symmetry. Each species has its own equilibrium equation coupled, however, to those of the other particles; the sum is the familiar Tolman-Oppenheimer-Volkoff (TOV) equation². The new "species TOV" or STOV equations were derived by making the asymptotic energy (or mass) of the fluid an extremum with respect to variations in the number densities of the individual species. A general criterion for electrical neutrality of the fluid was found.

In a second paper³ the theory was applied to a white dwarf star which is composed of two species of particles, one positively charged, the other negatively charged. This example of the theory was greatly facilitated by the assumption that the heavy nuclei formed a perfect Bose gas at zero temperature, thus having zero partial pressure. As a result, the complete set of equilibrium equations could be reduced to a single equation in one unknown function, for which the solution was determined numerically.

The purpose of this note is twofold: to interpret the equations of the first paper on a thermodynamic basis, and to show how the theory can be applied to cases more general than that of the second paper. As an example of the latter we use a neutron star conceived as a mixture of three perfect Fermi gases: neutrons, protons, and electrons. The internal structure of a neutron star is known to be far more complex than such a simple model; but the procedure outlined here could also be used with more sophisticated models provided the local energy density could be given as a function of the number densities of the constituents (i.e., provided an equation of state is available). As it is, we can obtain the various number densities, internal electric field, etc., as functions of the radius.

II. THE BASIC EQUATIONS FROM A THERMODYNAMIC POINT OF VIEW

As in Refs. 1 and 3 n_i is the number density of the i th species, $\rho_m(n_1, n_2, \dots)$ is the local mass density, and μ_i is the chemical potential excluding electrical effects:

$$\mu_i = \partial(\rho_m c^2) / \partial n_i. \quad (2.1)$$

The metric components involve $g_{00} = e^\nu$ and $g_{rr} = e^\lambda$. If q_i is the charge on a particle of the i th species then the STOV equations were found to be

$$\mu_i' \equiv \frac{d\mu_i}{dr} = q_i \mathcal{G} r^{-2} e^{\lambda/2} + \mu_i \left(\frac{1}{2} \lambda' - \frac{4\pi G}{c^2} r e^\lambda \sum_j n_j \mu_j \right), \quad (2.2)$$

where $\mathcal{G}(r)$ is the charge integral up to r . These equations were derived essentially by treating the energy integral at zero temperature (interpreted as the mass at infinity) as a functional of the densities n_i , and minimizing the respect to independent variations of the n_i .

To cast the equations into thermodynamic form, use Eqs. (2.2) and (2.3) of Ref. 1:

$$4\pi c^{-4} G r e^\lambda (\rho_m c^2 + p) = \frac{1}{2} (\lambda' + \nu'). \quad (2.3)$$

Then Eq. (2.2) above reduces to

$$\mu_i' + \mu_i \nu' / 2 - \mu_i 4\pi c^{-4} G r e^\lambda \left(\rho_m c^2 + p - \sum_j n_j \mu_j \right) = q_i r^{-2} \mathcal{G} e^{\lambda/2}. \quad (2.4)$$

We now use a local relation from thermodynamics⁴

$$U = TS - pV + \sum \mu_i N_i, \quad (2.5)$$

where p is the pressure, V is the volume, U is the internal energy, and N_i is the number of particles of type i . Divide this by V using

$$\rho_m c^2 = \frac{U}{V}, \quad n_i = \frac{N_i}{V}, \quad s = \frac{S}{V}. \quad (2.6)$$

Then Eq. (2.5) above becomes

$$\rho_m c^2 + p - \sum \mu_i n_i = -Ts. \quad (2.7)$$

At zero temperature, which we are considering here, the right-hand side is zero, which means that Eq. (2.4) can be written as

$$dM_i/dr = 0, \quad (2.8a)$$

$$M_i = \text{const}, \quad i = 1, 2, \dots, \quad (2.8b)$$

where

$$M_i = e^{\nu/2} \mu_i(r) - \int^r q_i r'^{-2} g(r') e^{(\lambda+\nu)/2} dr'. \quad (2.9a)$$

If the electrostatic potential ϕ_0 (the zero component of ϕ_a) and the four-velocity $u^a = dx^a/ds$ [whence $u^0 = \exp(-\nu/2)$] are introduced, then Eq. (2.9a) takes on the simpler form

$$\begin{aligned} M_i &= e^{\nu/2} \mu_i(r) + q_i \phi_0 \\ &= e^{\nu/2} [\mu_i(r) + q_i (\phi_a u^a)]. \end{aligned} \quad (2.9b)$$

M_i is the generalized chemical potential of the i th species at $T=0$, taking into account electrical and gravitational effects, and interactions between species. The form in Eq. (2.9b) is familiar in the literature for one-component systems.⁵ Equation (2.8) then states that the generalized chemical potential of each species in a spherically symmetric system in equilibrium at $T=0$ is constant throughout the fluid.⁶ This is the thermodynamic interpretation of the first integral of the STOV equations.

Equation (2.8) is an extension of a result found long ago by Klein.⁷ Klein found that $g_{00}^{1/2} \mu_i$ is constant throughout a fluid of any symmetry in thermal equilibrium at any temperature. His argument is based, however, on systems of independent (i.e., noninteracting) components, and did not take into account electrical effects.

Equations such as (2.8) are automatically consistent with the equations of chemical equilibrium. Multiply Eq. (2.8) by constants α_i and sum over i . If the α_i 's are chosen such that

$$\sum \alpha_i q_i = 0 \quad (2.10)$$

then we get

$$\sum \alpha_i M_i = \sum \alpha_i \mu_i e^{\nu/2} = \text{const.} \quad (2.11)$$

In this way the electrical effects are eliminated. Equation (2.11) reduces to the ordinary equation of chemical equilibrium for a reaction only if the α_i 's are the stoichiometric coefficients for the reaction, and the constant on the right in (2.11) is zero. The stoichiometric coefficients are always consistent with charge conservation Eq. (2.10), but they are fixed by the reaction and cannot be adjusted to set the constant on the right in (2.11) equal to zero. However, the M_i 's in Eq. (2.11) are the constants of integration of Eq. (2.8) and are so far arbitrary. Thus the M_i 's may always be chosen to make the constant on the right in (2.11) equal to zero when the α_i 's are the stoichiometric coefficients:

$$e^{-\nu/2} \sum \alpha_i M_i = \sum \alpha_i \mu_i = 0. \quad (2.12)$$

This, the classical equation of chemical equilibrium, is then consistent with Eqs. (2.8).

That Eq. (2.12) should be valid in any case follows directly from thermodynamic reasoning.⁸ We shall use Eq. (2.12) later in the paper.

It can be remarked that since Eq. (2.12) is a restriction on the constants M_i , chemical equilibrium plays the role of boundary conditions in solving the STOV equations. But more conditions than these are needed for a complete solution.

In the remainder of the paper we consider a system composed of electrons, protons, and neutrons. Equation (2.10) says that $\alpha_{\text{elec}} = \alpha_{\text{prot}}$, with α_{neut} left arbitrary. We can always choose α_{neut} to be such as to make the right-hand side of Eq. (2.11) zero at some point r (it will be zero everywhere then). But chemical equilibrium will require that the value that does this is $\alpha_{\text{neut}} = -\alpha_{\text{elec}}$.

III. SIMPLE MODEL CALCULATION FOR THE NEUTRON STAR

We consider here a simple model of a neutron star consisting of three ideal Fermi gases in equilibrium: the neutrons, protons, and electrons. There are three STOV equations each of the form of Eq. (4.1) of Ref. 1, or Eq. (2.2) above. Writing out the metric component $\exp(-\lambda)$ as $1 - 2m(r)/r$, we obtain (primes represent r derivatives)

$$\mu_i' = \frac{q_i \mathcal{S}}{r^2(1-2m/r)^{1/2}} + \frac{1}{1-2m/r} \mu_i \left[-\frac{m}{r^2} + \frac{1}{2} \frac{G\mathcal{S}^2}{c^4 r^3} + \frac{4\pi G r}{c^4} (\rho_m c^2 - n_e \mu_e - n_p \mu_p - n_n \mu_n) \right], \quad (3.1)$$

where i may be "e," "p," "n." To complete the equations we need Eqs. (3.14) and (2.6) of Ref. 1:

$$m' = 4\pi G e^{-2} r^2 [\rho_m + \mathcal{S}^2/(8\pi c^2 r^3)], \quad (3.2)$$

$$\mathcal{S}' = 4\pi r^2 (1 - 2m/r)^{-1/2} (q_e n_e + q_p n_p). \quad (3.3)$$

Equations (3.2), (3.3), and the sum of (3.1) constitute the Einstein-Maxwell equations for the problem. The remaining independent equations of (3.1) provide the new information from the STOV equations.

Equations (3.1)–(3.3) are five equations in the five unknowns n_e , n_p , n_n , m , and \mathcal{E} . It is understood that an equation of state will give us ρ_m as a function of the number densities n_i . Also, the μ_i are obtained from Eq. (2.1). Thus the system of equations appears determinate.

Rather than use all three Eqs. (3.1), we shall replace the $i=n$ equation with the equation of chemical equilibrium, Eq. (2.12). This is simpler to use since it has no gravitational or electrical functions in it, and can be solved in principle for n_n directly as a function of n_e and n_p :

$$n_n = g(n_e, n_p), \text{ solution of (2.12) ,} \quad (3.4)$$

where g is some function.

The plan of solution is to solve the $i=e$ and $i=p$ equations of (3.1) for m and \mathcal{E} and then substitute the results in Eqs. (3.2) and (3.3) to give us two equations in the two unknowns n_e and n_p . The solution of the $i=e, p$ equations of (3.1) for m and \mathcal{E} is

$$m = \frac{r}{2} \frac{b+a^2}{K+a^2}, \quad (3.5)$$

$$\mathcal{E} = c^2 G^{-1/2} r a \left(\frac{K-b}{K+a^2} \right)^{1/2}, \quad (3.6)$$

where the abbreviations a , b , and K —all known functions of n_e and n_p , once ρ_m is known and Eq. (3.4) is accepted—are

$$K = 1 - 2r \frac{q_p \mu'_e - q_e \mu'_p}{q_p \mu_e - q_e \mu_p}, \quad (3.7)$$

$$a = \frac{G^{1/2} r}{c^2} \frac{\mu_e \mu'_p - \mu_e \mu'_p}{q_p \mu_e - q_e \mu_p}, \quad (3.8)$$

$$b = 8\pi G c^{-4} r^2 (\rho_m c^2 - \mu_e n_e - \mu_p n_p - \mu_n n_n). \quad (3.9)$$

The constants a , b , and K are dimensionless; a and b are very much smaller than one.

Substitution of Eqs. (3.5) and (3.6) into Eqs. (3.2) and (3.3) brings the basic equations to solve

$$\frac{d}{dr} \left(r \frac{b+a^2}{K+a^2} \right) = \gamma_1 + a^2 \frac{K-b}{K+a^2}, \quad (3.10)$$

$$\frac{d}{dr} \left(r^2 a^2 \frac{K-b}{K+a^2} \right) = r \gamma_2, \quad (3.11)$$

where

$$\gamma_1 = 8\pi G c^{-4} r^2 \rho_m c^2, \quad (3.12)$$

$$\gamma_2 = 8\pi G^{1/2} c^{-2} a r^2 (q_e n_e + q_p n_p). \quad (3.13)$$

It can be seen by eliminating the general-relativistic effects that Eq. (3.10) corresponds to the usual equilibrium equation, whereas Eq. (3.11) would vanish if charge neutrality occurred. The purpose of the above manipulations was in fact to obtain equations that had this significance.

In principle Eqs. (3.10) and (3.11) could be solved numerically for any given equation of state.

The equations are valid in fact for any 3-component system with one component positively charged and one component negatively charged.

The plan for solving these equations is to make a perturbation expansion in terms of the charge imbalance, that is, of $(n_e - n_p)/n_e$. The effects turn out to be so small that even the computers of today would have a hard time seeing them, which is why we start by making such an expansion.

Thus in a first iteration Eq. (3.11) is ignored, and Eq. (3.10) is solved using $n_e = n_p$. With the solution thus obtained the left-hand side of Eq. (3.11) can be calculated, enabling us to solve for the right-hand side of Eq. (3.11) which contains the charge imbalance directly, $q_e n_e + q_p n_p$. At this point one can check to see if the original premise is verified, namely that $x \ll 1$, where $x = (n_e - n_p)/n_e$ and also that $(n_e - n_p)' \ll n_e'$, i.e., $x' \ll n_e'/n_e$. In the calculations made, these inequalities were verified; x was of the order 10^{-33} .

Without going into details,⁹ we write down here the energy density for the three perfect gases,

$$\rho_m c^2 = m_e c^2 \lambda_e^{-3} [f(s) + \epsilon^{-2} f(\epsilon u) + \theta^{-2} f(\theta t)], \quad (3.14)$$

where $f(s)$ is the function

$$f(s) = \pi \{ (2s^{3/2} + s^{1/2})(s+1)^{1/2} - \ln[s^{1/2} + (s+1)^{1/2}] \}, \quad (3.15)$$

and the constants are

$$\lambda_e = h/m_e c, \quad \epsilon = (m_e/m_p)^2, \quad \theta = (m_e/m_n)^2, \quad (3.16)$$

and the variables are

$$\begin{aligned} s &= (3n_e \lambda_e^3 / 8\pi)^{2/3}, \\ u &= (3n_p \lambda_e^3 / 8\pi)^{2/3}, \\ t &= (3n_n \lambda_e^3 / 8\pi)^{2/3}. \end{aligned} \quad (3.17)$$

The equation of chemical equilibrium (2.12) for the free gases gives us t in terms of s and u :

$$\theta t = \theta [(u + \epsilon^{-1})^{1/2} + (s+1)^{1/2}]^2 - 1. \quad (3.18)$$

From here on the calculation consists of substituting these expressions into Eq. (3.10) identifying negligibly small quantities (i.e., setting $n_e = n_p$, or $s = u$), and solving for s , then substituting the results in Eq. (3.11) to compute the charge.

The boundary conditions at the origin are that $s(r=0)$ is arbitrary, and $s'(r=0) = 0$. Thus the solutions will have one free parameter, $s(0)$, or $\rho_m(0)$.

IV. RESULTS OF THE CALCULATION

The results obtained for the 3-component neutron star are displayed in five figures. Figures 1 and 2 refer to functions (total mass and radius of the

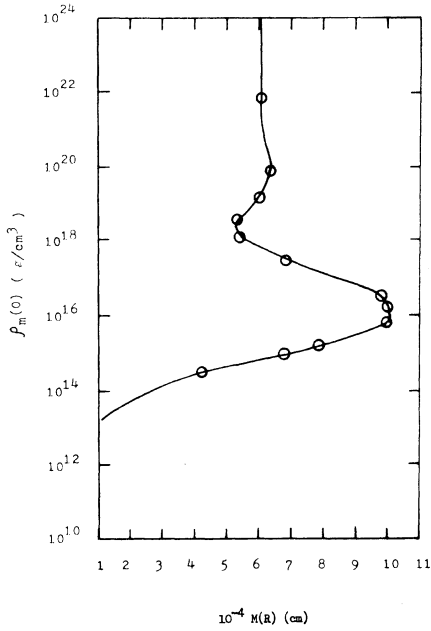


FIG. 1. Star mass as a function of central density. Circles represent calculation without charge effects.

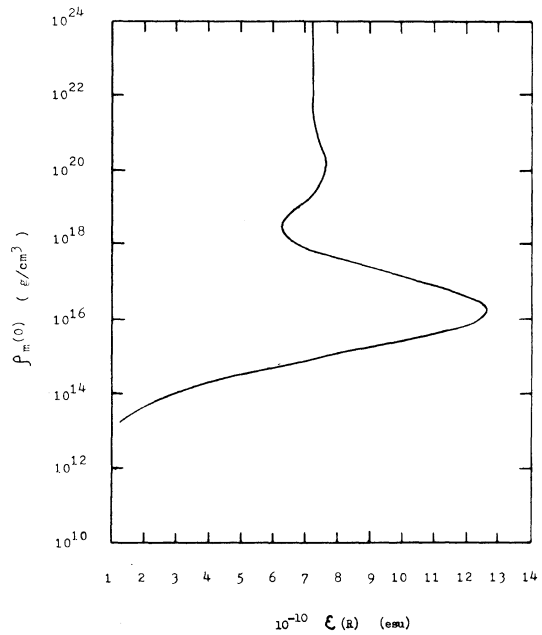


FIG. 3. Charge integral as a function of central density.

star) that are not expected to change much as a result of adding the electrical or multicomponent effects into account. These figures show the present calculation (solid line) compared with the uncharged model (circles). To within computational

error these are the same. The two figures are included to demonstrate the insensitivity of these quantities. The same type of result occurred in the white dwarf star.³

Figures 3, 4, and 5 refer to quantities that are directly affected by charge: the total charge of the star, the central charge density, and the electric field. Figure 3 shows the total charge as a function of central density. This also resembles the corresponding function in the white dwarf star.

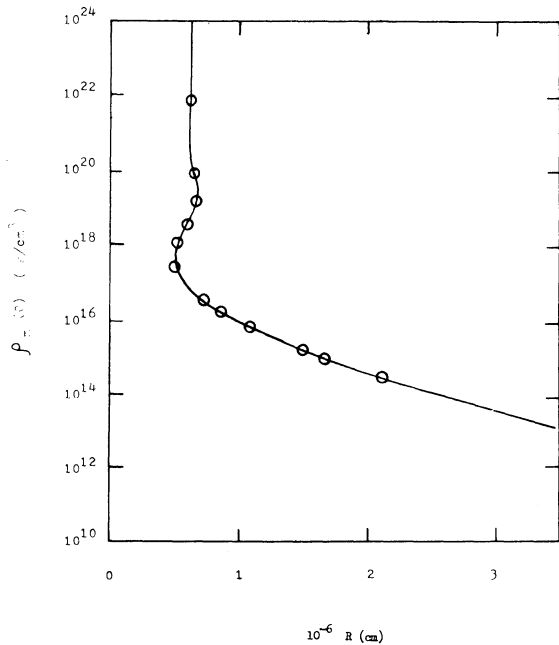


FIG. 2. Star radius as a function of central density. Circles represent calculation without charge effects.

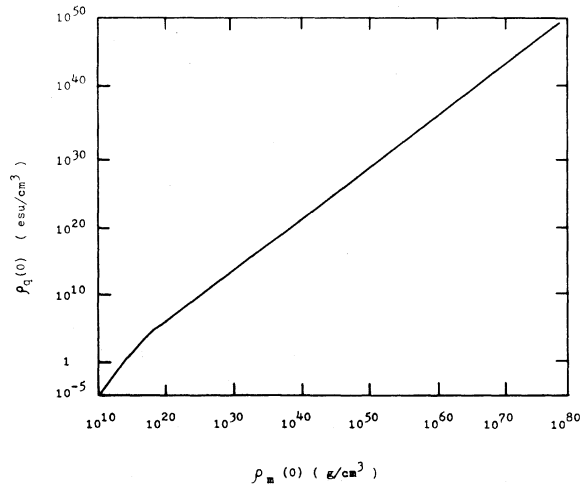


FIG. 4. Central charge density as a function of central energy density.

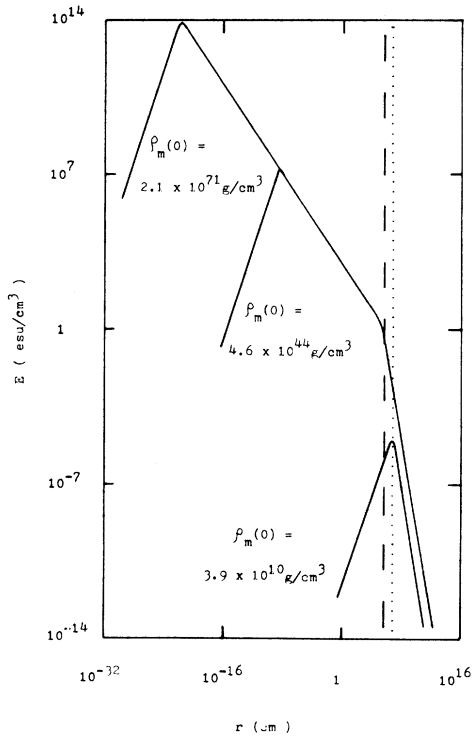


FIG. 5. Electric field as a function of radius for various central densities. The dotted line is the boundary radius for the smallest central density, the dashed line is the same for the largest central density.

The total charge of the white dwarf star is, however, uniformly about a factor 10 larger than that of the neutron star. The maximum charge for the neutron star is about 12×10^{10} esu (about 50 C), and is about 1.2×10^{12} esu for the white dwarf. This maximum is a very small charge considering the mass involved. The charge density, $n_p - n_e$, inside the star has been verified to be extremely small compared to the number density, $\sum n_i$. This small charge density is what is needed

to maintain equilibrium.

In an effort to get some feeling for the possible charge effects on this model the charge density at the origin was plotted as a function of the central density in Fig. 4, and the electric field was plotted in Fig. 5 as a function of radius for several central densities. The central densities were extended to unphysically large values to see how the equations behave asymptotically.

Figure 4 shows that the central charge density increases directly with increasing central mass density, and increases linearly with a slope of $\frac{3}{4}$ beyond a central density of about 10^{23} g/cm³.

Figure 5 shows the general behavior that the electric field increases from the origin, reaches a maximum somewhere close to but below the radius, then falls off until the radius is reached. Then it varies as $1/r^2$ into free space. As the central density gets larger the maximum of the electric field moves toward the center of the star.

The lower density (physically reasonable) neutron stars are characterized by the 4×10^{10} g/cm³ central density plot in Fig. 5, the dotted line indicating the radius of the star. The asymptotic limit is approximated by the 2×10^{71} g/cm³ central density plot in Fig. 5, the radius indicated by the dashed line. In the latter case the field variation from its maximum out to the radius goes as $r^{-(1/2)}$.

It has been suggested¹⁰ that electron-positron pair production should occur for electric fields greater than about 3×10^{13} esu/cm². Such fields can in fact be reached very close to the origin in the models calculated near the asymptotic limit, "very close" meaning at distances of the order of 10^{-22} cm from the origin. Thus the densities and distances required take us out of the physically meaningful regions. Nevertheless, extrapolation of the model to high central densities does suggest that charge effects will be pronounced in regions of high mass density.

¹E. Olson and M. Bailyn, Phys. Rev. D **12**, 3030 (1975).

²See, for example, S. Weinberg, *Gravitation and Cosmology* (Wiley, N.Y., 1972), pp. 300 and 301.

³E. Olson and M. Bailyn, Phys. Rev. D **13**, 2204 (1976).

⁴H. Callen, *Thermodynamics* (Wiley, N.Y., 1960). See Eq. (3.6).

⁵L. Landau and E. Lifshitz, *Statistical Physics* (Addison-Wesley, Reading, Mass., 1958). The equation above (25.2) gives the form of the electric field effect, and p. 76 contains the $(g_{00})^{1/2}$ factor. Combining the two gives the conservable energy, top of p. 76, as $E_0 = E(-g_{00})^{1/2} + Q \varphi_0$, where Q is the charge of the "small part of the body." One then proceeds to Eq. (27.3) which will read $\mu(-g_{00})^{1/2} + q \varphi_0 = \text{const}$, where q is dQ/dN , i.e., is the charge per particle.

⁶It has been suggested that the validity of Eqs. (2.8) and (2.9) for each species of a multifluid can be established independent of the STOV equations by essentially the same procedure that was used for the one-component case in Ref. 6. But local interactions between species seem to be neglected in such a derivation.

⁷O. Klein, Rev. Mod. Phys. **21**, 531 (1949).

⁸The argument in R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford Univ. Press, England, 1934), Secs. 126, 127 may be adapted, with inclusion of an electric field, for this purpose.

⁹The calculation can be found in the thesis by E. Olson (Northwestern University, 1977) (unpublished), or on the Xerox Microfilm.

¹⁰J. Bekenstein, Phys. Rev. D **4**, 2187 (1971).

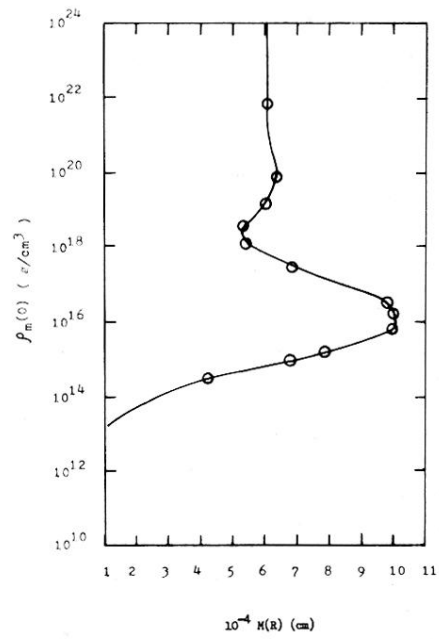


FIG. 1. Star mass as a function of central density. Circles represent calculation without charge effects.

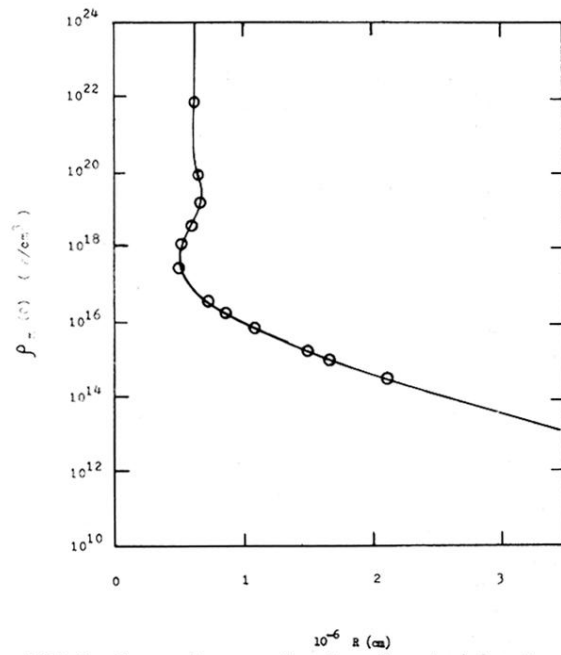


FIG. 2. Star radius as a function of central density. Circles represent calculation without charge effects.

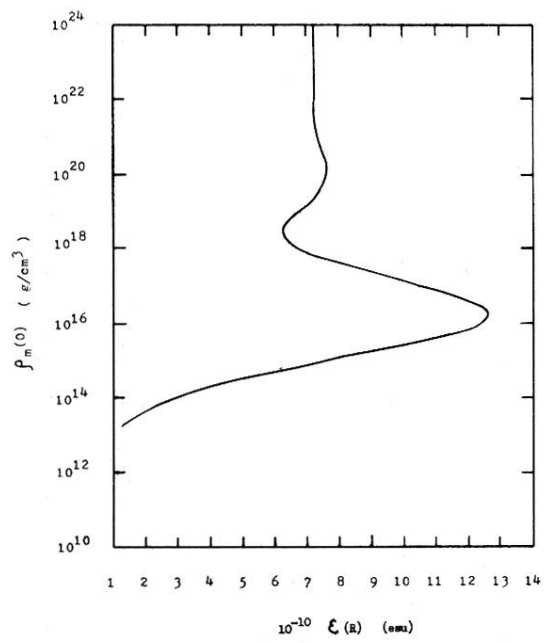


FIG. 3. Charge integral as a function of central density.

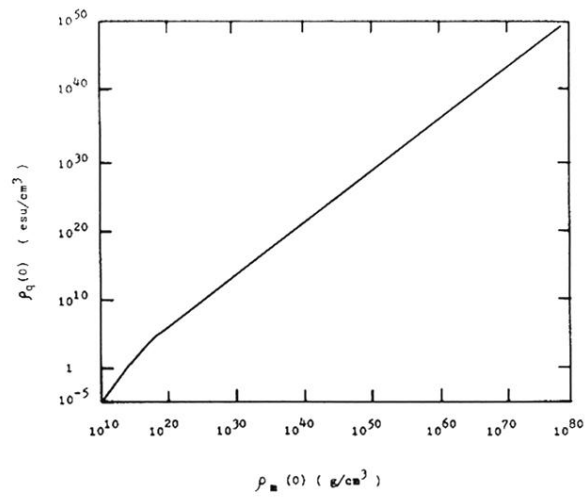


FIG. 4. Central charge density as a function of central energy density.

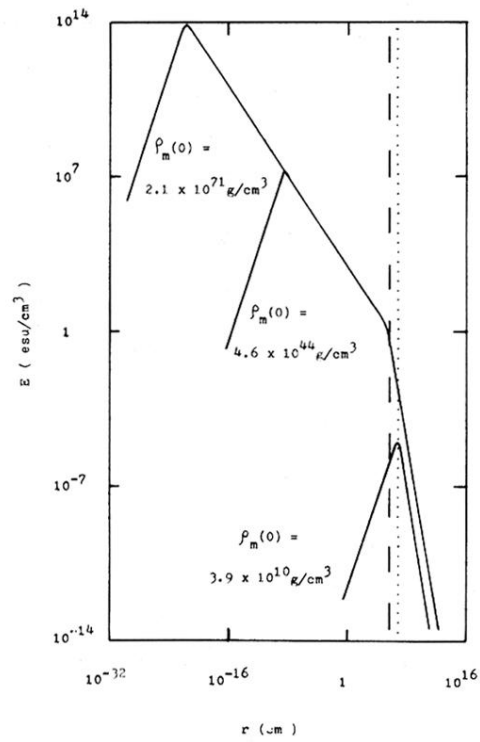


FIG. 5. Electric field as a function of radius for various central densities. The dotted line is the boundary radius for the smallest central density, the dashed line is the same for the largest central density.