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Interference between transition and Čerenkov radiation*

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General explicit expressions for the spectral intensity of the radiation emitted by a charge crossing the interface between two dielectric media have been derived by a source-theoretical method: This direct approach obviates the angular integrations. Below the Čerenkov threshold the results yield a compact formula for the transition radiation spectrum which is valid for *all* velocities ($\leq c$) and in *all* spectral ranges where dielectric functions can be defined. Above the Čerenkov threshold, new contributions appear in the spectrum: These may be interpreted as interference terms between transition and Čerenkov radiation. The nature of this interference and its experimental implications are considered in some detail.

I. INTRODUCTION

In many cases of practical interest there is no possibility of confusing an interface effect such as transition radiation with a volume effect such as Čerenkov emission. However, it was emphasized many years ago by Frank[1] that, for very-high-energy particles, transition radiation is not localized at the dielectric interface, but rather occurs over a "formation length" l_a which can assume macroscopic values. Specifically for a medium with index of refraction $n_a(\omega)$, the formation length corresponding to the emission of transition radiation of wavelength $\lambdabar$ is given by[2]

$$l_a \sim \frac{\lambdabar}{|1-[n_a v/c]^2|}, \tag{1.1a}$$

where v is the speed of the charge. In the special case of an ultrarelativistic electron ($E \gg mc^2$) incident on a vacuum-dielectric surface, the formation length required for the radiation of an optical photon may be enormous, e.g., for $\lambdabar \sim 10^{-5}$ cm and $n_a = 1$, (1.1a) implies

$$l_{\text{vac}} \sim \lambdabar \left(\frac{E}{mc^2}\right)^2 \to 1 \text{ km for } E \simeq 100 \text{ GeV}. \tag{1.1b}$$

In accelerator applications this relativistic enhancement often appears in conjunction with large values of $\lambdabar$ ($\gtrsim 1$ cm) corresponding to the emission of microwaves. Under these circumstances transition radiation becomes so "delocalized" that it merges into diffraction radiation.[3-5]

High energies also lead to problems in identifying "pure" transition radiation on the dielectric side of an interface. Under these conditions the Čerenkov threshold criterion

$$n_a(\omega)v/c > 1 \tag{1.2}$$

will be satisfied for some range of frequencies and the corresponding emission, originating from the dielectric formation zone l_a, will actually be comprised of a mixture of Čerenkov and transition radiation.[6,7] It is easy to check that this mixing can give rise to nontrivial interference effects: The differential photon number spectrum for Čerenkov radiation by a charge e traversing the formation length l_a is given by

$$dN_c \sim \frac{\alpha}{[n_a(\omega)\beta]^2}\frac{d\omega}{\omega}, \tag{1.3}$$

where $\alpha = e^2/(4\pi\hbar c) \simeq \frac{1}{137}$, and $\beta = v/c$. The corresponding number of transition-radiation photons is given by [cf. (3.21a)]

$$dN_{\text{tr}} \sim \frac{\alpha}{\pi}\mathfrak{F}_{\text{tr}}(\epsilon_a(\omega),\beta)\frac{d\omega}{\omega}; \tag{1.4}$$

and clearly whenever the parity

$$(n\beta)^{-2} \sim \mathfrak{F}_{\text{tr}} \to O(1) \tag{1.5}$$

prevails for the auxiliary function $\mathfrak{F}_{\text{tr}}$ we expect competition between the two radiation mechanisms. The detailed calculations confirm the existence of these interference effects and also lead to simple closed-form expressions for the transition amplitude $\mathfrak{F}_{\text{tr}}$.

Previous discussions of the interference between Čerenkov and transition radiation have been hampered by technical difficulties stemming from the complicated space-time structure of the electromagnetic fields. For instance, if one treats these processes by classical boundary-value methods it is easy to identify transition radiation by expansion in terms of asymptotically spherical waves, and similarly the Čerenkov components can be distinguished by their cylindrical symmetry; however, the residual field terms which describe the interference effects are analytically intractable.[8] Analogous problems appear if the fields are represented by Fourier integrals: The general quadratures are too complicated, and simplifications have to be imposed with special devices such as saddle-point approximations.[9] These methods lead to elegant rederivations of known results such as the high-energy behavior of transition radiation,[10] but they do not appear to be sufficiently flexible to handle interference effects or allied phenomena such as subthreshold Čerenkov radiation.[11]

In contrast, computations utilizing source theory have already proven their value in describing synergic radiation phenomena arising from a combination of synchrotron and Čerenkov processes.[7] In the present paper we show that these techniques can easily be extended to include transition radiation as well as interference effects between Čerenkov and transition radiation. Since the angular integrals are bypassed in this approach we obtain explicit results for the spectral intensities. As a further consequence, special cases corresponding to nonrelativistic and relativistic limits, subthreshold and superthreshold Čerenkov effects, and various dielectric interface combinations can all be considered within a unified framework. In analogy with results obtained previously for synchrotron-Čerenkov radiation,[12] the interference between Čerenkov and transition radiation can be utilized in "separated function" counters where the overall radiation rates and the energy sensitive thresholds are controlled by distinct combinations of dielectric functions.

The general source theory formulation of the problem is given in Sec. II. The Čerenkov and transition-radiation spectra are derived in Secs. III A and III B. Various limiting cases of transition radiation are considered in Sec. III C. The interference effects and their experimental implications are discussed in Sec. IV.

II. GENERAL EXPRESSION OF ČERENKOV-TRANSITION RADIATION

A. General Formalism

It is convenient to begin by recalling some essential elements of source theory. The quantity of basic interest is the vacuum persistence amplitude[7,13]

$$\langle 0_+|0_-\rangle^J = \exp\left(\frac{i}{\hbar} W\right) \tag{2.1}$$

which corresponds to the probability amplitude that during the action of an electromagnetic current, or source J^μ, the vacuum remains undisturbed. The action W which appears in (2.1) is given by

$$W = \frac{1}{2c^2}\int (dx)(dx')\, J^\mu(x) D_+(x,x')_{\mu\nu} J^\nu(x')\,, \tag{2.2}$$

where $D_+(x,x')_{\mu\nu}$ denotes the photon propagator from x' to x. The vector potential $A_\mu(x)$ is defined in terms of the response of the system to a small perturbation of the sources, i.e.,

$$\delta W = \frac{1}{c^2}\int (dx)\, \delta J^\mu(x) A_\mu(x)\,; \tag{2.3}$$

and this implies

$$A_\mu(x) = \int (dx') D_+(x,x')_{\mu\nu} J^\nu(x')\,. \tag{2.4}$$

Consequently the action (2.2), can be rewritten in the form

$$W = \frac{1}{2c^2}\int (dx)\, J^\mu(x) A_\mu(x)\,. \tag{2.5}$$

Finally, the vacuum persistence probability is given by

$$|\langle 0_+|0_-\rangle^J|^2 = \exp[-(2/\hbar)\mathrm{Im}\, W]\,. \tag{2.6}$$

from which we infer that the total probability for photon emission is simply $(2/\hbar)\mathrm{Im}W$. The energy radiated per unit frequency interval, or intensity spectrum $I(\omega)$, is then related to the action by

$$2\,\mathrm{Im}\, W = \int_0^\infty \frac{d\omega}{\omega} I(\omega)\,. \tag{2.7}$$

As emphasized in the Introduction this approach has the advantage of detouring the angular integrations and allows us to compute the energy spectrum directly from the imaginary part of the relevant components of the action (2.5).

B. Calculations

The simplest idealization of the physical processes leading to Čerenkov-transition radiation corresponds to a charge e moving with constant velocity $\vec{v}$ normal to the plane interface of two semi-infinite homogeneous media characterized by the scalar dielectric functions $\epsilon_a(\omega)$, $a=1,2$. We shall neglect absorption and *eo ipso* all conductive mechanisms involving allied effects such as diffraction radiation; these assumptions also imply[14] $\mathrm{Im}\,\epsilon_a(\omega)=0$. It is natural to choose a coordinate system with the interface located at the $z=0$ plane; the velocity of the charge is then parallel to the z axis. Finally, if we ignore any spatial structure of the source [see, however, Sec. IV (4.10a) and (4.10b)], the current for a point charge can be represented in the simple form[15]

$$J^\mu(\vec{r},t) = ec\beta^\mu \delta(x)\delta(y)\delta(z-vt)\,, \tag{2.8a}$$

where

$$\beta^\mu = (1,0,0,\beta)\,, \quad \beta = v/c\,. \tag{2.8b}$$

The explicit construction of the action (2.5) then depends on the solution of Maxwell's equations with appropriate boundary conditions for the vector potential A_μ. In rationalized cgs units, the macroscopic Maxwell equations for a material medium are

$$\vec{\nabla}\cdot\vec{D}(\vec{r},\omega)=\rho(\vec{r},\omega)\,, \tag{2.9a}$$

$$\vec{\nabla}\times\vec{H}(\vec{r},\omega)=(1/c)\,\vec{J}(\vec{r},\omega)-(i\omega/c)\vec{D}(\vec{r},\omega)\,, \tag{2.9b}$$

$$\vec{\nabla}\cdot\vec{B}(\vec{r},\omega)=0\,, \tag{2.9c}$$

$$\vec{\nabla}\times\vec{E}(\vec{r},\omega)=(i\omega/c)\vec{B}(\vec{r},\omega)\,, \tag{2.9d}$$

where we have introduced the temporal Fourier transform

$$f(\vec{r},t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}f(\vec{r},\omega)e^{-i\omega t}. \tag{2.10}$$

For simplicity (and practical relevance), we ignore the magnetic permeabilities so that $\vec{H}=\vec{B}$. The dielectric effects enter via the constitutive relations[14]

$$\vec{D}(\vec{r},\omega)=\epsilon(z,\omega)\vec{E}(\vec{r},\omega)\,, \tag{2.11a}$$

where, for the single interface case considered here,

$$\epsilon(z,\omega)=\begin{cases}\epsilon_1(\omega) & \text{for } z<0\,,\\ \epsilon_2(\omega) & \text{for } z>0\,.\end{cases} \tag{2.11b}$$

Equations (2.9c) and (2.9d) can, of course, be identically satisfied by introducing the potentials $\vec{A}$ and ϕ (= A^0):

$$\vec{B}(\vec{r},\omega)=\vec{\nabla}\times\vec{A}(\vec{r},\omega)\,, \tag{2.12a}$$

$$\vec{E}(\vec{r},\omega)=(i\omega/c)\,\vec{A}(\vec{r},\omega)-\vec{\nabla}\phi(\vec{r},\omega)\,. \tag{2.12b}$$

The gauge freedom then allows us to choose the Lorentz gauge which satisfies the condition

$$\vec{\nabla}\cdot\vec{A}_a(\vec{r},\omega)-(i\omega/c)\epsilon_a(\omega)\phi_a(\vec{r},\omega)=0\,,\quad z\neq 0\,, \tag{2.13}$$

where the indices $a=1,2$ correspond to the half-spaces $z<0$ or $z>0$ in accordance with (2.11b). Substituting (2.12a) and (2.12b) into (2.9a) and (2.9b) and taking into account the guage constraint (2.13) we obtain two equations for the vector potentials $A_a^\mu(\vec{r},\omega)$:

$$[\nabla^2+(\omega^2/c^2)\epsilon_a]A_a^\mu(\vec{r},\omega)=-\frac{1}{c}\{g^{\mu\nu}+[(\epsilon_a-1)/\epsilon_a]\eta^\mu\eta^\nu\}J_\nu(\vec{r},\omega)\,,\quad a=1,2\,, \tag{2.14a}$$

where

$$J^\nu=(c\rho,\vec{J})\,,\quad \eta^\mu=(1,\vec{0})\,, \tag{2.14b}$$

and $g^{\mu\nu}$ denotes the metric tensor

$$g^{00}=-1\,,\quad g^{0k}=0\,,\quad g^{kl}=\delta_{kl}\,. \tag{2.14c}$$

It is now convenient to incorporate the cylindrical symmetry of the problem by carrying out Fourier transforms with respect to the x and y coordinates, i.e., the $\vec{r}_\perp$ components:

$$A_a^\mu(\vec{r},\omega)=\int\frac{(d\vec{k}_\perp)}{(2\pi)^2}A_a^\mu(\omega,\vec{k}_\perp,z)e^{i(\vec{k}\cdot\vec{r})_\perp}\,. \tag{2.15}$$

If we combine this representation with (2.14a) and utilize the corresponding form of the current (2.8a) and (2.8b), we finally obtain the basic equation for the Fourier components of the vector potential, viz.,

$$\left(\frac{d^2}{dz^2}+\frac{\omega^2}{c^2}\epsilon_a-k_\perp^2\right)A_a^\mu(\omega,\vec{k}_\perp,z)=-\frac{e}{v}\left(\beta^\mu-\frac{\epsilon_a-1}{\epsilon_a}\eta^\mu\right)e^{(i\omega/v)z}\,,\quad a=1,2\,. \tag{2.16}$$

It is then a straightforward matter to verify that

$$\tilde{A}_a^\mu(\omega,\vec{k}_\perp,z)=\frac{e}{v\xi_a}\left(\beta^\mu-\frac{\epsilon_a-1}{\epsilon_a}\eta^\mu\right)e^{ik_zz} \tag{2.17}$$

is a particular solution of (2.16). The new symbols which enter here are defined by

$$k^2=k_\perp^2+k_z^2\,,\quad k_z=\omega/v \tag{2.18a}$$

and

$$\xi_a=k^2-\frac{\omega^2}{c^2}\epsilon_a-i\delta\,,\quad a=1,2\,. \tag{2.18b}$$

Since ξ_a appears in the denominator of (2.17) it is obvious that the dielectric light cone condition $\xi_a=0$ could be troublesome in subsequent integrations; however, the addition of the complex increment $-i\delta(\delta\to 0+)$ is required by the quantum-mechanical formulation of the expressions[13] (2.1) and (2.2).

The general solution of (2.16) can be written in the form

$$A_1^\mu(\omega,\vec{k}_\perp,z)=a^\mu\frac{e}{v}e^{-i\kappa_1z}+\tilde{A}_1^\mu(\omega,\vec{k}_\perp,z)\,,\quad z<0 \tag{2.19a}$$

and

$$A_2^\mu(\omega,\vec{k}_\perp,z)=b^\mu\frac{e}{v}e^{i\kappa_2z}+\tilde{A}_2^\mu(\omega,\vec{k}_\perp,z)\,,\quad z>0\,, \tag{2.19b}$$

where

$$\kappa_a=\begin{cases}[(\omega^2/c^2)\epsilon_a-k_\perp^2]^{1/2}\,, & k_\perp^2<(\omega^2/c^2)\epsilon_a\,, \quad (2.20a)\\ i[k_\perp^2-(\omega^2/c^2)\epsilon_a]^{1/2}\,, & k_\perp^2>(\omega^2/c^2)\epsilon_a\,. \quad (2.20b)\end{cases}$$

The exponential factors $e^{\pm i\kappa_az}$ which appear in the inhomogeneous terms satisfy the physical condition that the A_a^μ describe outgoing waves in time (*Ausstrahlungs-Bedingung*[16]), and are consistent with the constraint that the A_a^μ remain finite in the limits $z \to \pm\infty$. The coefficients a^μ and b^μ are functions of ω and $k_\perp^2$: Physically they correspond to the Fourier components of the Röntgen current which is induced at the dielectric interface by the external current (2.8a).[17] These functions are to be determined by the boundary conditions that A^μ and D_z are continuous across the interface at $z=0$, as well as the Lorentz gauge condition (2.13). One can easily check that the continuity of $\vec{E}_\perp$ and B_z at $z=0$ does not yield any further constraints.

From the Lorentz gauge condition, we obtain

$$\vec{k}_\perp \cdot \vec{a}_\perp - \kappa_1 a_z - (\omega/c)\epsilon_1 a^0 = 0 \tag{2.21a}$$

and

$$\vec{k}_\perp \cdot \vec{b}_\perp + \kappa_2 b_z - (\omega/c)\epsilon_2 b^0 = 0 , \tag{2.21b}$$

where $a^\mu = (a^0, \vec{a})$, etc. The continuity of A_a^μ at the interface $z=0$ then implies

$$\vec{a}_\perp = \vec{b}_\perp , \tag{2.22a}$$

$$a^0 + (\xi_1\epsilon_1)^{-1} = b^0 + (\xi_2\epsilon_2)^{-1} , \tag{2.22b}$$

$$a_z + \beta/\xi_1 = b_z + \beta/\xi_2 ; \tag{2.22c}$$

and finally from the continuity of D_z, we infer the condition

$$\epsilon_1[(\omega/c)a_z + \kappa_1 a^0] - (k_z/\xi_1)(1-\beta^2\epsilon_1) = \epsilon_2[(\omega/c)b_z - \kappa_2 b^0] - (k_z/\xi_2)(1-\beta^2\epsilon_2) . \tag{2.23}$$

Equations (2.21a), (2.21b), and (2.22a) may be combined in the form

$$\kappa_2 b_z + \kappa_1 a_z = (\omega/c)(b^0\epsilon_2 - a^0\epsilon_1) ; \tag{2.24}$$

and therefore we are left with four equations, i.e., (2.22b), (2.22c), (2.23), and (2.24), to determine the four coefficients a^0, a_z, b^0, and b_z. Since the basic expression for the action is proportional to $J^\mu A_\mu$, (2.5), and in the present instance $J^\mu \sim \beta^\mu \to (1,0,0,\beta)$ [cf. (2.8b)], it is obvious that the perpendicular components $\vec{a}_\perp$ and $\vec{b}_\perp$ do not enter the calculation (see below). The solutions for the relevant coefficients then are

$$a^0 = -\kappa_1\Lambda_a , \quad a_z = (\omega/c)\epsilon_1\Lambda_a , \tag{2.25a}$$

and

$$b^0 = \kappa_2\Lambda_b , \quad b_z = (\omega/c)\epsilon_2\Lambda_b , \tag{2.25b}$$

where the Λ factors are given by

$$\Lambda_a = \frac{1}{\kappa_1\epsilon_2 + \kappa_2\epsilon_1}\left[\frac{1}{\xi_1}\left(\frac{\epsilon_2}{\epsilon_1} - \frac{v}{\omega}\kappa_2\right) - \frac{1}{\xi_2}\left(1 - \frac{v}{\omega}\kappa_2\right)\right] \tag{2.26a}$$

and

$$\Lambda_b = \frac{-1}{\kappa_1\epsilon_2 + \kappa_2\epsilon_1}\left[\frac{1}{\xi_2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{v}{\omega}\kappa_1\right) - \frac{1}{\xi_1}\left(1 + \frac{v}{\omega}\kappa_1\right)\right] . \tag{2.26b}$$

All of these results can be summarized by noting that the basic expressions for the vector potential (2.19a) and (2.19b) can be rewritten as follows:

(i) For $z<0$, $\epsilon \to \epsilon_1(\omega)$, we have

$$A_1^0(\omega, \vec{k}_\perp, z) = a^0 \frac{e}{v} e^{-i\kappa_1 z} + \tilde{A}_1^0(\omega, \vec{k}_\perp, z) , \tag{2.27a}$$

$$A_1^3(\omega, \vec{k}_\perp, z) = a_z \frac{e}{v} e^{-i\kappa_1 z} + \tilde{A}_1^3(\omega, \vec{k}_\perp, z) , \tag{2.27b}$$

where a^0 and a_z are given by (2.25a) and (2.26a), κ_1 by (2.20a) and (2.20b), and

$$\tilde{A}_1^0(\omega, \vec{k}_\perp, z) = \frac{e}{v\epsilon_1}\frac{1}{\xi_1} e^{ik_z z} , \tag{2.27c}$$

$$\tilde{A}_1^3(\omega, \vec{k}_\perp, z) = \frac{e}{c}\frac{1}{\xi_1} e^{ik_z z} \tag{2.27d}$$

with ξ_1 defined in (2.18b).

(ii) Similarly for $z>0$, $\epsilon \to \epsilon_2(\omega)$, we obtain

$$A_2^0(\omega, \vec{k}_\perp, z) = b^0 \frac{e}{v} e^{i\kappa_2 z} + \tilde{A}_2^0(\omega, \vec{k}_\perp, z) , \tag{2.28a}$$

$$A_2^3(\omega, \vec{k}_\perp, z) = b_z \frac{e}{v} e^{i\kappa_2 z} + \tilde{A}_2^3(\omega, \vec{k}_\perp, z) , \tag{2.28b}$$

where b^0 and b_z are given by (2.25b) and (2.26b), κ_2 by (2.20a) and (2.20b), and

$$\tilde{A}_2^0(\omega, \vec{k}_\perp, z) = \frac{e}{v\epsilon_2}\frac{1}{\xi_2} e^{ik_z z} , \tag{2.28c}$$

$$\tilde{A}_2^3(\omega, \vec{k}_\perp, z) = \frac{e}{c}\frac{1}{\xi_2} e^{ik_z z} \tag{2.28d}$$

with ξ_2 defined in (2.18b).

The space-time representation of A^μ can then be constructed by combining the Fourier transforms (2.10) and (2.15):

$$A^\mu(\vec{r}, t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{(d\vec{k}_\perp)}{(2\pi)^2} e^{i(\vec{k}\cdot\vec{r})_\perp} A_a^\mu(\omega, \vec{k}_\perp, z) . \tag{2.29}$$

Finally, substituting (2.8a) and (2.29) into (2.5), we obtain an explicit form for the action, viz.,

$$W = \frac{e}{2}\int dt\,dx\,dy\,dz\ \delta(x)\delta(y)\delta(z-vt)\beta_\mu \times \int \frac{(d\vec{k}_\perp)}{(2\pi)^2}\frac{d\omega}{2\pi} \exp\left[-i\omega t + i(\vec{k}\cdot\vec{r})_\perp\right] A_a^\mu(\omega, \vec{k}_\perp, z) . \tag{2.30}$$

This can be more succinctly expressed as

$$W = \frac{e}{2v}\int \frac{(d\vec{k}_\perp)}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty} dz\, e^{-ik_z z}\beta_\mu A_a^\mu(\omega, \vec{k}_\perp, z) . \tag{2.31}$$

The spectral intensity of Čerenkov-transition radiation can then be computed from the imaginary part of W, by utilizing (2.7).

III. EXPLICIT RESULTS

A. Čerenkov Radiation

It is characteristic of source theory that the potentials A^μ enter in the action linearly when it is expressed in the form (2.5). It is therefore natural to split the action into two parts

$$W = W_C(\tilde{A}^\mu) + W_T(\hat{A}^\mu) \tag{3.1}$$

corresponding to the division of A^μ into homogeneous ($\tilde{A}^\mu$) and inhomogeneous ($\hat{A}^\mu$) components. For Čerenkov-transition radiation, this division can be read off from (2.19a) and (2.19b) with $\hat{A}_1^\mu = a^\mu(e/v)e^{-i\kappa_1 z}$, etc. Since the homogeneous components $\tilde{A}^\mu$ are independent of the Röntgen currents at the dielectric interface, it is plausible to associate them with Čerenkov emission. Indeed we will first demonstrate that the spectral intensity derived from $W_C(\tilde{A}_\mu)$ coincides with the standard Čerenkov spectrum. Specifically from (2.27a), (2.27b), (2.28a), (2.28b), (2.31), and (2.32), we obtain

$$\begin{aligned} W_C(\tilde{A}_\mu) &= \frac{e}{2v}\int\frac{(d\vec{k}_\perp)}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}dz\,e^{-ik_z z}\left[-\tilde{A}_a^0(\omega,\vec{k}_\perp,z)+\beta\tilde{A}_a^3(\omega,\vec{k}_\perp,z)\right] \\ &= \frac{e}{2v}\lim_{L\to\infty}\int\frac{(d\vec{k}_\perp)}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\int_{-L}^{0}dz\,e^{-ik_z z}\left[-\tilde{A}_1^0(\omega,\vec{k}_\perp,z)+\beta\tilde{A}_1^3(\omega,\vec{k}_\perp,z)\right]\right. \\ &\qquad\left.+\int_0^L dz\,e^{-ik_z z}\left[-\tilde{A}_2^0(\omega,\vec{k}_\perp,z)+\beta\tilde{A}_2^3(\omega,\vec{k}_\perp,z)\right]\right). \end{aligned} \tag{3.2}$$

In the second form, we have recognized that the infinite thickness of the dielectric medium, as reflected in the limits of the z integration, is an idealization of the more realistic physical situation in which the dielectric medium has finite thickness, large compared to the wavelength of the radiation. It is easy to check that this expression diverges linearly with L. This behavior is consistent with the identification of ImW with the total energy dissipation — which of course is infinite for an infinitely long Čerenkov radiator. For a finite radiator with $-L \le z \le +L$, (3.2) becomes

$$W_C = \frac{e^2}{2c^2}\int\frac{(d\vec{k}_\perp)}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\int_{-L}^{0}dz\,\xi_1^{-1}[1-(\beta^2\epsilon_1)^{-1}] + \int_0^L dz\,\xi_2^{-1}[1-(\beta^2\epsilon_2)^{-1}]\right), \tag{3.3}$$

where we have inserted the explicit forms of $\tilde{A}_a^\mu$ from (2.27c), (2.27d), (2.28c) and (2.28d). Taking into account that $\epsilon_a(\omega)$ is an even function of ω for a nonabsorptive medium,[14] and making use of (2.18b), we obtain

$$W_C = \frac{e^2 L}{2\pi v^2}\int_0^\infty d\omega\int\frac{(d\vec{k}_\perp)}{(2\pi)^2}\left(\frac{\beta^2-\epsilon_1^{-1}}{k^2-(\omega^2/c^2)\epsilon_1 - i\delta} + \frac{\beta^2-\epsilon_2^{-1}}{k^2-(\omega^2/c^2)\epsilon_2 - i\delta}\right). \tag{3.4}$$

The corresponding spectrum can then easily be identified by comparing (2.7) and (3.4):

$$I_C(\omega) = \frac{e^2\omega L}{v^2}\times\int\frac{(d\vec{k}_\perp)}{(2\pi)^2}\left[(\beta^2-\epsilon_1^{-1})\delta\left(k_\perp^2-\frac{\omega^2}{v^2}(\beta^2\epsilon_1-1)\right) + (\beta^2-\epsilon_2^{-1})\delta\left(k_\perp^2-\frac{\omega^2}{v^2}(\beta^2\epsilon_2-1)\right)\right], \tag{3.5}$$

where the δ functions appear in virtue of

$$\lim_{\delta\to 0+}\frac{1}{x-i\delta} = \mathrm{P}(1/x) + i\pi\delta(x). \tag{3.6}$$

The remaining integrations are trivial and lead to the well-known Čerenkov spectrum

$$I_C(\omega) = \frac{e^2\omega L}{4\pi c^2}\left[\left(1-\frac{1}{\beta^2 n_1^2(\omega)}\right)\eta\left[\beta^2 n_1^2(\omega)-1\right] + \left(1-\frac{1}{\beta^2 n_2^2(\omega)}\right)\eta\left[\beta^2 n_2^2(\omega)-1\right]\right], \tag{3.7a}$$

where we have reintroduced the indices of refraction,

$$\epsilon_a(\omega) = n_a^2(\omega) \equiv [1+\Delta n_a(\omega)]^2, \tag{3.7b}$$

and indicated the Čerenkov thresholds in both media with the help of the Heaviside function

$$\eta(x) = \tfrac{1}{2}[1+\mathrm{sgn}(x)] = \begin{cases} +1, & x>0, \\ 0, & x<0. \end{cases} \tag{3.7c}$$

For later reference it is also convenient to note that the photon number spectrum is given by [cf. (1.3)]

$$dN_C(\omega)=\alpha\frac{Z^2L}{\lambda\!\!\!^{-}_c}\left[\left(1-\frac{1}{\beta^2 n_1{}^2(\omega)}\right)\eta[\beta^2 n_1{}^2(\omega)-1]+\left(1-\frac{1}{\beta^2 n_2{}^2(\omega)}\right)\eta[\beta^2 n_2{}^2(\omega)-1]\right]\times\frac{d(\hbar\omega)}{mc^2}, \tag{3.8}$$

where Ze denotes the total charge and the energy and length scales are fixed by $\lambda\!\!\!^{-}_c mc^2\equiv(\hbar/mc)mc^2$.

B. Transition Radiation

There are a wide variety of experimental conditions corresponding either to low charge velocities $\beta\ll 1$, and/or indices of refraction near unity, i.e., tenuous media or dispersion in the x-ray portion of the spectrum $[\Delta n(\omega)\lesssim 0]$, where the inequalities

$$\beta^2 n_a{}^2(\omega)<1\,,\quad a=1,2 \tag{3.9}$$

are satisfied and the Čerenkov terms (3.7a) vanish identically. Under these circumstances we are justified in characterizing the contributions of the inhomogeneous action term, $\mathrm{Im}W(\hat{A}_\mu)$, as "pure" transition radiation. Formally this quantity is given by an expression analogous to (3.2) except that now $\tilde{A}^\mu$ is replaced by $\hat{A}^\mu$ everywhere; the specific forms of the inhomogeneous terms can then be read off from (2.27a), (2.27b), (2.28a), and (2.28b). The resulting action describing pure transition radiation is

$$W_{\mathrm{tr}}=\frac{e^2}{2v^2}\lim_{L\to\infty}\int\frac{(d\vec{k}_\perp)}{(2\pi)^2}\times\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\int_{-L}^{0}dz\,e^{-i(\kappa_1+k_z)z}(-a^0+\beta a_z)+\int_0^L dz\,e^{-i(k_z-\kappa_2)z}(-b^0+\beta b_z)\right). \tag{3.10}$$

In this case one can verify that the z integrations are unambiguous [cf. (2.18b) and the Appendix] and lead to[18]

$$W_{\mathrm{tr}}=\frac{ie^2}{2v^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{(d\vec{k}_\perp)}{(2\pi)^2}\left(\frac{-a^0+\beta a_z}{\kappa_1+k_z}+\frac{-b^0+\beta b_z}{\kappa_2-k_z}\right). \tag{3.11}$$

As before, the spectral intensity can be determined by comparing (2.7) and (3.11): It is convenient to write this in the form

$$I_{\mathrm{tr}}(\omega)=\frac{e^2}{4\pi^2 c}\mathrm{Re}\,\mathcal{J}(\omega)\,, \tag{3.12a}$$

where [cf. (2.18a)]

$$\mathcal{J}(\omega)=\frac{\omega}{2\beta^2 c}\int_0^\infty dk_\perp{}^2\left[\left(\frac{-a^0+\beta a_z}{\kappa_1+k_z}+\frac{-b^0+\beta b_z}{\kappa_2-k_z}\right)+(\omega\to-\omega)\right]. \tag{3.12b}$$

Substituting (2.25a) through (2.26b), and rationalizing the denominators, we obtain a sum of quadratures which can be grouped as follows:

$$\mathcal{J}(\omega)=\mathcal{J}_1(\omega)+\mathcal{J}_2(\omega)\,, \tag{3.13a}$$

where

$$\mathcal{J}_1(\omega)=\frac{\omega}{\beta^2 c}\int_0^\infty\frac{dk_\perp{}^2}{\xi_1(\kappa_1\epsilon_2+\kappa_2\epsilon_1)}\times\left(\frac{k_\perp{}^2(\epsilon_2/\epsilon_1)-\kappa_1\kappa_2(1-\beta^2\epsilon_1)}{\xi_1}+\frac{-k_\perp{}^2+\kappa_1\kappa_2(1-\beta^2\epsilon_1)}{\xi_2}\right) \tag{3.13b}$$

and

$$\mathcal{J}_2(\omega)=\frac{\omega}{\beta^2 c}\int_0^\infty\frac{dk_\perp{}^2}{\xi_2(\kappa_1\epsilon_2+\kappa_2\epsilon_1)}\times\left(\frac{-k_\perp{}^2+\kappa_1\kappa_2(1-\beta^2\epsilon_2)}{\xi_1}+\frac{k_\perp{}^2(\epsilon_1/\epsilon_2)-\kappa_1\kappa_2(1-\beta^2\epsilon_2)}{\xi_2}\right). \tag{3.13c}$$

It is easy to check that $\mathcal{J}_2$ can be obtained from $\mathcal{J}_1$ by the interchange $\epsilon_1\leftrightarrow\epsilon_2$; moreover, both terms are invariant under the "velocity reversal" $\beta\to-\beta$. These symmetries are slightly *stronger* than required by the obvious physical condition that the total energy dissipation in transition radiation remain invariant under the dual interchange $\epsilon_1\leftrightarrow\epsilon_2$, $\beta\leftrightarrow-\beta$. These features of course do not clash with the asymmetry of the angular distribution of the radiation which appears at relativistic energies.

Although the quadratures indicated in (3.13b) and (3.13c) are elementary, there are a great variety of special cases. Without loss of generality, all the interesting points of the remaining computation can be illustrated by choosing $\epsilon_2>\epsilon_1$, and specializing (3.9) to

$$1-\beta^2\epsilon_1(\omega)>1-\beta^2\epsilon_2(\omega)>0\,. \tag{3.14}$$

Since we are only interested in the real parts of (3.13b) and (3.13c), and since κ_1 and κ_2 can both become imaginary for sufficiently large values of $k_\perp{}^2$, we begin by decomposing the region of integration into three contiguous segments

$$\int_0^\infty dk_\perp{}^2=\left(\int_{\mathrm{I}}+\int_{\mathrm{II}}+\int_{\mathrm{III}}\right)dk_\perp{}^2\,, \tag{3.15a}$$

which are defined in terms of the following inequalities:

region I:

$$0 \leqslant k_\perp{}^2 \leqslant (\omega^2/c^2)\epsilon_1 , \tag{3.15b}$$

region II:

$$(\omega^2/c^2)\epsilon_1 \leqslant k_\perp{}^2 \leqslant (\omega^2/c^2)\epsilon_2 , \tag{3.15c}$$

region III:

$$(\omega^2/c^2)\epsilon_2 \leqslant k_\perp{}^2 \to \infty . \tag{3.15d}$$

In each region it is convenient to readjust the variable of integration by a distinct transformation [cf. (2.20a) and (2.20b)]:

region I:

$$\kappa_1/\kappa_2 = y , \tag{3.16a}$$

region II:

$$-i\kappa_1/\kappa_2 = y \to x^2 , \tag{3.16b}$$

region III:

$$|\kappa_1/\kappa_2| = y . \tag{3.16c}$$

After some elementary rearrangement, the integrals (3.13b) and (3.13c) can then be reduced to the form

$$\mathrm{Re}\mathscr{I}_1(\omega) = \beta^2 \int_{-\infty}^{a} dx \left(\frac{\epsilon_{21}}{1-x}\right)^{1/2} \frac{a(a+b)+x(1+ac)}{(a^2-x)(b+x)(1+cx)} -2\beta^2 \int_0^{\sqrt{a}} dy \left(\frac{\epsilon_{21}}{1-y^2}\right)^{1/2} \frac{y^2}{(a^2-y^2)(1+cy^2)} \left(\frac{a(1+ac)+(a-y^2)/a}{1+cy^2} + \frac{a(1+ac)+(a-y^2)}{b+y^2}\right) \tag{3.17a}$$

and

$$\mathrm{Re}\mathscr{I}_2(\omega) = \beta^2 \int_{-\infty}^{a} dx \left(\frac{\epsilon_{21}}{1-x}\right)^{1/2} \frac{1}{(b+x)(a^2-x)} \left(\frac{a(a-x)-x(a+b)}{1+cx} + \frac{a(a^2-x)-x(a^2+b)}{b+x}\right) + 2\beta^2 \int_0^{\sqrt{a}} dy \left(\frac{\epsilon_{21}}{1-y^2}\right)^{1/2} \frac{y^2}{(a^2-y^2)(b+y^2)} \frac{a(a+b)+(1+ac)y^2}{1+cy^2} . \tag{3.17b}$$

One can easily verify that the y integrals stem from region I and the x integrals from region II. The contributions from region III drop out because our initial restrictions (3.14) force them to be purely imaginary. The new auxiliary quantities that appear in (3.17a) and (3.17b) represent the following combinations of parameters:

$$\epsilon_{21} = \epsilon_2(\omega) - \epsilon_1(\omega) > 0 , \quad a = \epsilon_1(\omega)/\epsilon_2(\omega) , \quad b = -1 + \beta^2\epsilon_{21} , \quad c = -(1+\beta^2\epsilon_{21}) . \tag{3.18}$$

The remaining quadratures are elementary and lead to the following results:

$$\mathrm{Re}\mathscr{I}_1(\omega) = -\frac{\beta^2\epsilon_1\epsilon_2{}^2[\beta^2(\epsilon_1+\epsilon_2)-1]}{X_1X_2(\epsilon_1+\epsilon_2)^{1/2}} \mathcal{L}_1(\epsilon_1,\epsilon_2) - \epsilon_1{}^{-1/2} + (\epsilon_1{}^{-1} - \beta^2)\mathcal{L}_2(\epsilon_1) - \frac{2[\beta^2(\epsilon_1+\epsilon_2)-1]}{\beta^2\epsilon_{21}X_2} \mathcal{L}_3(\epsilon_1,\epsilon_2) + \frac{2(1+\beta^2\epsilon_{21})[\beta^2(\epsilon_1+\epsilon_2)-1]}{\beta^2\epsilon_{21}X_1} \mathcal{L}_2(\epsilon_1) - \frac{2(1-\beta^2\epsilon_1+\beta^4\epsilon_1\epsilon_2)}{\beta^2\epsilon_{21}X_1} \mathcal{L}_3(\epsilon_2,\epsilon_1) + \frac{2(1-\beta^2\epsilon_2+\beta^4\epsilon_1\epsilon_2)}{\beta^2\epsilon_{21}X_2} \mathcal{L}_2(\epsilon_2) , \tag{3.19}$$

where

$$X_a = \epsilon_1 + \epsilon_2 - \beta^2\epsilon_a{}^2 , \quad a = 1, 2 , \tag{3.20a}$$

$$\mathcal{L}_1(\epsilon_1,\epsilon_2) = 2\ln\left(\frac{\epsilon_2 + [\epsilon_2(\epsilon_1+\epsilon_2)]^{1/2}}{\epsilon_1 + [\epsilon_1(\epsilon_1+\epsilon_2)]^{1/2}}\right) , \tag{3.20b}$$

$$\mathcal{L}_2(\epsilon_a) = \frac{1}{2\beta} \ln\left(\frac{1+\beta\epsilon_a{}^{1/2}}{1-\beta\epsilon_a{}^{1/2}}\right) , \quad a = 1, 2 , \tag{3.20c}$$

$$\mathcal{L}_3(\epsilon_1,\epsilon_2) = \frac{(1-\beta^2\epsilon_{21})^{1/2}}{2\beta} \ln\left(\frac{(1-\beta^2\epsilon_{21})^{1/2} + \beta\epsilon_1{}^{1/2}}{(1-\beta^2\epsilon_{21})^{1/2} - \beta\epsilon_1{}^{1/2}}\right) . \tag{3.20d}$$

A similar expression can be derived for $\mathrm{Re}\mathscr{I}_2(\omega)$ and one can check *a posteriori* that this is equivalent to recalculating $\mathrm{Re}\mathscr{I}_1(\omega)$ with the interchange $\epsilon_1 \leftrightarrow \epsilon_2$. These symmetries are, of course, consistent with the invariance properties exhibited by (3.13b) and (3.13c) *et seq.*

The final result for the intensity spectrum of pure transition radiation then is

$$I_{tr}(\omega)=\frac{e^2}{4\pi^2 c}\mathfrak{F}_{tr}(\omega,\beta;\epsilon_1,\epsilon_2)\,, \tag{3.21a}$$

where[19]

$$\mathfrak{F}_{tr}(\omega,\beta;\epsilon_1,\epsilon_2)=F(\omega,\beta;\epsilon_1,\epsilon_2)+F(\omega,\beta;\epsilon_2,\epsilon_1) \tag{3.21b}$$

and

$$F(\omega,\beta;\epsilon_1,\epsilon_2)=-\epsilon_1^{-1/2}-\frac{\beta^2\epsilon_1\epsilon_2^2[\beta^2(\epsilon_1+\epsilon_2)-1]}{X_1X_2(\epsilon_1+\epsilon_2)^{1/2}}\mathfrak{L}_1(\epsilon_1,\epsilon_2)+\left(\frac{1-\beta^2\epsilon_2}{\epsilon_2}+\frac{2\epsilon_1\beta^2}{X_2}\right)\mathfrak{L}_2(\epsilon_2)$$
$$+\frac{2(1-\beta^2\epsilon_2)}{\beta^2\epsilon_{21}X_2}[(2-\beta^2\epsilon_2)\mathfrak{L}_2(\epsilon_2)+(2-\beta^2\epsilon_1)\mathfrak{L}_3(\epsilon_1,\epsilon_2)]\,. \tag{3.21c}$$

As indicated previously this result is constrained solely by the subthreshold Čerenkov conditions (3.9) and (3.14). In fact, (3.14) is not an essential restriction since (3.21a)–(3.21c) can easily be modified to allow for the replacement $\epsilon_2>\epsilon_1\to\epsilon_1>\epsilon_2$.

C. Discussion

1. Nonrelativistic Limit

It is interesting to consider transition radiation in the limit

$$\beta^2\ll 1\,. \tag{3.22}$$

Under these circumstances (3.21a)–(3.21c) can be simplified to yield a compact expression for the differential photon number spectrum, i.e.,

$$dN_{tr}\cong\frac{\alpha}{\pi}(\beta Z)^2F_1(\epsilon_1,\epsilon_2)\frac{d\omega}{\omega}\,, \tag{3.23a}$$

where Ze denotes the charge of the radiating object and[20]

$$\mathfrak{F}_{tr}\to F_1(\epsilon_1,\epsilon_2)=\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}(\epsilon_2^{1/2}-\epsilon_1^{1/2})\left[\frac{2}{3}+\frac{2(\epsilon_1\epsilon_2)^{1/2}}{\epsilon_1+\epsilon_2}+\frac{2\epsilon_1\epsilon_2}{(\epsilon_2^{1/2}-\epsilon_1^{1/2})(\epsilon_1+\epsilon_2)^{3/2}}\ln\left(\frac{\epsilon_2+[\epsilon_2(\epsilon_1+\epsilon_2)]^{1/2}}{\epsilon_1+[\epsilon_1(\epsilon_1+\epsilon_2)]^{1/2}}\right)\right]. \tag{3.23b}$$

This approximation can also be obtained directly from (3.17a) and (3.17b) by inserting the expansions corresponding to (3.22).

In the special case of nonrelativistic electrons, $Z\to 1$, and for crude numerical orientation, it is convenient to assume $\epsilon_2\gg\epsilon_1$, and $d\omega/\omega\sim 1$; these estimates imply $F_1\sim\frac{2}{3}n_2(\omega)$, or

$$dN_{tr}\sim\frac{2\alpha}{3\pi}n_2(\omega)\beta^2\frac{d\omega}{\omega}\to\frac{\beta^2}{100}\ \text{photons/electron}\,. \tag{3.24}$$

Experimentally this corresponds to the radiation first observed by Lilienfeld in 1919.[21,22] With present techniques intensities adequate for beam monitoring applications can be obtained for electron energies as low as 10–500 eV and currents in the range 10–50 μA.[23] The feasibility of these measurements of course hinges on the fact that transition radiation tends to predominate over optical (Coulomb) bremsstrahlung in thin foils.[24-26]

The analogy between transition radiation and bremsstrahlung suggests that the intensities are favorably enhanced for highly charged objects ($Z\gg 1$), and indeed this follows by inspection of (3.23a). For instance at the Bevalac one can currently obtain beams of fully stripped Fe ($Z=26$) nuclei with $\beta^2\sim\frac{8}{9}$. So if we stretch (3.22) just for numerical orientation, the optical photon yield per nucleus is expected to be of the order of

$$dN_{tr}\sim F_1(\epsilon_1,\epsilon_2)\frac{d\omega}{\omega}\sim O(1)\,. \tag{3.25}$$

Since the formation length criterion (1.1a) indicates that very thin ($\lesssim 0.1$ mil) Mylar sheets ought to be adequate radiators, and beam intensities of 10^3 Fe nuclei per pulse at the rate of 20 pulses per minute are available, there should be no practical difficulty in checking the validity of (3.23a) and (3.23b) and the exact formulas (3.21a)–(3.21c). Conditions are, of course, even more favorable for fully stripped uranium beams: In this case optical transition radiation should yield about 10 photons per nucleus per interface.

In this connection it is amusing to recall that Alvarez's interpretation[27] of the "monopole" event[28] of 1975 invoked a cosmic-ray object with $Z\approx 78$ and $\beta\sim 0.66$; one can easily verify that the transition radiation in this case is scaled by

$$\frac{\alpha}{\pi}(\beta Z)^2\sim 6\,. \tag{3.26}$$

Since there is agreement that this event occurred below the Čerenkov threshold (3.9), the "pure" transition radiation estimate (3.23a) and (3.23b) ought to be applicable. Depending on the precise values of (3.23b) for the interfaces bounding the "fast film" (cf. Fig. 1 of Ref. 28), some optical photon signals might have been recorded.

In principle the spectrum of transition radiation can extend far above the optical range. Even a "slow" nucleus such as $_{26}$Fe at $\beta \lesssim \frac{1}{2}$ has an energy of circa 10 GeV that could be radiated away. However, the specific nature of the radiation has to be considered on a case-by-case basis depending on the structure of the composite functions $F_1(\epsilon_1(\omega), \epsilon_2(\omega))$. Of course in the UV and soft x-ray regions of the spectrum absorptive corrections are important and our results would have to be modified.[25] At still higher photon energies (x-ray region), we have the general estimate [cf. (3.7b)]

$$\epsilon(\omega) \cong 1 + 2\Delta n(\omega) \lesssim 1\,, \qquad (3.27)$$

and the corresponding limit of F_1 can be computed in a model-independent way. Specifically for photon energies exceeding the characteristic binding or resonance energies of material media, the indices of refraction are essentially determined by the coherent Compton amplitude.[12] This implies

$$2\Delta n(\omega) = -\omega_p^2/\omega^2\,, \quad \text{for } \omega(\text{keV}) \gtrsim 0.054\tilde{Z}^2\,, \qquad (3.28a)$$

where the quantity

$$\omega_p(\text{eV}) = 28.82(\rho\tilde{Z}/A)^{1/2} \qquad (3.28b)$$

denotes the conventional plasma frequency: The parameters characterizing the medium then are the density $\rho(\text{g/cm}^3)$, the atomic charge $\tilde{Z}$, and atomic number A. In particular for a dielectric-vacuum interface the high-frequency limit of (3.23a) is

$$dN_{\text{tr}} \to \frac{2\alpha}{3\pi}(\beta Z)^2\left(\frac{\omega_p}{\omega}\right)^4\frac{d\omega}{\omega}\,. \qquad (3.29a)$$

The total number of high-frequency transition photons per (nonrelativistic) charge is therefore negligibly small, i.e.,

$$N_{\text{tr}}(\omega > .054\tilde{Z}^2) \sim 3\times 10^{-5}(\beta\rho Z/A\tilde{Z}^3)^2\,. \qquad (3.29b)$$

2. Relativistic Domain

These trends are drastically altered under relativistic conditions. However, the approach of v to c has to be restricted by the inequalities [cf. (3.9) and (3.14)]

$$2\Delta n_a(\omega) < 1-\beta^2 \to 0\,, \quad \Delta n_2(\omega) - \Delta n_1(\omega) > 0 \quad (3.30)$$

to insure that our results still correspond to "pure" transition radiation without any admixture of Čerenkov components. It is then easy to show that our general expressions (3.21a)–(3.21c) reduce to

$$dN_{\text{tr}} \cong \frac{\alpha}{\pi} Z^2 F_2(\beta;\epsilon_1,\epsilon_2)\frac{d\omega}{\omega}\,, \qquad (3.31a)$$

where [cf. (3.27)]

$$\mathfrak{F}_{\text{tr}} \to F_2(\beta;\epsilon_1,\epsilon_2) = \left(\frac{\Delta n_2(\omega)+\Delta n_1(\omega)}{\Delta n_2(\omega)-\Delta n_1(\omega)} - \frac{1-\beta^2}{\Delta n_2(\omega)-\Delta n_1(\omega)}\right) \times \ln\left(\frac{1-\beta^2-2\Delta n_2(\omega)}{1-\beta^2-2\Delta n_1(\omega)}\right) - 2\,. \qquad (3.31b)$$

If we specialize still further to a dielectric-vacuum interface with Δn_1 given by (3.28a) and (3.28b), we obtain the well-known Garibian limit[29]

$$F_2 \to \left[1+2\left(\frac{\omega}{\omega_0}\right)^2\right]\ln\left[1+\left(\frac{\omega_0}{\omega}\right)^2\right] - 2\,, \qquad (3.32a)$$

where ω_0 denotes the "boosted" plasma frequency[30]

$$\omega_0 = \gamma\omega_p\,, \quad \gamma = (1-\beta^2)^{-1/2}\,. \qquad (3.32b)$$

The corresponding total number of "hard" transition-radiation photons emitted per charge (Ze) per interface then is

$$N_{\text{tr}}(\omega > 0.054\tilde{Z}^2) \sim 10^{-3}(Z\ln\Omega)^2\,, \quad \Omega \simeq \frac{\gamma^2\rho}{3.7A\tilde{Z}^3} \gg 1\,, \qquad (3.32c)$$

and in contrast to (3.29b) this exhibits a logarithmic increase at ultrarelativistic energies.[10] Another obvious distinction arises from the $\gamma^2\rho$ combination in (3.32c): This implies that there is an intensity trade-off between high energies and "weak" discontinuities across the dielectric interface $\epsilon_1 \to \epsilon_2$. One practical consequence is that multiply-layered transition-radiation detectors tend to saturate at high energies,[31] i.e., the spectrum given by (3.31b) has an energy independent limit when $\beta \to 1$. Another experimental limit is imposed by the competition with high-energy Coulomb bremsstrahlung. Under conditions corresponding to (3.30), there is a nontrivial lower bound on the thickness of the transition radiators which is scaled by the formation length (1.1a). This minimum length in turn leads to the criterion [cf. (3.30)][32]

$$\omega(\text{keV})F_2(\beta;\epsilon_1,\epsilon_2) \gg 5\times 10^{-5}\gamma^2\,, \qquad (3.33)$$

which must be satisfied in order that the transition radiation may dominate over Coulomb bremsstrahlung. One can easily check from (3.31b) and (3.22a) that this implies severe constraints for the detection of transition radiation from high-energy particles.

IV. ČERENKOV-TRANSITION RADIATION

We have already seen that the general expression for the action describing Čerenkov-transition radiation can be decomposed into two parts [cf. (3.1)]

$$W = W_C(\tilde{A}^\mu) + W_T(\hat{A}^\mu).$$

In the case when

$$\beta^2 n_a^2(\omega) < 1\,, \quad a = 1, 2\,, \quad W_T \to W_{\text{tr}}\,, \tag{4.1a}$$

the only nonvanishing contribution arises from the second term, and experimentally this corresponds to "pure" transition radiation. However, in the converse instance

$$\beta^2 n_a^2(\omega) > 1 \quad \text{for either or both } a = 1, 2\,; \tag{4.1b}$$

the identification of W_C with "pure" Čerenkov radiation has only a formal significance. Under these conditions W_T does not vanish identically — in fact, it may interfere destructively with W_C — and experimentally we are justified in speaking of "pure" Čerenkov radiation only to the extent that the W_T contributions can be suppressed relative to W_C by the use of radiators which are much longer than a formation length [cf. (1.5)]. These idealizations do not apply to very thin foils and foam radiators — both of which have been considered as practical candidates for monitoring heavy-ion beams and detecting high-Z components in cosmic rays.[33]

In extending the calculation of W_T to cases where the Čerenkov threshold conditions (4.1b) are satisfied, we encounter a number of technicalities. Suppose for instance that only one of the inequalities, e.g., $\beta^2\epsilon_2 > 1$, is satisfied. Then one can easily check that the denominator ξ_2 in (3.13b) and (3.13c) passes through a zero in the region of integration, and this implies that there are both single and double poles present in (3.12b). These singularities must be evaluated by prescriptions consistent with the boundary conditions. We note first that the "dielectric light cone" condition ($\xi_2 = 0$) originates with the particular solution (2.17). Tracing this singularity forward to (3.7a) shows that it gives rise to the Čerenkov spectrum. The corresponding pole that occurs in the coefficients a^μ, b^μ [cf. (2.25a) and (2.25b)] may therefore be evaluated according to the prescription given in (2.18b). The other pole is associated with the denominator $\kappa_2 - k_z$ in (3.11), and this in turn originates from the z integral in (3.10). In the Appendix we show that the appropriate mathematical device for dealing with this case is simply

$$\int_0^\infty dz\, e^{-i(k_z - \kappa_2)z} \to i(\kappa_2 - k_z + i\delta)^{-1}. \tag{4.2}$$

The details of the double-pole evaluation are also relegated to the Appendix; cf. (A10).

With this formal machinery in hand it is then a straightforward matter to recalculate the basic integrals (2.13b) and (2.13c) without the sub-Čerenkov restrictions (3.9). There are two cases to consider. The simplest case corresponds to

$$\beta^2(\epsilon_2 - \epsilon_1) < 1 \quad \text{and } \epsilon_2 > \epsilon_1\,. \tag{4.3}$$

Under these circumstances the total spectral intensity of the radiation is given by

$$I(\omega) = I_C(\omega) + I_T^{(1)}(\omega)\,, \tag{4.4}$$

where $I_C(\omega)$ coincides with the usual Čerenkov spectrum (3.7a), and $I_T^{(1)}$ is obtained from the previous I_{tr}, (3.21a), by employing the absolute values of the arguments of all the logarithms in (3.20b) to (3.20d). This simple generalization is adequate since we are only interested in the real part of $\mathcal{G}$, (3.13a), and the additional imaginary term associated with the pole is irrelevant.

The other case, which is more interesting in practical applications, arises if the parameter b introduced in (3.18) becomes positive. Again assuming $\epsilon_2 > \epsilon_1$, one can show that when

$$b = \beta^2(\epsilon_2 - \epsilon_1) - 1 > 0\,, \tag{4.5}$$

further contributions arise from region III, (3.15d). Specifically if we trace through the calculations leading to the auxiliary function $\mathcal{L}_3$, defined in (3.20d), we find that $\mathcal{L}_3(\epsilon_2, \epsilon_1)$ still has precisely the functional form given in (3.20d) with the interchange $\epsilon_1 \leftrightarrow \epsilon_2$, but the object $\mathcal{L}_3(\epsilon_1, \epsilon_2)$ is now given by a different expression, viz.,

$$\mathcal{L}_3'(\epsilon_1, \epsilon_2) = \int_0^{(\epsilon_1/\epsilon_2)^{1/2}} dx \left(\frac{\epsilon_2 - \epsilon_1}{1 - x^2}\right)^{1/2} \frac{b}{b + x^2}$$
$$= \beta^{-1}[\beta^2(\epsilon_2 - \epsilon_1) - 1]^{1/2} \times \tan^{-1}\left(\frac{\beta\epsilon_1^{1/2}}{[\beta^2(\epsilon_2 - \epsilon_1) - 1]^{1/2}}\right). \tag{4.6}$$

Another new feature associated with (4.5) is that the Čerenkov threshold now occurs in region II of the integrations [cf. (3.16b)], where κ_1 becomes purely imaginary. The basic quadrature (3.19) therefore acquires additional contributions arising from the δ functions of (A9) and (A10), viz.,

$$\pi \frac{[\beta^2(\epsilon_2 - \epsilon_1) - 1]^{1/2}}{\beta^3(\epsilon_2 - \epsilon_1)X_2} (\beta^2\epsilon_2 - 1)(2 - \beta^2\epsilon_1)\,, \tag{4.7}$$

where X_2 is given by (3.20a).

At this point it is convenient to introduce a new auxiliary function

$$\mathcal{L}_3^{(2)}(\epsilon_1, \epsilon_2) = -\beta^{-1}(\beta^2\epsilon_{21} - 1)^{1/2}\tan^{-1}\left(\frac{(\beta^2\epsilon_{21} - 1)^{1/2}}{\beta\epsilon_1^{1/2}}\right)$$

$$= \frac{1}{2\beta}(1 - \beta^2\epsilon_{21})^{1/2} \times \ln\left(\frac{\beta\epsilon_1^{1/2} + (1 - \beta^2\epsilon_{21})^{1/2}}{\beta\epsilon_1^{1/2} - (1 - \beta^2\epsilon_{21})^{1/2}}\right); \quad (4.8)$$

where $\epsilon_2 - \epsilon_1 = \epsilon_{21}$, and we fix the convention that when $1 - \beta^2\epsilon_{21} < 0$ [cf. (4.5)] the radicals are changed to $(1 - \beta^2\epsilon_{21})^{1/2} \to i(\beta^2\epsilon_{21} - 1)^{1/2}$, and the principal branch of the logarithm is used for evaluation. It is then easy to check that the functional asymmetry between $\mathcal{L}_3(\epsilon_1, \epsilon_2)$ and $\mathcal{L}_3'(\epsilon_1, \epsilon_2)$ (4.6), as well as the new pole contributions (4.7) can both be concisely represented by $\mathcal{L}_3^{(2)}$. Specifically the total spectral intensity of Čerenkov-transition radiation above the threshold (4.5) is given by

$$I(\omega) = I_C(\omega) + I_T^{(2)}(\omega), \quad (4.9)$$

where $I_C(\omega)$ still corresponds to the Čerenkov spectrum, (3.7a) and $I_T^{(2)}$ is derived from I_{tr}, (3.21a), by (i) replacing $\mathcal{L}_3(\epsilon_a, \epsilon_b)$ by $\mathcal{L}_3^{(2)}(\epsilon_a, \epsilon_b)$, for *both* $a, b = 1, 2$ and $2, 1$ and (ii) employing absolute values for the arguments of the remaining logarithms, e.g., (3.20c).

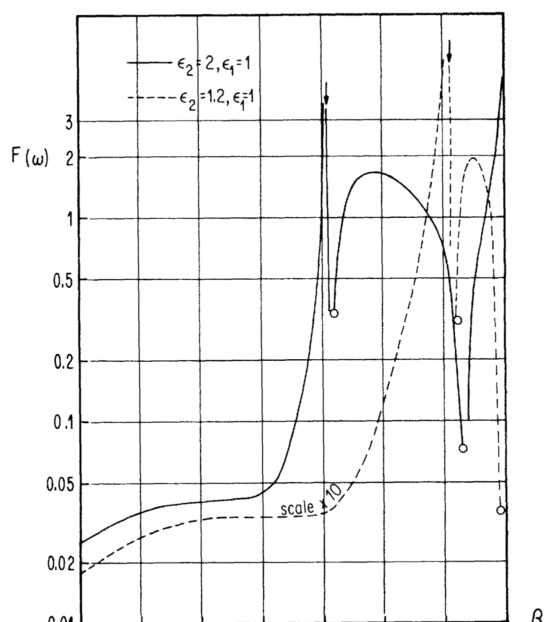

FIG. 1. Characteristics of the Čerenkov-transition amplitude $F(\omega) = (\pi/\hbar\alpha)I_T$, cf. (4.4) and (4.9). The variation of F is displayed as a function of β for a number of illustrative cases. The curve segments bounded by small circles represent negative values of $F(\omega)$.

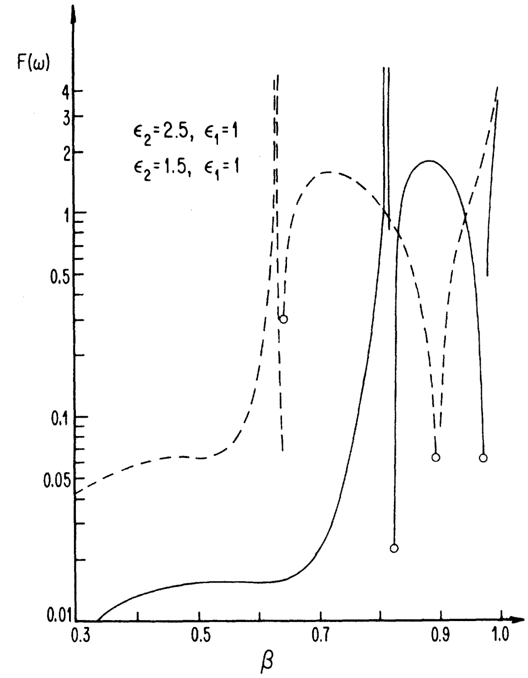

FIG. 2. Data display extending the ϵ_2 range of Fig. 1. Values of $\epsilon \sim 16$ are practicable in Ge for certain transparency windows in the infrared.

These special cases, e.g., (4.4) and (4.9), give a good indication of the general features of Čerenkov-transition radiation. The essential distinction vis-a-vis *synergic* radiation processes such as synchrotron-Čerenkov radiation[7] and Coulomb bremsstrahlung in condensed media[2] is that the Čerenkov spectrum $I_C(\omega)$ always preserves its identity in a mathematical sense even though, as we have emphasized earlier, its experimental isolation from $I_{tr}(\omega)$ may only be feasible in very long radiators. For this reason it is convenient to speak of an "interference" rather than a synergism between Čerenkov and transition radiation: In a strict sense matters are, of course, more complicated since "pure" transition radiation, i.e., the $I_{tr}(\omega)$ of (3.21a), is modified to $I_T^{(1)}$, (4.4), or $I_T^{(2)}$, (4.9), depending on the values of β and the interface parameters.

These relations can be clarified by considering a few numerical examples. For instance in Figs. 1 and 2, we show $F(\omega)$ [$= (\pi/\hbar\alpha)I_T$] as a function of β for $\epsilon_1 = 1$ and several values of ϵ_2. Figure 3 exhibits the variation of $F(\omega)$ as a function of ϵ_2 for fixed values of β and ϵ_1. These graphs confirm that there is a slight logarithmic singularity at the Čerenkov threshold due to the structure of $\mathcal{L}_2$ (3.20c). Another feature which can be read off these graphs is that $F(\omega)$ decreases sharply and becomes negative just above the Čerenkov

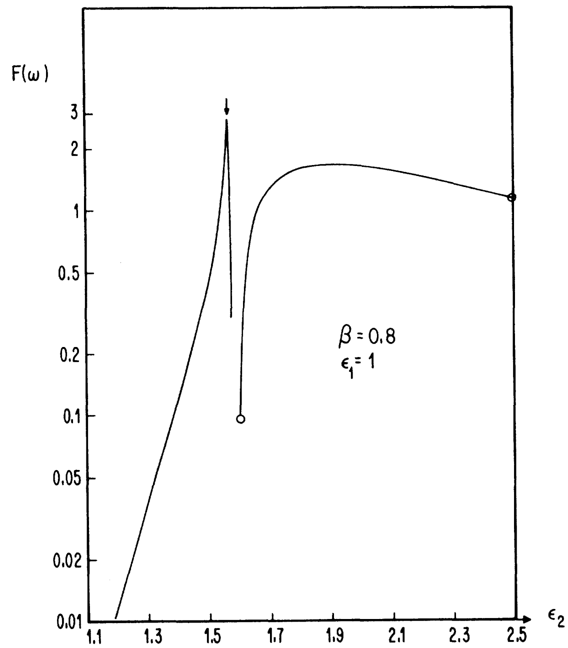

FIG. 3. Variation of the Čerenkov-transition amplitude as a function of ϵ_2 for fixed β and ϵ_1.

threshold. In this sense (4.9) can exhibit a destructive interference between the interface and volume effects. Of course, our calculations are constrained by a lower bound set by the formation length [cf. (1.1a) and the Appendix], and this insures that the total spectral intensity $I(\omega)$ remains non-negative.

A detailed survey of the experimental possibilities raised by the present results merits a separate discussion. After all, there are a variety of length dependent effects which can modify the "standard" spectrum (4.9) and lead to the possibility of constructing high-energy particle detectors. There are also exotic but scientifically sound possibilities for exploiting atomic or nuclear resonances to obtain indices of refraction with sharply defined thresholds in the x-ray region.[34,35] Among all these options it would appear that applications which are linked with *coherent* radiation processes offer the greatest promise. This coherence was already implicit in the derivation of (3.23a) since we tacitly assumed that the Z charges were localized within a bunch (="nucleus") of radius r_0 such that $c/\omega \gg r_0$ for all relevant frequencies in the Čerenkov-transition spectrum. Our (nonquantum mechanical) treatment of Čerenkov-transition radiation can easily be generalized to include emission from extended charge structures by modifying (3.8), (3.23a), etc. with the replacement[36]

$$Z \to \rho_k , \qquad (4.10a)$$

where ρ_k denotes the static form factor

$$\rho_k = \int d\vec{r}\, \rho(|\vec{r}|)\, e^{-i\vec{k}\cdot\vec{r}} \qquad (4.10b)$$

and the wave number is linked to the frequency and index of refraction by

$$k = (\omega/c)\, n(\omega) .$$

For instance a uniformly charged sphere of radius r_0 corresponds to the form factor

$$\rho_k = \frac{3(2\pi)^2}{2} Z (kr_0)^{-3/2} J_{3/2}(kr_0) , \qquad (4.11)$$

where $J_{3/2}$ is the Bessel function of $\frac{3}{2}$ order. In the point-charge limit $kr_0 \to 0$, we recover the previous cases $\rho_k \to Z$; however, in the high frequency limit $kr_0 \to \infty$, the form factor damps the emission according to $\rho_k \to (kr_0)^{-1} \cos kr_0$.

We have already indicated that in accelerators the coherence associated with charge bunching is manifested in beam loading losses: For instance if the 20-GeV accelerator at SLAC is operated with $\sim 10^9$ electrons/bunch, approximately 40 MeV *per electron* is dissipated in diffraction radiation during the traversal of circa 86000 iris sections.[3] One can easily check that this corresponds to a loss of $\gtrsim 10^5$ rf photons per electron per iris section; and this enormous enhancement over the nominal orders of magnitude implied by (3.21a) is clearly due to the collective bunching effects, i.e., $Z^2 \sim 10^{18}$. It is problematical whether there are areas of application in which this coherent enhancement is beneficial rather than detrimental, but one can speculate that collective Čerenkov-transition radiation might play a role in coupling electron or ion energies into target pellets in fusion schemes. An approximate criterion for the threshold of coherent radiation can easily be inferred from (4.11), i.e.,

$$\rho^{-1/3} \lesssim \lambda . \qquad (4.12)$$

In the "Proto 2" e^--beam fusion experiment at Sandia, 1.5-MeV electrons will converge on a millimeter-diameter fuel pellet: The electron densities in the vicinity of the pellet surface will be 10^{14} cm^{-3}. The coherence criterion (4.12) then indicates charge bunching enhancements of the order of $Z_{eff}^2 \gtrsim 10^4$ for Čerenkov-transition radiation in the micron wavelength region of the spectrum.

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APPENDIX: SINGULAR INTEGRALS

One of the key difficulties in carrying out the angular integrations over the differential transition radiation spectrum is the appearance of singularities associated with the Čerenkov cone, as well as the critical and Brewster-Malus angles at the dielectric interface.[25] Although the source theory formulation permits us to detour these angular integrals, some technical difficulties originating from singularities remain in the calculation. However, it is easy to resolve these problems if we relax the idealization of radiators of infinite extent and consider the radiation occurring over a finite length L. For definiteness we will consider the case when the dielectric ϵ_2 is above the Čerenkov threshold, i.e., $\beta^2\epsilon_2 > 1$, while ϵ_1 lies below, $\beta^2\epsilon_1 < 1$. Then if L is large in the sense $kL \gg 1$, the only source of difficulty arises from values of $k_\perp{}^2$ in the vicinity of the zero of ξ_2, i.e.,

$$\xi_2 = 0;\ k_\perp{}^2 = \frac{\omega^2}{v^2}(\beta^2\epsilon_2 - 1) < \frac{\omega^2}{c^2}\epsilon_2 . \tag{A1}$$

Of course, this occurs in a region of $k_\perp{}^2$ values for which κ_2 is real. In the vicinity of this region it is convenient to introduce an auxiliary variable u defined in terms of

$$k_\perp{}^2 = (\omega^2/c^2)\epsilon_2(1-u^2), \tag{A2}$$

so that

$$\kappa_2 = (|\omega|/c)\epsilon_2{}^{1/2}u \tag{A3a}$$

and

$$\xi_2 = (\omega^2/c^2)\epsilon_2(\beta^{-1}\epsilon_2{}^{-1/2} + u) \times (\beta^{-1}\epsilon_2{}^{-1/2} - u - i\delta), \quad \delta \to 0+. \tag{A3b}$$

Čerenkov radiation occurs when $\xi_2 = 0$, or

$$u = u_0 = \beta^{-1}\epsilon_2{}^{-1/2}, \tag{A4}$$

and this implies that u_0 corresponds to the cosine of the Čerenkov angle. Now the term in (3.11) that gives rise to the pole $(\kappa_2 - k_z)^{-1}$ after the z integration actually has the form

$$\int du \int_0^L dz\, e^{-i(k_z-\kappa_2)z}\left(f(u) + \frac{g(u)}{\beta^{-1}\epsilon_2{}^{-1/2} - u - i\delta}\right), \tag{A5}$$

where $f(u)$ and $g(u)$ represent analytic functions of u in the neighborhood of u_0. The z integral presents no difficulty, and leads to the expression

$$i\int du(1 - e^{i(u-u_0)N})\frac{1}{u-u_0}\left(f(u) - \frac{g(u)}{u-u_0+i\delta}\right), \tag{A6}$$

which exhibits the double-pole structure latent in (3.11). The object N is defined by

$$N = (\omega/c)\epsilon_2{}^{1/2}L = L/\lambda\!\!\!^{-}, \tag{A7}$$

where $\lambda\!\!\!^{-}$ denotes the reduced wavelength of the Čerenkov-transition radiation in the medium. If we now introduce the formation length constraint [cf. (1.1a)]

$$L \gg l_a \to N \gg 1; \tag{A8}$$

we can evaluate (A6) by standard means. Specifically

$$\int du(1 - e^{i(u-u_0)N})\frac{f(u)}{u-u_0} \to \left(\int^{u_0-\delta} + \int_{u_0+\delta}\right)\frac{du\, f(u)}{u-u_0} - i\pi f(u_0) \tag{A9}$$

and

$$\int du(1 - e^{i(u-u_0)N})\frac{g(u)}{(u-u_0)(u-u_0+i\delta)} \to \left(\int^{u_0-\delta} + \int_{u_0+\delta}\right)\frac{du\, g(u)}{(u-u_0)^2} - \frac{2g(u_0)}{\delta} - i\pi g'(u)\big|_{u=u_0}. \tag{A10}$$

Equation (A9) corresponds to the standard principal-value prescription (3.6), and (A10) represents the generalization applicable to a double pole. In particular, the idealization (A8) leads to the identification

$$\int_0^\infty dz\, e^{-i(k_z-\kappa_2)z} = i(\kappa_2 - k_z + i\delta), \quad \delta \to 0+ \tag{A11}$$

and this confirms that the denominators ξ_a in (3.13b) and (3.13c) have the structure indicated in (2.18b).

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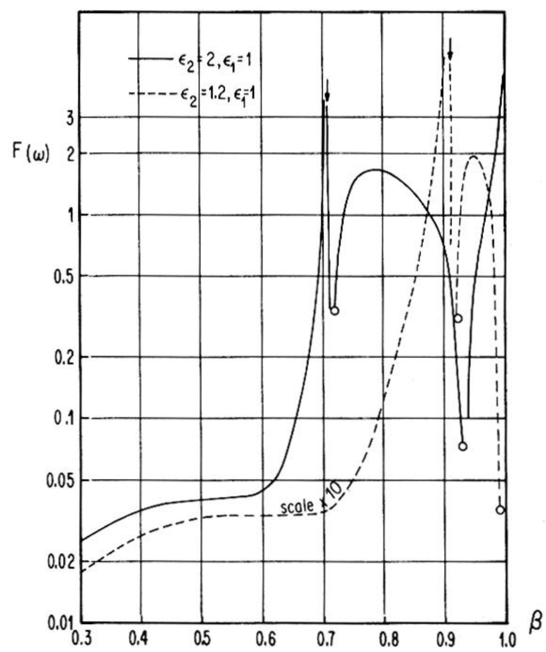

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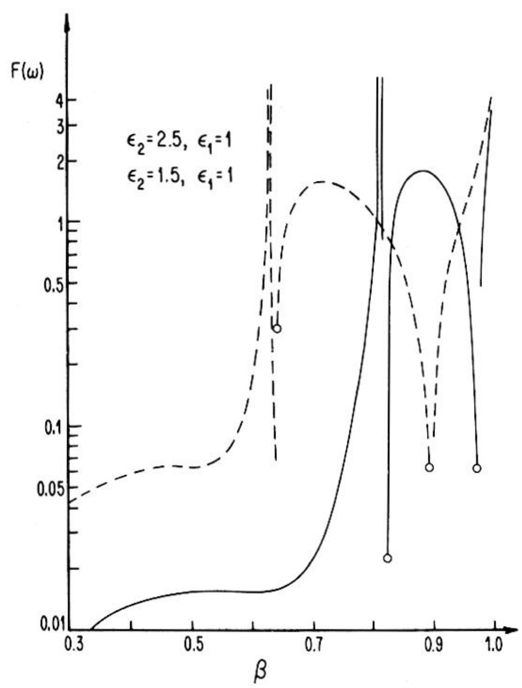

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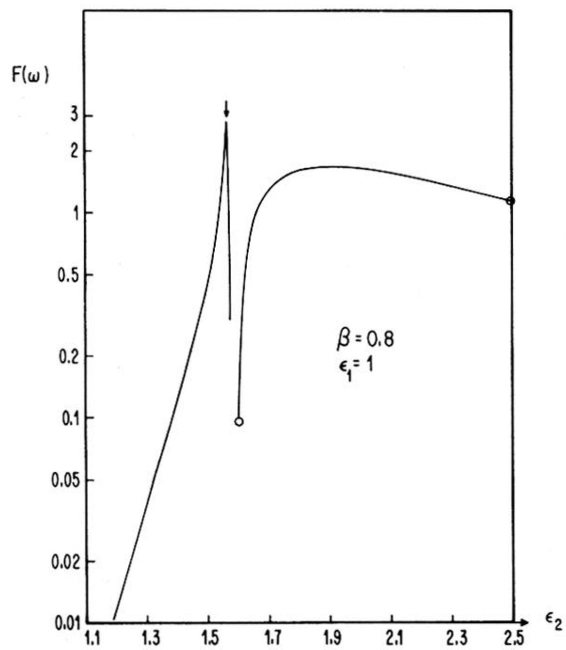

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