PHYSICAL REVIEW D VOLUME 18, NUMBER 6 15 SEPTEMBER 1978

Differential equations with respect to a coupling constant: An approach to Čerenkov and stimulated radiations

Josip Šoln

Harry Diamond Laboratories, Nuclear Radiation Effects Laboratory, Adelphi, Maryland 20783

(Received 12 May 1978)

The PDECC (partial differential equations with respect to coupling constants) formalism is adapted to derive two, physically equivalent, S matrices for calculating the power spectrum of stimulated radiation semiclassically, and the power spectrum of Čerenkov radiation either semiclassically or quantum mechanically (to the lowest order in the perturbation theory). The stimulated radiation resulting from the interaction between electrons and an incident electromagnetic field (a laser, for example) could have frequencies significantly upshifted from those of the incident electromagnetic field. This suggests a possible practical way of generating radiation at higher frequencies than currently available from lasers, as well as submillimeter radiation from microwaves.

I. INTRODUCTION

Recently there has been a renewed interest in electromagnetic processes in a refractive medium. This can be attributed at least partially to the fact that the existence of the Čerenkov effect enriches the usual electromagnetic processes already existing in a vacuum. For example, in addition to the normal (or vacuumlike) electron-photon scattering, one also expects an anomalous, in the terminology of Frank[1] (or Čerenkov-effect mediated), electron-photon scattering, which in the limit of vanishing matter density disappears much in the same way as the Čerenkov effect itself. In fact, Schneider and Spitzer[2] believe that the Čerenkov effect will mediate the generation of intense radiation, frequency upshifted, with respect to an incident coherent electromagnetic wave which is colliding head-on with an electron beam. This process they call SESR (stimulated electromagnetic shock radiation).[3]

In this article we arrive at two equivalent S matrices from the physical point of view for treating semiclassical and quantum (to the lowest order in e) radiation processes in a medium, using the PDECC (partial differential equations with respect to coupling constants) formalism.[4] The resultant S matrices allow us to establish a bridge between the recently proposed source-theory approach[5] and more conventional approaches for the treatment of the radiation in a medium. The unitarity of the S matrix plays a vital role in establishing this correspondence, which we finally verify in calculating the quantum Čerenkov effect to the lowest order in e. On the semiclassical level, on the other hand, we analyze the possibility of the generation of frequency-upshifted radiation with respect to an incident coherent electromagnetic wave which is colliding head-on with an electron beam, as originally proposed by Schneider and Spitzer.[2,3] In this connection we establish that the actual process can be split into two "branches," a vacuum branch and a Čerenkov branch. Of course, in the limit of a vanishing matter density, a vacuum branch is the only one that survives. For each of these two branches, we derive expressions for the frequencies of outgoing radiation, as well as the corresponding power spectra.

The SESR of Schneider and Spitzer[2,3] should correspond to our Čerenkov branch solution of the electron-beam–coherent-electromagnetic-wave collision, except that their calculated frequency of outgoing radiation does not agree with ours. This discrepancy we attribute to the fact that they deduce the frequency from the nonasymptotic scattered electric field, while our frequency corresponds to the asymptotically free radiation field.

In Sec. II the differential equations with respect to a coupling constant for the S matrix and the interacting fields (corresponding to electrons and the radiation in a medium) are given.[4] The solution for the S matrix is given in a form suitable for calculating the amplitudes with any number of photons in either initial or final state.

Section III is devoted to establishing the connection between Schwinger's definition of power spectrum of emitted radiation with the more usual definitions. Here the unitarity of the S matrix is fully employed, and the result is that semiclassically Schwinger's definition is equivalent to the usual definitions, while quantum mechanically, however, it is equivalent only up to the lowest order in the perturbation theory.

The actual calculations of power spectra are given in Sec. IV. On the quantum level (Sec. IV A) we treat the Čerenkov effect itself, while on the semiclassical level (Sec. IV. B) the interaction between an electron beam and an external electromagnetic field is treated.

Section V is devoted to discussion and concluding remarks.

In Appendix A some details of the quantization of a free electromagnetic field in a moving medium are given. This appendix also contains a brief discussion of relevant singular functions needed in the text.

A brief derivation of the change in the electron's trajectory due to the interaction with an external electromagnetic field (a laser) is given in Appendix B.

Appendix C is devoted to derivation by means of the PDECC formalism of the S matrix, which, while formally different, nevertheless physically is equivalent to the S matrix derived in the text.

In addition to using the rationalized Heaviside system of units, we also set $\hbar = c = 1$ throughout the text. For convenience we also assume an infinite medium of unit magnetic permeability ($\mu = 1$), so that the index of refraction n and the real dielectric constant ϵ are related as $n^2 = \epsilon$.

II. THE S MATRIX

It has been shown on various occasions that the differential equations with respect to coupling constants for the S matrix, interacting fields, and even "out" fields can be quite useful.[4,6,7] For some models of quantum field theory, they enabled one to reduce the S matrix to the closed normal form in free-field "in" operators from which then the S-matrix elements can be read off quite straightforwardly.

Although we are dealing here with the electromagnetic processes in a medium, the differential equations for the S matrix and the interacting fields are assumed to be the same as in a vacuum[4,7]:

$$\frac{1}{i}\frac{d}{d\lambda}S(\lambda) = S(\lambda)\int d^4x\,\mathcal{L}_{\rm int}(x)\,, \tag{1}$$

$$\frac{1}{i}\frac{d}{d\lambda}S(\lambda)A_\mu(x) = \int d^4y\,S(\lambda)TA_\mu(x)\mathcal{L}_{\rm int}(y)\,, \tag{2}$$

$$\frac{1}{i}\frac{d}{d\lambda}S(\lambda)\psi(x) = \int d^4y\,S(\lambda)T\psi(x)\mathcal{L}_{\rm int}(y)\,, \tag{3}$$

where λ, the dimensionless coupling constant, is varied between 0 and 1. T is the time-ordering operator, $A^\mu(x)$ is the quantized electromagnetic potential corresponding to the radiation in a medium, and $\psi(x)$ is the quantized field operator corresponding to the electrons (or some other charged particles, if so desired). Since we introduced the dimensionless λ, the electromagnetic current is now defined as (with e the usual electromagnetic coupling constant)

$$J^\mu(x) = e\bar{\psi}(x)\gamma^\mu\psi(x)\,, \tag{4}$$

with which $\mathcal{L}_{\rm int}$ is given as

$$\mathcal{L}_{\rm int}(x) = J_\mu(x)A^\mu(x)\,. \tag{5}$$

Relations (1) through (5) are now the basis of the quantum theory of radiation in a medium. A semiclassical theory of radiation is one where J^μ is a classical (nonquantized) quantity. This theory can be easily obtained as a special case from the quantum theory of radiation.

Now in relations (1) through (3), T will act on both the fermion ($\psi_{\rm in}$) and photon ($A^\mu_{\rm in}$) free-field operators. Consequently we write[8]

$$T = T_\psi T_A\,, \tag{6}$$

where T_ψ and T_A act on $\psi_{\rm in}$'s and $A^\mu_{\rm in}$'s, respectively. Denoting with S' the S matrix in which only the time ordering of $A^\mu_{\rm in}$'s is carried out, we have

$$S = T_\psi S'\,. \tag{7}$$

Furthermore, introducing a new interacting field $A'_\mu(x)$ by relation

$$S'A'_\mu(x) = T_A S'A^{\rm in}_\mu(x)\,, \tag{8}$$

we get

$$SA_\mu(x) = TSA^{\rm in}_\mu(x) = T_\psi T_A S'A^{\rm in}_\mu(x) = T_\psi S'A'_\mu(x)\,. \tag{9}$$

However, the similarly defined $\psi'(x)$ turns out to be equal to $\psi_{\rm in}(x)$,

$$S'\psi'(x) = T_A S'\psi_{\rm in}(x) = S'\psi_{\rm in}(x)\,, \tag{10}$$

thus giving

$$S\psi(x) = T_\psi S'\psi_{\rm in}(x)\,. \tag{11}$$

It is not difficult to see now that for S', A'_μ, and $\psi' = \psi_{\rm in}$, the following set of differential equations hold:

$$\begin{aligned}
&\frac{1}{i}\frac{d}{d\lambda}S'(\lambda) = S'(\lambda)\int d^4x\,\mathcal{L}'_{\rm int}(x)\,,\\
&\frac{1}{i}\frac{d}{d\lambda}S'(\lambda)A'_\mu(x) = \int d^4y\,S'(\lambda)T_A A'_\mu(x)\mathcal{L}'_{\rm int}(y)\,,\\
&\frac{1}{i}\frac{d}{d\lambda}\psi'(x) = \frac{1}{i}\frac{d}{d\lambda}\psi_{\rm in}(x) = 0\,,
\end{aligned} \tag{12}$$

where

$$\mathcal{L}'_{\rm int}(x) = J^{\rm in}_\mu(x)A'_\mu(x)\,, \tag{13}$$

and the fact that $\psi_{\rm in}$ is assumed to be independent of λ, was assumed. The last equation in (12) implies that fermion field $\psi_{\rm in}(x)$ can be formally treated in (7) to (12) as an anticommuting c-number field. Consequently, in (12) $J^{\rm in}_\mu(x)$ is formally a c-number current, and we can integrate these equations at once (see Refs. 6 and 9 where similar situations of a pion field coupled to the c-number source are discussed):

$$A'^{\mu}(x) = A^{\mu}_{\text{in}}(x) + \lambda \int d^4y\, D^{\mu\nu}_R(x-y) J^{\text{in}}_{\nu}(y)\,, \tag{14}$$

$$S'(\lambda) = :\exp\left[i\lambda \int d^4x A^{\text{in}}_{\mu}(x) J^{\mu}_{\text{in}}(x)\right]: \times \exp\left[i\frac{\lambda^2}{2}\int d^4x\, d^4y J^{\text{in}}_{\mu}(x) D^{\mu\nu}_F(x-y) J^{\text{in}}_{\nu}(y)\right], \tag{15}$$

where $D^{\mu\nu}_R$ and $D^{\mu\nu}_F$ are given in Appendix A. In view of (7), the total S matrix is given as

$$S(\lambda) = T_{\psi}:\exp\left[i\lambda \int d^4x\, A^{\text{in}}_{\mu}(x) J^{\mu}_{\text{in}}(x)\right]: \times \exp\left[i\frac{\lambda^2}{2}\int d^4x\, d^4y J^{\text{in}}_{\mu}(x) D^{\mu\nu}_F(x-y) J^{\text{in}}_{\nu}(y)\right]. \tag{16}$$

The first term in (16) contributes to external photon lines, while the second term contributes to the internal photon lines because the normal ordering (: :) applies only to A^{in}_{μ} fields.

III. POWER SPECTRUM OF EMITTED RADIATION

In discussing the power spectrum it will prove useful to use the identity (see Appendix A)

$$D^{\mu\nu}_F = \bar{D}^{\mu\nu} + \tfrac{1}{2} i\, D^{\mu\nu}_{(1)}\,. \tag{17}$$

Let us suppose that in (16) the current is a c-number quantity (now, of course, T_{ψ} drops out). Utilizing (17) we see that the part of the S matrix containing $\bar{D}^{\mu\nu}$ becomes an irrelevant phase factor, and it can be left out. This suggests defining a new $\tilde{S}(\lambda)$ matrix, when J^{μ} is also a q-number current, as

$$\tilde{S}(\lambda) = T_{\psi}:\exp\left[i\lambda \int d^4x A_{\mu}(x) J^{\mu}_{\text{in}}(x)\right]: \times \exp\left[-\frac{\lambda^2}{2}\int d^4x\, d^4y J^{\mu}_{\text{in}}(x) D^{(1)}_{\mu\nu}(x-y) J^{\nu}_{\text{in}}(y)\right], \tag{18}$$

and, as seen in Appendix C, it is also unitary.

It is quite clear that S matrices (16) and (18) are generally different. However, for the processes which we wish to consider in this article, they are equivalent. Namely, in Feynman diagrammatic language, the quantum Čerenkov effect to lowest order in perturbation theory corresponds to a simple electron-photon vertex. Unitary S matrices (16) and (18) clearly should give the same result, since the terms of the order of λ^2 do not contribute to it. As far as the possibility of generating submillimeter waves is concerned, we intend to analyze it on the semiclassical level, i.e., by using a suitable c-number current $J^{\mu}(x)$, which simply means that the S matrix (18) (without T_{ψ}) will be used.

Finally, utilizing the relationship of $D^{(1)}_{\mu\nu}$ to $D^{(+)}_{\mu\nu}$ and $D^{(-)}_{\mu\nu}$ and taking into account that $D^{(-)}_{\mu\nu}(x-y) = -D^{(+)}_{\mu\nu}(y-x) = -D^{(+)}_{\nu\mu}(y-x)$ (see Appendix A), in expressions (16) and (18) we can write

$$\int d^4x\, d^4y J^{\mu}_{\text{in}}(x) D^{(1)}_{\mu\nu}(x-y) J^{\nu}_{\text{in}}(y) = 2i\int d^4x\, d^4y J^{\mu}_{\text{in}}(x) D^{(+)}_{\mu\nu}(x-y) J^{\nu}_{\text{in}}(y)\,, \tag{19}$$

a relationship that will be very useful in practical calculations.

A. Power spectrum when $J^{\mu}(x)$ is a classical current

When $J^{\mu}(x)$ is a classical current, we define the "excitation" amplitude of a medium to be [see relation (16)]

$$\langle 0|S(\lambda)|0\rangle = \exp\left[i\frac{\lambda^2}{2}\int d^4x\, d^4y J_{\mu}(x) D^{\mu\nu}_F(x-y) J_{\nu}(y)\right]. \tag{20}$$

It is quite clear that

$$|\langle 0|S(\lambda)|0\rangle| = |\langle 0|\tilde{S}(\lambda)|0\rangle|\,.$$

Excitation amplitude (20) is exactly the same as what Schwinger *et al.*[5] call the vacuum persistence amplitude in the language of source theory. Thus following Schwinger *et al.*,[5] the decay rate $\Gamma(t)$ is defined through the excitation probability

$$|\langle 0|S(\lambda)|0\rangle|^2 = |\langle 0|\tilde{S}(\lambda)|0\rangle|^2 = \exp\left[-\lambda^2\int dt\, \Gamma(t)\right], \tag{21}$$

which in turn defines the power spectrum $P(\omega, t)$ by

$$\Gamma(t) = \int d\omega \frac{P(\omega, t)}{\omega}\,. \tag{22}$$

We wish to point out that throughout this paper Γ is actually independent of t and that $\int dt\, \Gamma(t)$ is written in place of $2\pi\delta(0)\Gamma$. The δ function at the origin, of course, always drops out in calculations. Consequently, $P(\omega, t)$ is also t independent.

Now $\Gamma(t)$ can also be defined from a production amplitude of one photon with momentum k and polarization α as follows [with $d\tilde{\sigma}(k)$ defined in Appendix A]:

$$\lambda^2\int dt\, \Gamma(t) = \sum_{\alpha}\int d\tilde{\sigma}(k)\, |\langle k, \alpha|S(\lambda)|0\rangle|^2\,, \tag{23}$$

where we find

$$\int dt\, \Gamma(t) = i\int d^4x\, d^4y\, D^{(+)}_{\mu\nu}(x-y) J^{\mu}(x) J^{\nu}(y)\,. \tag{24}$$

Since J^μ is a c-number quantity, the emission of photons of any momenta and polarizations will occur in a statistically independent way (see Ref. 9, where the pion production in high-energy collisions with a similar S matrix is discussed). Thus, probability W_m for emission of any kind of m photons is simply the Poisson distribution function

$$W_m = \frac{\langle m\rangle^m e^{-\langle m\rangle}}{m!}, \quad (25)$$

where $\langle m\rangle$, the average multiplicity of emitted photons, is

$$\langle m\rangle = \int dt\, \Gamma(t). \quad (26)$$

From here we learn that definition (21) for the decay rate $\Gamma(t)$ is always valid no matter what the strength of the interaction. However, we have to be aware of the fact that our semiclassical theory of radiation, described by the above formalism, will be valid only to the extent that the emitted radiation does not alter appreciably the source of the radiation (one is ignoring the recoil suffered by the electrons in the course of emission of photons).

B. Power spectrum when $J^\mu(x)$ is a q-number current

Although, with $J^\mu(x)$ as a q-number current, we shall discuss specifically only the Čerenkov radiation, nevertheless, we wish to see to which extent the power spectrum via the transition amplitude and the power spectrum defined in the manner of Schwinger et al.[5] coincide.

To arrive at the definition of the power spectrum in the spirit of Schwinger *et al.*,[5] we start with the "elastic" electron-to-electron amplitude, $\langle p', s'|S(\lambda)|p, s\rangle$, where p, s and p', s' are the momenta and polarizations of the electrons in initial and final states, respectively. Now, in analogy to (21), the decay rate $\Gamma(t;\lambda)$ in the manner of Schwinger *et al.*[5] is defined as

$$\sum_{s'} |\langle p', s'|S(\lambda)|p, s|^2 = \left(\sum_{s'} |\langle p', s'|p, s\rangle|^2\right) \times \exp\left[-\int dt\, \Gamma(t;\lambda)\right], \quad (27)$$

which in turn again defines the power spectrum $P(\omega, t; \lambda)$ as

$$\Gamma(t;\lambda) = \int d\omega \frac{P(\omega, t;\lambda)}{\omega}. \quad (28)$$

The usual definition of $\Gamma(t)$, however, utilizes a production amplitude of one photon in a process $e\to e+\gamma$, with which $\Gamma(t;\lambda)$ can be defined as $[d\sigma(p) = d^3p/E(p),\ E(p) = (\vec{p}^{\,2}+m^2)^{1/2}]$

$$\sum_s \langle p, s|p, s\rangle \int dt\, \Gamma(t;\lambda) = \int d\sigma(p')d\tilde{\sigma}(k) \times \sum_{s,s',\alpha} |\langle p', s'; k, \alpha|S(\lambda)-1|p, s\rangle|^2. \quad (29)$$

What we wish to see now is to what extent relation (27) is consistent with (29). First of all, the unitarity of the S matrix implies

$$[S^\dagger(\lambda)-1]+[S(\lambda)-1]+[S^\dagger(\lambda)-1][S(\lambda)-1]=0. \quad (30)$$

Now according to (27),

$$\langle p, s|[S^\dagger(\lambda)-1]+[S(\lambda)-1]|p, s\rangle = \langle p, s|p, s\rangle\left\{\exp\left[-\int dt\, \Gamma(t;\lambda)\right]-1\right\}, \quad (31)$$

while according to (29),

$$\langle p, s|[S^\dagger(\lambda)-1][S(\lambda)-1]|p, s\rangle = \langle p, s|p, s\rangle \int dt\, \Gamma(t;\lambda) + (\cdots), \quad (32)$$

where the terms denoted as $(\cdots)$ have intermediate states with at least three particles including processes such as $e\to e+2\gamma$ etc. Clearly, (31) and (32) are generally going to be consistent with (30) only to lowest order in perturbation theory, since the terms denoted as $(\cdots)$ are at least of $O(\lambda^4)$. This then means that definitions for $\Gamma(t;\lambda)$ in the spirit of Schwinger *et al.*[5] [relation (27)] and the more usual definition (29) are the same only to lowest order in perturbation theory.

To lowest order in perturbation theory it is irrelevant whether S matrices (16) or (18) are used, since only the mass-shell part of the photon propagator, $D^{(1)}_{\mu\nu}$, contributes. In this sense, we could say that the Čerenkov effect (to lowest order in perturbation theory) is an on-mass-shell effect. The theory, however, which reproduces the $\tilde{S}$ matrix (18) regardless of perturbation theory considerations is described briefly in Appendix C.

IV. APPLICATIONS

With the theory thus far developed, we find it suitable to describe the "quantum" Čerenkov effect [with $J^\mu(x)$ as a q-number current] and the stimulated photon emission [with $J^\mu(x)$ as a c-number current]. As argued in the preceding section either of the S matrices (16) or (18) can be used for calculations of these processes (providing that the quantum Čerenkov effect is of interest to us to lowest order in perturbation theory).

A. The quantum theory of Čerenkov radiation

In deriving the power spectrum of Čerenkov radiation to lowest order in perturbation theory, we could start with Γ either from (27) which we defined in the manner of Schwinger *et al.*[5] or from (29) which is a more usual definition. We will find, however, relation (29) to be more practical for calculations of the power spectrum since it allows us to utilize the trace properties of various products of Dirac γ matrices.

The S-matrix element entering into (29) is (where we take $\lambda = 1$, now)

$$\langle p', s'; k, \alpha | S - 1 | p, s\rangle = \delta^{(4)}(p - p' - k)\, \frac{ie2\pi}{4\pi^{3/2}} \times \bar{u}(p', s')\gamma^{\mu} u(p, s)\epsilon_{\mu}(k, \alpha)\,. \tag{33}$$

Taking into account (see Appendix A)

$$\sum_{\alpha} \epsilon_{\mu}(k, \alpha)\epsilon_{\nu}(k, \alpha) = \epsilon_{\mu\nu}(k)$$

and evaluating the appropriated traces of products of γ matrices, we get from (29)

$$\Gamma = \frac{2\alpha}{\pi E(p)} \int d^4k\, \delta_{(+)}(\eta^{\mu\nu} k_{\mu} k_{\nu})[\epsilon_{\mu\nu}(k) p^{\mu} p^{\nu} + (p\cdot k)] \times \delta(k^2 - 2p\cdot k)\,, \tag{34}$$

where $\alpha = e^2/4\pi$ and the $\delta_{(+)}$ function and $\eta^{\mu\nu}$ are defined in Appendix A. Taking the refractive medium with the index of refraction n to be at rest and denoting k^4 with ω ($\omega > 0$), we have (see also Appendix A)

$$\eta^{\mu\nu} k_{\mu} k_{\nu} = k^2 + (1 - n^2)\omega^2\,,$$

$$\epsilon_{\mu\nu} p^{\mu} p^{\nu} + p\cdot k = \vec{p}(1 - \cos^2\theta) + \frac{k^2}{2}\,,$$

where θ is the angle between momenta of the incident electron and the radiated photon. With this, (34) reduces to

$$\Gamma = v\alpha \int_{\omega>0} d\omega\, d\cos\theta \left[(1 - \cos^2\theta) + \frac{(n^2 - 1)\omega^2}{2v^2E^2}\right] \times \delta\left(\cos\theta - \frac{1}{nv}\left[1 + \frac{(n^2 - 1)\omega}{2E}\right]\right), \tag{35}$$

where v is the velocity of incident electron. From (35) the power spectrum is

$$P(\omega) = v\alpha\omega \int d\cos\theta \left[(1 - \cos^2\theta) + \frac{(n^2 - 1)\omega^2}{2v^2E^2}\right] \times \delta\left(\cos\theta - \frac{1}{nv}\left[1 + \frac{(n^2 - 1)\omega}{2E}\right]\right). \tag{36}$$

From (35) or (36) we immediately read off the usual Čerenkov threshold criterion with quantum corrections:

$$nv > 1 + \frac{(n^2 - 1)\omega}{2E}\,. \tag{37}$$

So far we have not said much about n. In fact, it is easiest to assume throughout the derivations that n corresponds to an isotropic medium and that it is independent of ω. At the end one may then specify how n depends on θ and ω, and then try to evaluate the integral in (36). If n is independent of θ (isotropic medium), (36) reduces to

$$P(\omega) = v\alpha\omega \left\{1 - \frac{1}{n^2v^2}\left[1 + (n^2 - 1)\frac{\omega}{E} + \frac{(1 - n^4)}{4}\frac{\omega^2}{E^2}\right]\right\}, \tag{38}$$

where we still may assume n to depend on ω (the case of dispersion).

Our result for $P(\omega)$ is exactly the same as the one derived by Sokolov.[10] Schwinger *et al.*,[5] however, invoke approximation $(\omega | E) \ll 1$, so their result does not have the last term in the square brackets of (38). In all known applications of Čerenkov radiation, however, one finds that

$$|n(\omega) - 1|\,\frac{\omega}{E} \ll 1 \tag{39}$$

is well satisfied. Consequently, the quantum corrections in the square brackets in (38) are negligible, reducing (38) to

$$P(\omega) \approx v\alpha\omega\left(1 - \frac{1}{n^2v^2}\right), \tag{40}$$

which is simply the semiclassical expression for the power spectrum of Čerenkov radiation due to the classical current

$$\vec{J}(\vec{x}, t) = e\vec{v}\delta(\vec{x} - \vec{v}t)\,, \quad J^4(\vec{x}, t) = \rho(\vec{x}, t) = e\delta(\vec{x} - \vec{v}t)\,. \tag{41}$$

Expression (40) would follow either from (21) or (23) utilizing (41) and, of course, (22).

Now on the semiclassical level we have derived that the mean multiplicity of emitted photons [relation (26)] is $\langle m\rangle = \int dt\, \Gamma(t)$. Since Γ is independent of t, $\langle m\rangle$ turns out to be infinite if the integral is evaluated over an infinite time interval. However, the emission of photons lasts only while an electron is passing through a dielectric medium, which is, of course, finite. Thus, if a path transversed by an electron with an average velocity $\bar{v}$ is l, then the time T during which photons are emitted is

$$T = l/\bar{v}\,, \tag{42}$$

and instead of (26) we should write

$$\langle m\rangle = \frac{l}{\bar{v}}\,\Gamma\,. \tag{43}$$

Of course, from a practical point of view we can approximate $\bar{v}$ with the initial electron velocity v. As to decay rate Γ, its finiteness is ensured by inequality (37) as long as a medium is at least weakly dispersive. Namely, condition (37) for a dispersive medium $[n=n(\omega)]$ will set an upper limit to the frequencies that can be radiated. It is worthwhile noting that (37) also becomes an implicit equation for the threshold frequency.[5]

To end this subsection, we note that on the semiclassical level, relation (43) is very useful in deriving the number of photons emitted by an electron within a spectral region λ_1 and λ_2. Assuming $\bar{v}\approx v$ and that n does not change much between λ_1 and λ_2, from (43), (40), and (28) we get

$$\begin{aligned}\langle m\rangle_{\lambda_1,\lambda_2} &= \frac{l}{v}\, v\alpha\int_{\omega_1}^{\omega_2} d\omega\left(1-\frac{1}{n^2v^2}\right)\\ &\approx 2\pi\alpha l\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\left(1-\frac{1}{n^2v^2}\right)\\ &= 2\pi\alpha l\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\sin^2\theta\,, \end{aligned}\tag{44}$$

which agrees with the results from Ref. 11.

B. Semiclassical theory of stimulated radiation

Here we wish to find the power spectrum of stimulated radiation by an electron beam colliding head-on with, say, an intense laser field in a refractive medium, as schematically shown in Fig. 1. This specific problem, as already mentioned in the Introduction, has been discussed on various occasions by Schneider and Spitzer,[2] while on a more general level the stimulated radiation by electrons scattered by photons in a refractive medium has been already discussed by Frank.[1]

In this subsection we shall assume exclusively that a medium is at rest.

Initially let the electron beam have a velocity $\vec{v}$ along the z axis. Then, before it collides with a laser field, its current density will be

$$\vec{J}_{(0)}(\vec{x},t)=e\vec{v}\delta(\vec{x}-\vec{v}t)\,,\quad \vec{v}=\vec{z}v\,. \tag{45}$$

Now let the electron collide head-on with a laser field described as

$$\begin{aligned}&\vec{E}(\vec{x},t)=\vec{E}_0\sin(\omega_0 t-\vec{x}\cdot\vec{k}_0)\,,\\ &\vec{H}(\vec{x},t)=\vec{H}_0\sin(\omega_0 t-\vec{x}\cdot\vec{k}_0)\,,\\ &\vec{E}_0=E_0\hat{y},\quad \vec{H}_0=H_0\hat{x}\,,\quad \vec{k}_0=-|\vec{k}_0|\hat{z}\,,\end{aligned}\tag{46}$$

where

$$\begin{aligned}&|\vec{k}_0|=n_0\omega_0\,,\\ &H_0=\frac{|\vec{k}_0|}{\omega_0}E_0=n_0E_0\,,\end{aligned}\tag{47}$$

and n_0 means $n(\omega_0)$.

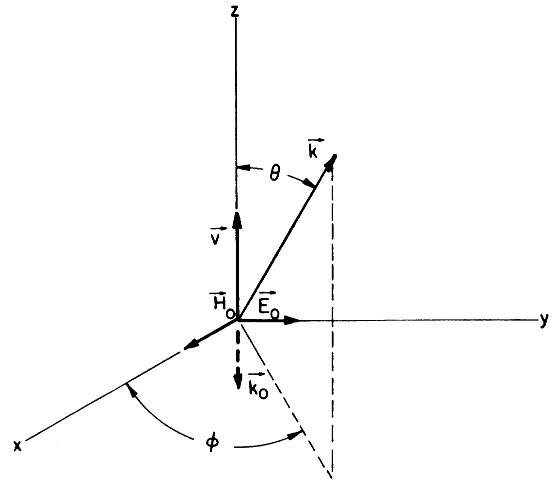

FIG. 1. Schematics of head-on electron-laser field collision for stimulated radiation from Sec. IV B. Velocity $\vec{v}$ of the incident electron and wave vector $\vec{k}_0$ of the sinusoidal laser field are antiparallel and chosen to be along the z axis. The constant electric ($\vec{E}_0$) and magnetic ($\vec{H}_0$) fields define the polarization of a laser field. The momentum of radiated photon is denoted by $\vec{k}$.

Now, owing to the laser-electron interaction, current density (45) is changed into

$$\begin{aligned}&\vec{J}(\vec{x},t)=e\vec{V}(t)\delta(\vec{x}-\vec{R}(t))\,,\\ &\vec{R}(t)=\vec{v}t+\vec{r}(t),\quad \vec{V}(t)=\dot{\vec{R}}(t)=\vec{v}+\dot{\vec{r}}(t)\,,\end{aligned}\tag{48}$$

where $\vec{r}(t)$ is the change in the trajectory of the electron due to the laser field. We shall work in the approximation

$$|\vec{r}(t)|\approx O(e)\,, \tag{49}$$

so that a laser field can be taken at

$$\vec{R}_0(t)=\vec{v}t\,, \tag{50}$$

in the Lorentz force equation from which $\vec{r}(t)$ is then evaluated. As shown in Appendix B, $\vec{r}(t)$ is then formally given as $[\gamma_0^2=1/(1-v^2)]$

$$\vec{r}(t)=\frac{e\vec{E}_0(1+vn_0)}{m\gamma_0}\lim_{\mu\to 0}\int dt'\mathfrak{D}_R(t-t';\mu)\times\sin\omega_0 t'(1+vn_0)\,, \tag{51}$$

where $\mathfrak{D}_R(t;\mu)$ is a one-dimensional retarded Green's function, as defined in Appendix B. Differentiating both sides of (51) with respect to t, one gets easily $\dot{\vec{r}}(t)$.

Taking into account approximation (49), the current $\vec{J}(\vec{x},t)$ from (48) can now be written as

$$\vec{J}(\vec{x},t)\approx\vec{J}_{(0)}(\vec{x},t)+\vec{J}_{(1)}(\vec{x},t)+\vec{J}_{(2)}(\vec{x},t)\,, \tag{52}$$

where $\vec{J}_{(0)}(\vec{x},t)$ is given by (45) and

$$\vec{J}_{(1)}(\vec{x},t)=e\dot{\vec{r}}(t)\delta(\vec{x}-\vec{v}t)\,, \tag{53a}$$

$$\vec{J}_{(2)}(\vec{x},t)=-e\vec{v}[\vec{r}(t)\cdot\nabla(x)]\,\delta(\vec{x}-\vec{v}t)\,. \tag{53b}$$

The current density $\vec{J}_{(0)}$ is the familiar source of the Čerenkov radiation, while terms represented by (53a) and (53b) are sources of stimulated radiations. Ordinarily, for each of the terms in (52) there is a corresponding $J^4(\vec{x}, t)$, which can be obtained in principle from $\nabla \cdot \vec{J} + \dot{J}^4 = 0$. However, since we take a medium to be at rest, J^4's will not be necessary in calculations of power spectra.

In what follows it will be convenient to have the Fourier transforms of $\vec{J}_{(0)}$, $\vec{J}_{(1)}$ and $\vec{J}_{(2)}$. Utilizing (51) and the Fourier representation of $\mathfrak{D}_R(t;\mu)$ from Appendix B, we get (with $\mu \to 0$)

$$\tilde{J}^i_{(0)}(\vec{k}, k^4) = 2\pi e v^i \delta(k^4 - \vec{k}\cdot\vec{v}) , \tag{54a}$$

$$\tilde{J}^i_{(1)}(\vec{k}, k^4) = -\frac{e^2 \pi E_0^i}{\gamma_0 m \omega_0}[\delta(k^4 - \vec{k}\cdot\vec{v} + \omega_0(1+vn_0)) + \delta(k^4 - \vec{k}\cdot\vec{v} - \omega_0(1+vn_0))], \tag{54b}$$

$$\tilde{J}^i_{(2)}(\vec{k}, k^4) = \frac{e^2 E_0 \pi v^i k_2}{\gamma_0 m \omega_0^2(1+vn_0)} \times [\delta(k^4 - \vec{k}\cdot\vec{v} + \omega_0(1+vn_0)) - \delta(k^4 - \vec{k}\cdot\vec{v} - \omega_0(1+vn_0))]. \tag{54c}$$

One notices the invariance of relations (54) under $k^\mu \to -k^\mu$. The important thing to notice is that while (54b) and (54c) are "kinematically" similar, (54a) is different from both of them. This simply means that while there might be cross contribution (coherence) between (54b) and (54c) to the power spectrum, there cannot be such a thing between (54a) and (54b) or (54c). Consequently, (54a), (54b) and (54c) can be treated separately.

From (24) we can express the decay rate also as

$$\int dt\, \Gamma(t) = \frac{i}{(2\pi)^4}\int d^4k\, \tilde{J}_\mu(-k)\tilde{J}_\nu(k)\tilde{D}^{\mu\nu}_{(+)}(k) , \tag{24$'$}$$

where, as can be seen from Appendix A, for a medium at rest

$$\tilde{D}^{\mu\nu}_{(+)}(k) = -2\pi i \epsilon^{\mu\nu}(k)\theta_+(k^4)\delta(\eta^{\mu\nu}k_\mu k_\nu) \to -\frac{i\pi}{|\vec{k}|}\delta(nk^4 - |\vec{k}|)\theta_+(k^4)\epsilon^{ij}(k) \tag{55}$$

with $\epsilon^{ij} = g^{ij} - \hat{k}^i\hat{k}^j$.

Clearly $\vec{J}_{(1)}$ and $\vec{J}_{(2)}$ are responsible for stimulated radiation. Denoting with Γ_s the decay rate for stimulated radiation, we write

$$\Gamma_s = \Gamma_{s(1)} + \Gamma_{s(2)} + \Gamma_{s(1,2)} , \tag{56}$$

where (denoting now k^4 with ω, $\omega > 0$),

$$\int dt\, \Gamma^{(t)}_{s(r)} = \frac{\pi}{(2\pi)^4}\int d^3k\, d\omega\, \tilde{J}^i_{(r)}(-\vec{k}, -\omega)(g^{ij} - \hat{k}^i\hat{k}^j)\tilde{J}^j_{(r)}(\vec{k}, \omega)\frac{1}{|\vec{k}|}\delta(n\omega - |\vec{k}|), \quad r = 1, 2 , \tag{57}$$

$$\int dt\, \Gamma_{s(1,2)}(t) = \frac{\pi}{(2\pi)^4}\int d^3k\, d\omega [\tilde{J}^i_{(1)}(-\vec{k}, -\omega)(g^{ij} - \hat{k}^i\hat{k}^j)\tilde{J}^j_{(2)}(\vec{k}, \omega) + \tilde{J}^i_{(2)}(-\vec{k}, -\omega)(g^{ij} - \hat{k}^i\hat{k}^j)\tilde{J}^j_{(1)}(\vec{k}, \omega)]\frac{1}{|\vec{k}|}\delta(n\omega - |\vec{k}|) , \tag{58}$$

where we may assume that $n = n(\omega)$, i.e., that we have dispersion. At this point we wish to define the ϕ-dependent power spectrum $P(\omega, \phi)$. Choosing

$$\vec{k} = |\vec{k}|(\hat{x}\cos\phi \sin\theta + \hat{y}\sin\theta \sin\phi + \hat{z}\cos\theta) , \tag{59}$$

relation (22) is then generalized into

$$\Gamma = \int d\omega\, d\phi \frac{P(\omega, \phi)}{\omega} , \tag{60}$$

and the ϕ dependent power spectrum for the stimulated radiation, as suggested by (56), can also be written as

$$P_s(\omega, \phi) = P_{s(1)}(\omega, \phi) + P_{s(2)}(\omega, \phi) + P_{s(1,2)}(\omega, \phi) . \tag{61}$$

Taking into account that Γ's are actually time independent [$\int dt\, \Gamma = 2\pi\delta(0)\Gamma$], from (57), (58), and (60) we get after some manipulations ($\omega, \omega_0 > 0$)

$$P_{s(1)}(\omega, \phi) = \frac{\alpha^2 E_0^2 \omega}{2m^2\gamma_0^2 v\omega_0^2}\int d\cos\theta (1 - \sin^2\theta \sin^2\phi)\left[\delta\left(\cos\theta - \frac{1}{nv} - \frac{\omega_0(1+vn_0)}{nv\omega}\right) + \delta\left(\cos\theta - \frac{1}{nv} + \frac{\omega_0(1+vn_0)}{nv\omega}\right)\right] , \tag{62}$$

$$P_{s(2)}(\omega,\phi)=\frac{\alpha^2E_0^{\,2}v\omega^3}{2m^2\gamma_0^{\,2}\omega_0^{\,4}(1+vn_0)^2}\int d\cos\theta n^2\sin^4\theta\sin^2\phi\left[\delta\left(\cos\theta-\frac{1}{nv}-\frac{\omega_0(1+vn_0)}{nv\omega}\right)+\delta\left(\cos\theta-\frac{1}{nv}+\frac{\omega_0(1+vn_0)}{nv\omega}\right)\right], \tag{63}$$

$$P_{s(1,2)}(\omega,\phi)=\frac{\alpha^2E_0^{\,2}\omega^2}{\gamma_0^{\,2}m^2\omega_0^{\,3}(1+vn_0)}\int d\cos\theta n\sin^2\theta\cos\theta\sin^2\phi\left[\delta\left(\cos\theta-\frac{1}{nv}-\frac{\omega_0(1+vn_0)}{nv\omega}\right)-\delta\left(\cos\theta-\frac{1}{nv}+\frac{\omega_0(1+vn_0)}{nv\omega}\right)\right]. \tag{64}$$

If one assumes that n is a function of frequency only (an isotropic medium), then relation (62) through (64) can be integrated in principle. The presence of δ functions indicates that the stimulated radiation will occur at fixed angle θ. However, in angle ϕ the radiation will vary as indicated. $P_{s(1)}$ will have maxima at $\phi=0$ and π i.e., in the direction of the Lorentz force acting on the electron. Although angle θ is fixed, it can vary as we vary n, ω, ω_0, etc.

As n is generally a function of ω, and as such can satisfy $n\lessgtr 1$, $n=1$, Eqs. (62) through (64), although integrable in principle, it must, nevertheless, be handled with care. Here we shall assume that n and v are such that we can have $nv>1$ and $nv<1$, respectively. Consequently, we shall say that the power spectrum of stimulated emission belongs to the Čerenkov branch if $nv>1$ and that it belongs to the vacuum branch if $nv<1$. It can be easily seen that both δ functions will contribute to the Čerenkov branch of the power spectrum while only

$$\delta\left(\cos\theta-\frac{1}{nv}+\frac{\omega_0(1+vn_0)}{nv\omega}\right) \tag{65}$$

will contribute to the vacuum branch of the power spectrum. Consequently, integrating over θ as indicated in relations (62) through (64) and taking into account (61), we get for the vacuum branch

$$P_s(\omega,\phi;nv<1)=\frac{\alpha^2E_0^{\,2}}{m^2\gamma_0^{\,2}\omega_0 2v}\left\{\left(\frac{\omega}{\omega_0}\right)\left[1+\frac{\sin^2\phi}{n^2v^2}(1-n^2v^2)\right]+\left(\frac{\omega}{\omega_0}\right)^2\frac{2(n^2v^2-1)}{n^2v^2(1+vn_0)}\sin^2\phi+\left(\frac{\omega}{\omega_0}\right)^3\frac{(1-n^2v^2)^2}{n^2v^2(1+vn_0)^2}\sin^2\phi\right\}. \tag{66}$$

Repeating the same thing for the Čerenkov branch, we get for each of the δ-function contributions to the power spectrum

$\delta\left(\cos\theta-\frac{1}{nv}-\frac{\omega_0(1+vn_0)}{nv\omega}\right)$:

$$P_s^+(\omega,\phi;nv>1)=\frac{\alpha^2E_0^{\,2}}{m^2\gamma_0^{\,2}\omega_0 2v}\left\{\left(\frac{\omega}{\omega_0}\right)\left[1+\frac{\sin^2\phi}{n^2v^2}(1-n^2v^2)\right]+\left(\frac{\omega}{\omega_0}\right)^2\frac{2(1-v^2n^2)}{n^2v^2(1+vn_0)}\sin^2\phi+\left(\frac{\omega}{\omega_0}\right)^3\frac{(1-n^2v^2)^2}{n^2v^2(1+vn_0)^2}\sin^2\phi\right\} \tag{67}$$

and

$\delta\left(\cos\theta-\frac{1}{nv}+\frac{\omega_0(1+vn_0)}{vn\omega}\right)$:

$$P_s^-(\omega,\phi;nv>1)=\frac{\alpha^2E_0^{\,2}}{m^2\gamma_0^{\,2}\omega_0 2v}\left\{\left(\frac{\omega}{\omega_0}\right)\left[1+\frac{\sin^2\phi}{n^2v^2}(1-n^2v^2)\right]+\left(\frac{\omega}{\omega_0}\right)^2\frac{2(n^2v^2-1)}{n^2v^2(1+vn_0)}\sin^2\phi+\left(\frac{\omega}{\omega_0}\right)^3\frac{(1-n^2v^2)^2}{n^2v^2(1+vn_0)^2}\sin^2\phi\right\}. \tag{68}$$

The superscripts + and − refer to the fact that the δ functions in (67) and (68) divide the Čerenkov branch into forward (+) and backward (−) Čerenkov branches. The two branches meet at the angle θ where the usual (spontaneous) Čerenkov radiation occurs:

$$\cos\theta=1/nv\,. \tag{69}$$

If n were not a function of ω (no dispersion at all), then at θ satisfying (69), we would get an infinite ω. Realistically, this will not happen because the dispersion is always present, and even for weak dispersions, (69) can be satisfied for finite ω's. Consequently, we conclude that at angle θ satisfying (69) we should not have stimulated radiation, as the δ functions imply. As a consequence, we do not expect to observe those frequencies ω in the stimulated radiation for which Eq. (69) can be satisfied (which may involve more than one angle θ). With this in mind, we can relate the "frequency upshift" (ω/ω_0) and $\cos\theta$ as follows:

$$\frac{\omega}{\omega_0}=\frac{1+vn_0}{vn\cos\theta-1}\,,\quad \cos\theta>\frac{1}{nv} \tag{70a}$$

for the forward Čerenkov branch, and

$$\frac{\omega}{\omega_0}=\frac{1+vn_0}{1-vn\cos\theta}\,,\quad \cos\theta<\frac{1}{nv} \tag{70b}$$

for the backward Čerenkov branch. Of course, from Eqs. (70a) and (70b) we can read off (ω/ω_0) only when n is independent of ω. When $n=n(\omega)$, which is usually the case, (70a) or (70b) are simply equations from which (ω/ω_0)'s follow as solutions. The question now is whether (70a) and (70b) have any solutions. Specifically, can we satisfy (70a) for $\theta=0$, when

$$\frac{\omega}{\omega_0}=\frac{1+vn_0}{vn-1}\,? \tag{71}$$

As far as we can tell, we do not see any particular reason that (71) could not be satisfied, since now $\cos\theta>1/nv$ and $nv>1$ become the same condition. Equation (71), of course, predicts a possibility of a rather large frequency upshift.

For vacuum branch $(nv<1)$, we get

$$\frac{\omega}{\omega_0}=\frac{1+vn_0}{1-vn\cos\theta}\,, \tag{72}$$

where again it must be emphasized that from (72) we read off ω/ω_0 only when n is independent of ω. Otherwise (72) is an equation from which ω/ω_0 is sought as a solution. We notice that relation (72) is not associated with any restriction on $\cos\theta$. Consequently, we can vary θ at will. For $\theta=\pi/2$, we get

$$\frac{\omega}{\omega_0}=1+vn_0\,, \tag{73}$$

that is, the frequency upshift is determined unambiguously (the same thing is true for the backward Čerenkov branch, since $\theta=\pi/2$ is clearly allowed). Case $\theta=0$ is also quite interesting, since now we get (note $vn<1$)

$$\frac{\omega}{\omega_0}=\frac{1+vn_0}{1-vn}\,, \tag{74}$$

suggesting again a possibility of a rather large frequency upshift.

It is quite clear that if one chooses ω_0 at a microwave frequency ($\omega_0\approx10^{10}$ Hz), then it should be quite easy to achieve experimentally ω at a submillimeter frequency ($\omega\approx10^{12}$ Hz) either through the Čerenkov branch $(nv>1)$ or the vacuum branch $(nv<1)$.

V. DISCUSSION AND CONCLUSION

We have demonstrated the possibility of generating rather large frequency-upshifted radiations through interaction of an electron beam with an external electromagnetic field in a medium. This frequency-upshifted stimulated radiation appears to go either through a vacuum branch $(nv<1)$ or the forward and backward Čerenkov branches $(nv>1)$. The stimulated electromagnetic shock radiation (SESR) of Schneider and Spitzer[2] seems to be most closely related to our stimulated radiation through the forward Čerenkov branch. In the forward direction we have relation (71), while in Ref. 2, one concludes that

$$\frac{\omega}{\omega_0}=\frac{1+vn_0}{(v^2n^2-1)}\,.$$

Although this result disagrees with ours, it nevertheless predicts the frequency upshift. The discrepancy actually is easy to understand. Namely, in Ref. 2 the frequency upshift (ω/ω_0) is deduced from the nonasymptotic electric field, while our results for (ω/ω_0) correspond formally to asymptotic electromagnetic fields of the radiation. Let us point out that the results we give for (ω/ω_0) [relations (70a), (70b), and (72)] are consistent with expressions for (ω/ω_0) given by Frank in his discussion of electron-photon scattering in a refractive medium.[1]

APPENDIX A

In this appendix we wish to give a collection of formulas associated with a free electromagnetic field in a refractive medium. The free electromagnetic field is defined in the usual way through the potential $A^{\mu}_{\rm in}(x)$, which satisfies the equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu A^{\rho}_{\rm in}(x)=0\,, \tag{A1}$$

and the Lorentz condition

$$\eta^{\mu\nu}\partial_\mu A^{\rm in}_{\nu}(x)=0\,, \tag{A2}$$

where

$$\eta^{\mu\nu}=g^{\mu\nu}+(1-n^2)u^\mu u^\nu \tag{A3}$$

is the metric tensor of a medium moving with a four-velocity u^μ [in the rest frame of a medium $u^\mu=(\vec{0},1)$]. The index of refraction n is assumed to be independent of frequency ω. However, it will be easy to generalize final results to n dependent on ω.

The presence of a medium modifies the mass-shell condition for the free photon momentum from $k^2=0$ into

$$k_\mu k_\nu \eta^{\mu\nu}=0\ . \tag{A4}$$

Relation (A4) can be solved, giving k^4 (Ref. 12)

$$k_\pm^4=\frac{1}{N^2}\{u^4(\vec{k}\cdot\vec{u})(n^2-1)\pm[N^2\vec{k}^2-(n^2-1)(\vec{k}\cdot\vec{u})^2]^{1/2}\}\ , \tag{A5}$$

$$N^2=1+(n^2-1)(u^4)^2\ .$$

These two energy solutions, k_+^4 and k_-^4, are neither positive nor negative definite, respectively, except for $\vec{u}=0$ (a medium at rest). However, let us define another four-vector as

$$\tilde{k}^\mu=\eta^{\mu\nu}k_\nu . \tag{A6}$$

Then we see that, in general,

$$\tilde{k}^4=N^2k^4-(n^2-1)u^4(\vec{k}\cdot\vec{u})\ , \tag{A7}$$

giving

$$\begin{aligned}\tilde{k}_\pm^4 &\equiv \pm\epsilon(k)=N^2k_\pm^4-(n^2-1)u^4(\vec{k}\cdot\vec{u})\\ &=\pm[N^2\vec{k}^2-(n^2-1)(\vec{k}\cdot\vec{u})^2]^{1/2}\\ &=\pm\tfrac{1}{2}N^2(k_+^4-k_-^4)\ .\end{aligned} \tag{A8}$$

It is not difficult to verify now that, for example, $\tilde{k}_+^4$ is indeed positive-definite. Writing $A_\mu^{\text{in}}(x)$ as a sum of the positive- ($A_\mu^{\text{in}(+)}$) and negative- ($A_\mu^{\text{in}(-)}$) frequency parts which are defined as

$$\begin{aligned}A_\mu^{\text{in}(+)}(x)&=\frac{1}{4\pi^{3/2}}\int d\tilde{\sigma}(k)a_\mu(k)e^{i(\vec{k}\cdot\vec{x}-k_+^4t)}\ ,\\ A_\mu^{\text{in}(-)}(x)&=A_\mu^{\text{in}(+)\dagger}(x),\quad d\tilde{\sigma}(k)=\frac{d^3k}{\epsilon(k)}\ ,\end{aligned} \tag{A9}$$

the quantization, valid in any reference frame with respect to which a medium is moving a four-velocity u^μ, is carried out as follows:

$$\begin{aligned}[a^\mu(k),a^{\dagger\nu}(k')]&=\epsilon^{\mu\nu}(k)\tilde{\delta}(k-k')\ ,\\ \tilde{\delta}(k-k')&=\delta^{(3)}(\vec{k}-\vec{k}')\epsilon(k)\ .\end{aligned} \tag{A10}$$

In (A10) $\epsilon^{\mu\nu}$ is defined as

$$\epsilon^{\mu\nu}(k)=g^{\mu\nu}+u^\mu u^\nu-\tilde{k}_0^\mu\tilde{k}_0^\nu\ , \tag{A11}$$

where unit vector $\tilde{k}_0^\mu$ is given with the expression

$$\tilde{k}_0^\mu=\frac{\tilde{k}^\mu+(\tilde{k}\cdot u)u^\mu}{[\tilde{k}^2+(\tilde{k}\cdot u)^2]^{1/2}}\ . \tag{A12}$$

One verifies the consistency of (A10) with (A2) by noting that

$$\tilde{k}^\mu\epsilon_{\mu\nu}(k)=\eta^{\mu\rho}k_\rho\epsilon_{\mu\nu}(k)=0\ . \tag{A13}$$

One verifies also these relations:

$$\begin{aligned}&\tilde{k}_0^\mu|_{\vec{u}=0}=(\hat{k}^{(i)},0),\quad \tilde{k}_0\cdot u=0\ ,\\ &u_\mu\epsilon^{\mu\nu}(k)=u_\nu\epsilon^{\mu\nu}(k)=0,\\ &\tilde{k}_0^\mu\epsilon_{\mu\nu}(k)=\tilde{k}_0^\nu\epsilon_{\mu\nu}(k)=0\ ,\end{aligned} \tag{A14}$$

where $\hat{k}^i$ is denoting the space components of a photon unit vector in the rest frame of a medium. In this appendix, in order to avoid confusion, an index of a vector will be written in parentheses whenever it is denoting the components in the rest frame of a medium. For example, $u^{(\mu)}$ means $(\vec{0},1)$. Also, if we have a tensor, then with respect to a particular index, this tensor can be evaluated in the rest frame of a medium.

One easily verifies

$$\epsilon_\lambda^\mu\epsilon^{\nu\lambda}=\epsilon^{\mu\nu}. \tag{A15}$$

Let us now denote

$$\epsilon^\mu(k;\alpha)=\epsilon^\mu(k)_{(\alpha)}\ , \tag{A16}$$

where as mentioned above, the "tensor" $\epsilon_{(\alpha)}^\mu$ is evaluated with respect to index α in the rest frame of a medium. Taking into account (A14) and choosing

$$\hat{k}^{(i)}=\delta_{i3}\ , \tag{A17}$$

we get

$$\epsilon^\mu(k;3)=0,\quad \epsilon^\mu(k;4)=0\ . \tag{A18}$$

In view of (A15) we notice the important relation

$$\sum_{\alpha=1,2}\epsilon^\mu(k;\alpha)\epsilon^\nu(k;\alpha)=\epsilon^{\mu\nu}(k)\ , \tag{A19}$$

Expanding $a^\mu(k)$ as

$$a^\mu(k)=\sum_{\alpha=1,2}\epsilon^\mu(k;\alpha)a(k;\alpha)\ , \tag{A20}$$

we clearly have

$$[a(k;\alpha),a^\dagger(k';\alpha')]=\delta_{\alpha\alpha'}\tilde{\delta}(k-k')\ , \tag{A21}$$

where $a(k;\alpha)$ and $a^\dagger(k;\alpha')$ are the annihilation and creation photon operators with polarization α and α', respectively.

In the rest frame of a medium, $\epsilon^{\mu\nu}$ clearly reduces to

$$\epsilon^{\mu\nu}(k)|_{\vec{u}=0} \to \epsilon^{(i)(j)}(k) = g^{(i)(j)} - \hat{k}^{(i)}\hat{k}^{(j)}, \quad \text{(A22)}$$

a result used in the text (written without parentheses around the indices). Our quantization procedure selects explicitly the radiation gauge in the rest frame of a medium (quantization frame), since for $\vec{u}=0$, $\epsilon^{\mu}(k;\alpha) = (\epsilon^{(i)}(k;\alpha), 0)$. Such a quantization procedure for the case of a vacuum can be found, for example, in Ref. 13.

Now let us summarize singular functions. Consistent with (A15), we have

$$[A_\mu^{\text{in}(+)}(x), A_\nu^{\text{in}(-)}(y)] = i\, D_{\mu\nu}^{(+)}(x-y) = -i\, D_{\nu\mu}^{(-)}(y-x)\,, \quad \text{(A23)}$$

with

$$D_{\mu\nu}^{(\pm)}(x) = \frac{1}{(2\pi)^4}\int d^4k\, \tilde{D}_{\mu\nu}^{(\pm)}(k) e^{ikx}\,,$$
$$\tilde{D}_{\mu\nu}^{(\pm)}(k) = \mp\, 2\pi i\, \epsilon_{\mu\nu}(k)\delta_{(\pm)}(\eta^{\rho\sigma}k_\rho k_\sigma)\,, \quad \text{(A24)}$$

$$\delta_{(\pm)}(\eta^{\mu\nu}k_\mu k_\nu) = \theta_\pm(\tilde{k}^4)\delta(\eta^{\mu\nu}k_\mu k_\nu)\,. \quad \text{(A25)}$$

In evaluating (A24) with (A25), one should notice that [see (A7) and (A8)]

$$\delta(\eta^{\mu\nu}k_\mu k_\nu) = \frac{N^2}{2\epsilon(k)}\left[\delta(\tilde{k}^4 - \epsilon(k)) + \delta(\tilde{k}^4 + \epsilon(k))\right],$$
$$dk^4 = \frac{1}{N^2}\, d\tilde{k}^4\,. \quad \text{(A26)}$$

The retarded, advanced, Feynman, and anti-Feynman Green's functions are defined as usual:

$$D_{R,A}^{\mu\nu}(x) = \mp\, \theta_\pm(t) D^{\mu\nu}(x), \quad \text{(A27)}$$

$$D_{F,F^*}^{\mu\nu}(x) = \theta_-(t) D_{-,+}^{\mu\nu}(x) - \theta_+(t) D_{+,-}^{\mu\nu}(x)\,, \quad \text{(A28)}$$

with

$$D_{\mu\nu}(x) = D_{\mu\nu}^{(+)}(x) + D_{\mu\nu}^{(-)}(x)\,. \quad \text{(A29)}$$

In terms of Fourier integrals, the singular functions (A27) and (A28) can be written as

$$D_c^{\mu\nu}(x) = \frac{1}{(2\pi)^4}\int_c d^4k\, \frac{\epsilon^{\mu\nu}(k)e^{ikx}}{\eta^{\rho\sigma}k_\rho k_\sigma}$$
$$\equiv \int d^3k\, d\tilde{k}^4 \tilde{D}_c^{\mu\nu}(k) e^{ikx}, \quad c = R, A, F, F^* \quad \text{(A30)}$$

where

$$\tilde{D}_{R,A}^{\mu\nu}(k) = \frac{\epsilon^{\mu\nu}(k)}{[\epsilon(k) - (\tilde{k}^4 \pm i\sigma)][(\tilde{k}^4 \pm i\sigma) + \epsilon(k)]}\,,$$
$$\tilde{D}_{F,F^*}^{\mu\nu}(k) = \frac{\epsilon^{\mu\nu}(k)}{[\epsilon(k) - \tilde{k}^4][\tilde{k}^4 + \epsilon(k)] \mp i\sigma}\,, \quad \sigma \to +0. \quad \text{(A31)}$$

In the text we also had

$$\bar{D}_{\mu\nu}(x) = \tfrac{1}{2}[D_{\mu\nu}^R(x) + D_{\mu\nu}^A(x)]$$
$$= \tfrac{1}{2}[D_{\mu\nu}^F(x) + D_{\mu\nu}^{F*}(x)]\,,$$
$$D_{\mu\nu}^{(1)}(x) = i\,[D_{\mu\nu}^{(+)}(x) - D_{\mu\nu}^{(-)}(x)]$$
$$= i\,[D_{\mu\nu}^{F*}(x) - D_{\mu\nu}^F(x)]\,, \quad \text{(A32)}$$

$$\tilde{D}_{\mu\nu}^{(1)}(k) = 2\pi\epsilon_{\mu\nu}(k)\delta(\eta^{\rho\sigma}k_\rho k_\sigma)\,. \quad \text{(A33)}$$

properties such as

$$D_{\mu\nu}^{(+)}(x) = -D_{\mu\nu}^{(-)}(-x)\,,$$

etc. can be easily verified.

APPENDIX B

Here we wish to derive briefly result (51) used in Sec. IV B.

In order to determine $\vec{r}(t)$, a change in the electron trajectory due to the interaction with, say, laser field (46), we start with the Lorentz force equation

$$m\frac{d}{dt}[\gamma\dot{\vec{R}}(t)] = e[\vec{E}(\vec{R}(t), t) + \dot{\vec{R}}(t)\times\vec{H}(\vec{R}(t), t)]\,,$$
$$\gamma^2 = (1 - \dot{\vec{R}}^2(t))^{-1}\,. \quad \text{(B1)}$$

Writing

$$\vec{R}(t) = \vec{R}_0(t) + \vec{r}(t)\,,$$
$$\vec{R}_0(t) = \vec{v}\,t\,, \quad \vec{v} = \hat{z}v\,, \quad \text{(B2)}$$

where $\vec{v}$ is the velocity of an unperturbed electron, we get to the lowest order in $e\,[|\tilde{\gamma}| \sim O(e)]$

$$m\gamma_0[\gamma_0^2 v\ddot{z}(t)\vec{v} + \ddot{\vec{r}}(t)]$$
$$= e[\vec{E}(\vec{R}_0(t), t) + \vec{v}\times\vec{H}(\vec{R}_0(t)t)]\,, \quad \gamma_0^2 = (1-v^2)^{-1}. \quad \text{(B3)}$$

In view of (B3), we notice that

$$\ddot{z}(t) = \ddot{x}(t) = 0 \quad \text{(B4)}$$

and

$$\ddot{y}(t) = \frac{e}{m\gamma_0}E_0(1 + vn_0)\sin\omega_0 t(1 + vn_0)\,. \quad \text{(B5)}$$

In order to solve these equations, we introduce "one-dimensional" retarded Green's function

$\mathfrak{D}_R(t;\mu)$ satisfying

$$\left(\frac{d^2}{dt^2}+\mu^2\right)\mathfrak{D}_R(t;\mu)=\delta(t)\,. \qquad \text{(B6)}$$

The solution of (B6) can be cast in the form

$$\mathfrak{D}_R(t;\mu)=\frac{1}{2\pi}\int d\omega\,\frac{e^{-i\omega t}}{\mu^2-(\omega+i\epsilon)^2}\,,\quad \epsilon\to+0 \qquad \text{(B7)}$$

from which one can easily show that

$$\mathfrak{D}_R(t;\mu)=\theta(t)\frac{\sin\mu t}{\mu}\,. \qquad \text{(B8)}$$

The solution of (B4) and (B5) is seen to be

$$\vec{\gamma}(t)=\hat{y}y(t)=\frac{e}{m\gamma_0}\,\vec{\mathrm{E}}_0(1+vn_0)\lim_{\mu\to 0}\int dt'\,\mathfrak{D}_R(t-t';\mu)\times\sin\omega_0 t'(1+vn_0)\,, \qquad \text{(B9)}$$

a result used in the text.

APPENDIX C

Here we give a brief derivation by means of the PDECC formalism of the unitary $\tilde{S}$ matrix [relation (18)], which can be used in place of the S matrix [relation (16)] for calculations in this article.

The differential equations involving the S matrix are similar to those involving the S matrix [relations (1) through (3)], except that instead of relation (2) now we have

$$\frac{1}{i}\frac{d}{d\lambda}\tilde{S}(\lambda)\tilde{A}_\mu(x)=\int d^4y\tilde{S}(\lambda)\,\frac{e}{2}\,[\tilde{A}_\mu(x)J_\nu(y)\tilde{A}^\nu(y)+J_\nu(y)\tilde{A}^\nu(y)\tilde{A}_\mu(x)]\,, \qquad \text{(C1)}$$

where, for the sake of clarity, we changed the notation for the quantized electromagnetic potential from $A_\mu(x)$ to $\tilde{A}_\mu(x)$. Relation (C1) immediately suggests that the T_A operator from Sec. II be replaced with symmetrization operator τ_A (acting on $A_\mu^{\rm in}$'s). In view of this, now by definition

$$T=T_\psi\tau_A\,. \qquad \text{(C2)}$$

Going through the same exercise as we did with the equations involving the S matrix [relations (6) through (11)], we arrive now at

$$\frac{1}{i}\frac{d}{d\lambda}\tilde{S}'(\lambda)=\tilde{S}'(\lambda)\int d^4x\,\tilde{\mathfrak{L}}'_{\rm int}(x),$$

$$\frac{1}{i}\frac{d}{d\lambda}\tilde{S}'(\lambda)\tilde{A}'_\mu(x)=\int d^4y\,\tilde{S}'(\lambda)\tau_A\tilde{A}'_\mu(x)\,\tilde{\mathfrak{L}}'_{\rm int}(x)\,, \qquad \text{(C3)}$$

$$\frac{1}{i}\frac{d}{d\lambda}\psi'(x)=\frac{1}{i}\frac{d}{d\lambda}\psi_{\rm in}(x)=0\,,$$

where

$$\tilde{\mathfrak{L}}'_{\rm int}(x)=J_\mu^{\rm in}(x)\tilde{A}'^\mu(x)\,. \qquad \text{(C4)}$$

Now since

$$\tau_A A_\mu^{\rm in}(x)A_\nu^{\rm in}(y)=\tfrac{1}{2}[A_\mu^{\rm in}(x)A_\nu^{\rm in}(y)+A_\nu^{\rm in}(y)A_\mu^{\rm in}(x)]=T_A A_\mu^{\rm in}(x)A_\nu^{\rm in}(y)|_{\theta\to1/2}\,, \qquad \text{(C5)}$$

we see that the solutions of (C3) can be obtained formally by setting $\theta\to\frac{1}{2}$ into the solutions of (12). This replacement rule immediately gives

$$\tilde{A}'_\mu(x)=A_\mu^{\rm in}(x)-\tfrac{1}{2}\lambda\int d^4y\,D_{\mu\nu}(x-y)J^\nu_{\rm in}(y)\,, \qquad \text{(C6)}$$

where (see Appendix A)

$$D_R^{\mu\nu}(x)|_{\theta\to1/2}=-D^{\mu\nu}(x)$$

was taken into account. Now it is not difficult to see that the replacement of T_A with τ_A immediately yields the $\tilde{S}$ matrix (18) from the S matrix (16) since

$$\bar{D}^{\mu\nu}(x)|_{\theta\to1/2}=0\,,$$

as can be easily seen from (A27) and (A32).

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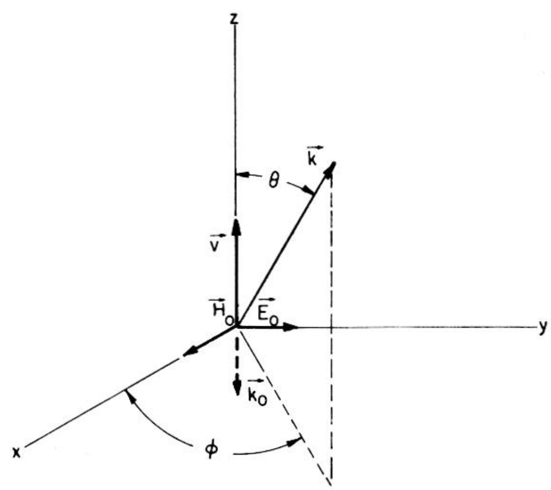

FIG. 1. Schematics of head-on electron-laser field collision for stimulated radiation from Sec. IV B. Velocity $\vec{v}$ of the incident electron and wave vector $\vec{k}_0$ of the sinusoidal laser field are antiparallel and chosen to be along the z axis. The constant electric ($\vec{E}_0$) and magnetic ($\vec{H}_0$) fields define the polarization of a laser field. The momentum of radiated photon is denoted by $\vec{k}$.