Renormalization group and slope parameter

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The renormalization-group method has been used to give the asymptotic behavior of the physical fixed-angle scattering amplitude in ϕ^4 field theory. This method when coupled with a rigorous inequality and additional assumptions leads to a new lower bound on the slope parameter. A new lower bound on the total cross section is also obtained.

It has been proved that the renormalization-group method can be effectively used in studying the high-energy behavior of semileptonic processes. This method gives[1] the asymptotic behavior in energy of the fixed-angle elastic-scattering amplitude in ϕ^4 field theory:

$$F(s,\theta)\sim s^{-2\gamma(g_\infty)}h(\theta)\,,\quad \theta\neq 0,\pi \tag{1}$$

where s is the c.m. energy squared, θ is the c.m. scattering angle, and $\gamma(g_\infty)$ is the anomalous dimension of the field ϕ at the ultraviolet fixed point $g=g_\infty$. On the basis of Eq. (1) and two assumptions with respect to the convergence of the integrals in $\cos\theta$, Khuri[2] obtained the following upper bound on the total cross section:

$$\sigma_{\text{tot}}(s)\leqslant Cs^{-2\gamma(g_\infty)}(\ln s)^2 \tag{2}$$

for at least one sequence of $s\to\infty$ under the assumption of the crossing-even property of the scattering amplitude. Here C refers only to a certain positive constant and should not be taken as the same value. He also used the rigorous inequality of Bessis[3] and Singh[4]

$$\frac{\text{Im}F(s,t)}{\text{Im}F(s,0)}\geqslant 1+\frac{t}{16m^2}\left[\ln\left(\frac{s}{s_0{}^2\sigma_{\text{tot}}}\right)\right]^2, \tag{3}$$

and the Jin-Martin lower bound[5]

$$\sigma_{\text{tot}}(s)\geqslant Cs^{-6} \tag{4}$$

for at least one sequence of $s\to\infty$ under the assumption of the crossing-even property of the amplitude. In this paper we obtain the new lower bound of the slope parameter. Also, we can improve the condition "for at least one sequence of $s\to\infty$" of the bound (2) as "for any sequence of $s\to\infty$", because it is unnecessary to use inequalities (3) and (4).

Now let us derive a new lower bound on the slope parameter. We know[6] the lower bound on the imaginary part of the scattering amplitude is

$$\frac{\text{Im}F(s,t)}{\text{Im}F(s,0)}\geqslant J_0(R(-t)^{1/2}/\sqrt{2}\,)\quad\text{as}\quad s\to\infty\,, \tag{5}$$

where

$$\frac{R^2}{8}\equiv B\equiv\frac{(d/dt)\,\text{Im}F(s,t)|_{t=0}}{\text{Im}F(s,0)}\,. \tag{6}$$

This bound holds in $t_1<t<4m_0{}^2$. Here t_1 is the value such that $R(-t_1)^{1/2}/\sqrt{2}$ is the minimum of $J_0(R(-t)^{1/2}/\sqrt{2})$, i.e., $R(-t_1)^{1/2}/\sqrt{2}=3.83$; and $2m_0$ is the threshold energy. So we obtain

$$\begin{aligned}\int_{1-\rho}^{1}|F(s,\theta)|\,d\cos\theta&=\int_{1-\rho}^{1}|\text{Im}F(s,t)|\,d\cos\theta=2s^{-1}\int_{-s\rho/2}^{0}|\text{Im}F(s,t)|\,dt\\&\geqslant 2s^{-1}\int_{-a/R^2}^{0}|\text{Im}F(s,t)|\,dt\\&\geqslant 2s^{-1}R^{-2}\,\text{Im}\,F(s,0)\int_{-a/R^2}^{0}J_0(R(-t)^{1/2})R^2dt\\&=CaR^{-2}\sigma_{\text{tot}}(s)\,.\end{aligned} \tag{7}$$

Here ρ is a certain constant such that $0<\rho<1$, and a is bounded as $0<a\leqslant(3.83)^2$ and $aR^{-2}\leqslant s\rho/2$. From Eq. (1) we have

$$\int_{1-\rho}^{1}|F(s,\theta)|\,d\cos\theta=s^{-2\gamma(g_\infty)}\int_{1-\rho}^{1}|h(\theta)|\,d\cos\theta+Cs^{-2\gamma(g_\infty)} \tag{8}$$

It was noted by Khuri[2] that the assumption of the admissibility of interchanging the large-s limit with the integration over $d(\cos\theta)$ was necessary in the estimate (8). Hence in the case $\lim_{s\to\infty} sB \neq 0$, we get the lower bound on the slope parameter

$$B \geq C\sigma_{tot}(s)s^{2\gamma(g_\infty)} \text{ as } s\to\infty, \tag{9}$$

because in our case we can take $h(s)$ as constant. If the integral of $|h(\theta)|$ in (8) is infinite, then we get $C=0$ in (9) and so our bound is trivial. If this integral is finite, we have a positive constant C in (9). This bound (9) is the main result of our research. For scalar particles in a positive-metric theory, $\gamma_\phi \geq 0$ follows from the Källên-Lehmann representation of the two-point function.

Also, we know that the upper bound on the slope parameter is

$$B(s) \leq C(\ln s)^2 \text{ as } s\to\infty. \tag{10}$$

This is easily obtained[7] by the cutting of the partial waves up to $L \leq C\sqrt{s}\ln s$. This cutting is also used in order to derive the Froissart bound.[8] Then we get the upper bound (2) "for any sequence of $s\to\infty$".

In the case $\lim_{s\to\infty} s\, B \neq \infty$ and $B\neq 0$, we obtain the bound

$$\sigma_{tot}(s) \leq C s^{-1-2\gamma(g_\infty)} \text{ as } s\to\infty, \tag{11}$$

because we can take a as CsB. It is noted that our upper bound (11) holds even if $sB\to 0$ as $s\to\infty$.

Lastly let us derive the lower bound on the total cross section. From Eq. (1) we have

$$\begin{aligned}\sigma_{tot}(s) \geq \sigma_{el}(s) &= \int dt \frac{d\sigma}{dt} = \int dt \left|\frac{F(s,\theta)}{s}\right|^2 \\ &= s^{-4\gamma(g_\infty)-1}\int d\cos\theta\,|h(\theta)|^2 \\ &= Cs^{-4\gamma(g_\infty)-1} \text{ as } s\to\infty. \end{aligned} \tag{12}$$

This is a new lower bound on the total cross section in comparison with the Jin-Martin lower bound,[5,9] when $\gamma(g_\infty) < \frac{5}{4}$.

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[1]M. Creutz and L.-L. Wang, Phys. Rev. D 10, 3749 (1974); C. G. Callan, Jr. and D. J. Gross, *ibid.* 11, 2904 (1975); G. Tiktopoulos, *ibid.* 11, 2252 (1975).
[2]N. N. Khuri, Phys. Rev. D 14, 2665 (1976).
[3]J. D. Bessis, Nuovo Cimento 45A, 974 (1966).
[4]V. Singh, Phys. Rev. Lett. 26, 530 (1971).
[5]Y. S. Jin and A. Martin, Phys. Rev. 135, B1369 (1964); H. Cornille, Nuovo Cimento 4A, 549 (1971).
[6]T. Uchiyama, Phys. Rev. D 10, 999 (1974).
[7]See also T. Kinoshita, *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, 1965), Vol. 7B; K. L. Kowalski, Phys. Rev. 137, B1350 (1965).
[8]M. Froissart, Phys. Rev. 123, 1053 (1961); A. Martin, *ibid.* 129, 1432 (1963); T. Uchiyama, Prog. Theor. Phys. 45, 1960 (1971).
[9]T. Uchiyama, J. Math. Phys 19, 409 (1978). T. Uchiyama, Prog. Theor. Phys. 55, 1871 (1976).