PHYSICAL REVIEW D VOLUME 18, NUMBER 6 15 SEPTEMBER 1978

Landau gauge formalism for non-Abelian gauge theories

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(Received 14 June 1977)

A local operator formulation of non-Abelian gauge theories in the Landau gauge is presented and discussed. The formalism involves the usual gauge fields $\vec{A}_\mu$, matter fields, unphysical ghost fields, and a further multiplet of unphysical local scalar fields $\vec{B}$. The gauge-fixing term in the Lagrangian is $\vec{B}\cdot(\partial\cdot\vec{A})$, which replaces the usual term $(1/\alpha)(\partial\cdot\vec{A})^2$ characteristic of the generalized Lorentz gauges. The $\vec{B}$ field, formally the limit of $(1/\alpha)\partial\cdot\vec{A}$ for $\alpha\to 0$, thus provides a local momentum operator which is canonically conjugate to $\vec{A}_0$, and generates the Landau-gauge relation $\partial\cdot\vec{A}=0$ as a field equation. Both operator and functional methods are used to deduce the transversality conditions, Slavnov identities, and renormalization-group equations obeyed by the Green's functions. A functional formalism for vertex functions is presented, and it is shown that these functions are well defined in spite of the fact that the *AA* propagator has no inverse and the *BB* propagator vanishes. The gauge-field vertex functions are shown to be the $\alpha\to 0$ limits of those in the Lorentz gauges.

I. INTRODUCTION

It is now widely believed that non-Abelian gauge theories (NAGT's) provide the framework for a possibly unified description of the strong, electromagnetic, and weak interactions. One distinctive feature of gauge theories is the freedom to perform calculations in various gauges, with all physical results being gauge-invariant. The most important are the Lorentz gauges, obtained by appending the gauge-fixing term

$$-\frac{1}{2\alpha}(\partial\cdot\vec{A})^2$$

to the classical Lagrangian. This results in a bare gauge-field propagator

$$i\Delta_{\mu\nu}^{(0)ab}=\left[\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+\alpha\,\frac{k_\mu k_\nu}{k^2}\right]\frac{1}{k^2}\delta^{ab}\ . \qquad (1.1)$$

Although the Lagrangian itself appears to be singular for $\alpha\to 0$, the Green's functions are clearly well-defined in that limit. Indeed, the Landau gauge ($\alpha=0$) is particularly simple, corresponding to a transverse propagator. This feature survives higher-order corrections and renormalization, so that in the Landau gauge the effective gauge parameter $\bar{\alpha}$, in the sense of the renormalization group (RG), remains zero. The RG analysis, normally involving two coupling constants (the gauge coupling, g, and α) reduces to the much more tractable single-coupling case. Thus the Landau gauge is extremely useful for the understanding of ultraviolet (UV) and infrared (IR) asymptotic behavior.

On the other hand, the equations of motion together with the canonical equal-time commutators (ETC's) provide an important complementary tool to the computation of Green's functions via Feynman graphs. Asymptotic freedom (AF) sufficiently softens the short-distance singularities to permit a study of certain aspects of the exact (rather than the order-by-order) behavior of the theory by this method.[1] Indeed it has been shown that the results so obtained agree with the more conventional RG analysis.[2]

Symmetries arising from renormalization, such as R invariance,[1,3] can be conveniently studied using the equations of motion. The low-energy theorems due to R invariance may contradict the existence of an IR fixed point and thus afford a proof of quark confinement. However, this picture is marred by the fact that the RG analysis is tractable only in the Landau gauge, whereas the equations of motion derived from the conventional Lagrangian do not formally have a well-defined $\alpha\to 0$ limit.

It is the purpose of this paper to study the equations of motion and their consequences for NAGT's in the Landau gauge. Here we follow the work of Symanzik[5] in the Abelian case and make use of a first-order Lagrangian, in which $(1/\alpha)(\partial\cdot\vec{A})$ is replaced by a new multiplet of scalar fields $\vec{B}$. Unlike the Abelian case, however, $\vec{B}$ is not a free field, and yet it effectively drops out from the problem, as intuitively it must. The $\vec{B}$ field provides a local momentum operator canonically conjugate to $\vec{A}_0$. Its presence will be seen to be crucial in obtaining a Lorentz-invariant theory and in defining one-particle-irreducible (1PI) vertex functions.

The generating-functional formalism for Green's functions and the Feynman rules used in this paper are not new. The primary new results are the *local*-field-equation formalism and the generating-functional formalism for 1PI *vertex* functions. In previous treatments, local field equations and vertex functions were not discussed. For example, in Ref. 6, one of the first papers to discuss the Feynman rules, the local $\vec{B}$ field is, unlike ours, a free field and the field equations involve another $\vec{B}'$ field which is a nonlocal function of $\vec{B}$. The other important early treatments[7,8] and the more recent discussions of renormalization[9-12] also did not consider local field equations or define vertex functions for the Landau gauge.

We present the first-order formalism in detail in Sec. II. We exhibit there the Lagrangian, field equations, commutation relations, gauge transformations, and Slavnov[11] transformations. In Sec. III we deduce properties of the Green's functions. The Feynman rules, transversality conditions, consequences of Slavnov invariance, and renormalization-group equations are discussed. The functional formalism for Green's functions is given in Sec. IV. The generating functional is shown to be equivalent to that of Ref. 6. Some further properties of the Green's functions are also deduced. In Sec. V we present a functional formalism for vertex functions. It is shown that, because of the coupling to the B fields, the vertex functions are well-defined in spite of the fact that the AA propagator has no inverse and the BB propagator vanishes. The gauge-field vertex functions are shown to be the $\alpha \to 0$ limits of those in the Lorentz gauges and the vertex functions involving at least one B field, except for $\Gamma(AB)$, are shown to vanish. We conclude in Sec. VI with a summary of our results.

II. FIRST-ORDER FORMALISM

The quantization of NAGT's in the Lorentz gauges is usually discussed in terms of the second-order Lagrangian[6-12]

$$\mathcal{L}_2 = -\tfrac{1}{4}\vec{G}_{\mu\nu}\cdot\vec{G}^{\mu\nu} - \frac{1}{2\alpha}(\partial\cdot\vec{A})^2 + \mathcal{L}_G + \mathcal{L}_F \,. \tag{2.1}$$

Here $\vec{A}_\mu$ are the gauge fields, and the field strength $\vec{G}_{\mu\nu}$ is defined by

$$\vec{G}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\vec{A}_\mu\times\vec{A}_\nu \,, \tag{2.2}$$

where the cross product $\vec{a}\times\vec{b}$ has components $f^{abc}a^b b^c$ defined by the structure constants f^{abc}. The ghost and fermion parts of the Lagrangian are

$$\mathcal{L}_G = -\partial_\mu\vec{C}_2\cdot\mathfrak{D}^\mu\vec{C}_1 \,, \tag{2.3}$$

$$\mathcal{L}_F = i\bar{\psi}\gamma_\mu D^\mu\psi \,, \tag{2.4}$$

where $\vec{C}_2, \vec{C}_1$ are ghost fields, ψ is the fermion field multiplet, and

$$\mathfrak{D}^\mu_{ab} = \delta^{ab}\partial^\mu + gf^{acb}A^{c\mu} \,, \tag{2.5}$$

$$D^\mu = 1 - igA^{a\mu}T^a \,, \tag{2.6}$$

T^a being the fermion representation matrices.

For $\alpha \neq 0$, (2.1) is equivalent to the first-order Lagrangian

$$\mathcal{L}_1 = -\tfrac{1}{2}\vec{G}_{\mu\nu}\cdot(\partial^\mu\vec{A}^\nu - \partial^\nu\vec{A}^\mu + g\vec{A}^\mu\times\vec{A}^\nu) + \tfrac{1}{4}\vec{G}_{\mu\nu}\cdot\vec{G}^{\mu\nu} - \vec{B}\cdot\partial\vec{A} + \tfrac{1}{2}\alpha\vec{B}^2 + \mathcal{L}_G + \mathcal{L}_F \,, \tag{2.7}$$

where $\vec{G}_{\mu\nu}$, $\vec{A}_\mu$, and $\vec{B}$ are regarded as independent dynamical variables. This can be easily seen by functional methods (see Sec. IV) or simply by noting that (2.1) and (2.7) lead to identical equations of motion.

The first-order Lagrangian (2.7) has the advantage that one can formally take the $\alpha \to 0$ limit, wherein

$$\mathcal{L} = -\tfrac{1}{2}\vec{G}_{\mu\nu}\cdot(\partial^\mu\vec{A}^\nu - \partial^\nu\vec{A}^\mu + g\vec{A}^\mu\times\vec{A}^\nu) + \tfrac{1}{4}\vec{G}_{\mu\nu}\cdot\vec{G}^{\mu\nu} - \vec{B}\cdot\partial\vec{A} + \mathcal{L}_G + \mathcal{L}_F \,. \tag{2.8}$$

Equation (2.8) is the basis of the Landau gauge formalism to be studied in this paper. It will be used to deduce the equations of motion, commutation relations and to define, as functional integrals, Green's functions and vertex functions. Useful properties of these functions will be deduced and the functions will be seen to be the $\alpha \to 0$ limits of the corresponding Lorentz gauge functions defined from (2.7).

Variation of $\vec{G}_{\mu\nu}$, $\vec{A}_\mu$, and $\vec{B}$ in (2.8), respectively, leads to the following equations of motion:

$$\vec{G}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\vec{A}_\mu\times\vec{A}_\nu \,, \tag{2.9}$$

$$\mathfrak{D}^\mu\vec{G}_{\mu\nu} + \partial_\nu\vec{B} = g(\partial_\nu\vec{C}_2\times\vec{C}_1 - \bar{\psi}\gamma_\nu\vec{T}\psi) \,, \tag{2.10}$$

$$\partial\cdot\vec{A} = 0 \,. \tag{2.11}$$

Equation (2.10) can also be written in the form

$$\partial^\mu(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu) + \partial_\nu\vec{B} = -\vec{K}_\nu \tag{2.12}$$

where

$$\vec{K}_\nu = g[\partial^\mu(\vec{A}_\mu\times\vec{A}_\nu) + \vec{A}^\mu\times\vec{G}_{\mu\nu} - \partial_\nu\vec{C}_2\times\vec{C}_1 + \bar{\psi}\gamma_\nu\vec{T}\psi] \,. \tag{2.13}$$

Thus $\vec{K}_\nu$ (or rather $\vec{K}_\nu + \partial_\nu\vec{B}$) is in the conventional sense the source of the gauge field. The gauge-invariant divergence of (2.10) gives another equation of interest:

$$\Box\vec{B}=g(\partial^{\mu}\vec{B}\times\vec{A}_{\mu}+\partial_{\mu}\vec{C}_{2}\times\mathfrak{D}^{\mu}\vec{C}_{1}) , \quad (2.14)$$

showing that $\vec{B}$ is *not* a free field. Since the right-hand side of (2.14) is a group-theoretic cross product, it is clear that in the Abelian case B would be a free field.[5] The $\vec{B}$ field plays a more interesting role in the non-Abelian case.

The momenta conjugate to $\vec{A}_{\mu}$ are

$$\vec{\Pi}^{\mu}=\frac{\delta\mathfrak{L}}{\delta\partial_{0}\vec{A}_{\mu}} , \quad (2.15)$$

and obey the canonical ETC

$$[A_{\mu}^{a}(x),\ \Pi^{b\nu}(y)]\,\delta(x_{0}-y_{0})=i\delta^{ab}\delta_{\mu}^{\nu}\delta^{4}(x-y) . \quad (2.16)$$

Evaluating (2.15) we find

$$\vec{\Pi}^{i}=-\vec{G}_{0i} , \qquad \vec{\Pi}^{0}=-\vec{B} , \quad (2.17)$$

so that (2.16) gives

$$[A_{i}^{a}(x),G_{0j}^{b}(y)]\,\delta(x_{0}-y_{0})=i\delta^{ab}\delta_{ij}\delta^{4}(x-y) , \quad (2.18a)$$

$$[A_{i}^{a}(x),B^{b}(y)]\delta(x_{0}-y_{0})=0 , \quad (2.18b)$$

$$[A_{0}^{a}(x),G_{0j}^{b}(y)]\delta(x_{0}-y_{0})=0 , \quad (2.18c)$$

$$[A_{0}^{a}(x),B^{b}(y)]\delta(x_{0}-y_{0})=-i\delta^{ab}\delta^{4}(x-y) . \quad (2.18d)$$

Another ETC of interest is $[\partial_{0}B,\ B]$, which can be obtained by commuting the zeroth component of Eq. (2.10) with $\vec{B}$. This gives

$$[\partial_{0}B^{a}(x),\ B^{b}(y)]\delta(x_{0}-y_{0})=0 . \quad (2.19)$$

The ghost and fermion equations of motion and the ETC's are the same as in the second-order formalism and will not be recorded. See, for example, Ref. 12.

The Lagrangian (2.8) is invariant under the global gauge transformations

$$\begin{aligned}\delta\vec{A}_{\mu}&=\vec{A}_{\mu}\times\vec{\omega} ,\\ \delta\vec{G}_{\mu\nu}&=\vec{G}_{\mu\nu}\times\vec{\omega} ,\\ \delta\vec{B}&=\vec{B}\times\vec{\omega} ,\\ \delta\vec{C}&=\vec{C}\times\vec{\omega} ,\\ \delta\psi&=i\vec{\omega}\cdot\vec{T}\psi ,\end{aligned} \quad (2.20)$$

where $\vec{\omega}$ is a constant c-number vector. The corresponding Noether currents

$$\vec{J}_{\mu}=\vec{G}_{\mu\nu}\times\vec{A}^{\nu}+\vec{B}\times\vec{A}_{\mu}+\bar{\psi}\gamma_{\mu}\vec{T}\psi+\partial_{\mu}\vec{C}_{2}\times\vec{C}_{1}+\vec{C}_{2}\times\mathfrak{D}_{\mu}\vec{C}_{1} \quad (2.21)$$

are therefore conserved:

$$\partial^{\mu}\vec{J}_{\mu}=0 .$$

From the canonical ETC's (2.18), we find that $\vec{J}_{0}$ satisfies

$$[J_{0}^{a}(x),\ J_{0}^{b}(y)]\delta(x_{0}-y_{0})=if^{abc}J_{0}^{c}(x)\delta^{4}(x-y) . \quad (2.22)$$

This conserved current is of course not renormalized, so that (2.22) implies that $\vec{J}_{\mu}$ has dimension 3 in any scale-invariant limit, in particular at an IR fixed point, if such a fixed point should exist. Note that the conserved Noether current $\vec{J}_{\mu}$ is fundamentally distinct from the (nonconserved) source current $\vec{K}_{\mu}$ defined in Eq. (2.13). Equation (2.12) gives

$$\partial^{\mu}\vec{K}_{\mu}=-\Box\vec{B} , \quad (2.23)$$

which can be evaluated from Eq. (2.14). All this is in contrast to the situation in Abelian gauge theories, where $\Box B=0$, and J_{μ} and K_{μ} are essentially equal and differ from $\partial_{\mu}B$ only by a conserved current with zero charge.[12]

As in the Lorentz ($\alpha\neq 0$) gauges, the Lagrangian (2.8) is *not* invariant to the *local* non-Abelian gauge transformations which leave the classical Lagrangian

$$\mathfrak{L}_{\rm cl}=-\tfrac{1}{4}\vec{G}_{\mu\nu}\cdot\vec{G}^{\mu\nu}+\mathfrak{L}_{F}$$

invariant.[12] It is instead invariant to the "Slavnov" transformation

$$\vec{A}_{\mu}\to\vec{A}_{\mu}+\omega\mathfrak{D}_{\mu}\vec{C}_{1} , \quad (2.24a)$$

$$\vec{C}_{1}\to\vec{C}_{1}-\tfrac{1}{2}g\omega\vec{C}_{1}\times\vec{C}_{1} , \quad (2.24b)$$

$$\vec{C}_{2}\to\vec{C}_{2}+\omega\vec{B} , \quad (2.24c)$$

$$\psi\to e^{-ig\omega\vec{T}\cdot\vec{C}_{1}}\psi , \quad (2.24d)$$

introduced by Becchi, Rouet, and Stora.[11] Some of the consequences of this will be discussed in the following sections. The more complete Lorentz gauge treatments (given formally in Ref. 12 and rigorously in Ref. 11) can be readily generalized to the Landau gauge with the replacement of $(1/\alpha)\partial\cdot\vec{A}$ by $\vec{B}$.

The renormalized version of the equations of motion and the ETC's can be obtained with the substitution[13]

$$\begin{aligned}\vec{A}^{\mu}&=Z_{3}^{1/2}\vec{A}_{R}^{\mu} ,\\ \psi&=Z_{2}^{1/2}\psi_{R} ,\\ \vec{C}&=\tilde{Z}_{3}^{1/2}\vec{C}_{R} ,\\ \vec{B}&=Z_{B}^{1/2}\vec{B}_{R} ,\\ g&=\frac{Z_{1}}{Z_{3}^{3/2}}g_{R} ,\end{aligned} \quad (2.25)$$

where the subscript R denotes renormalized quantities. At this stage, an independent renormalization constant Z_{B} has to be introduced. Since $\vec{B}$ plays the role of $(1/\alpha)(\partial\cdot\vec{A})$ and in the usual

($\alpha \neq 0$) formalism α and α_R are related by $\alpha = Z_3\alpha_R$, we expect $Z_B = Z_3^{-1}$. This will be proven in the next section. The renormalized forms of the global gauge transformations and the (more subtle) Slavnov transformation are simply obtained from the Lorentz gauge treatment given in Ref. 12 and will not be discussed here in detail.

The renormalized gauge-field equation has the form

$$\Box\vec{A}_{R\mu} + \partial_\mu \vec{B}_R = \vec{K}_{R\mu} , \tag{2.26}$$

just as in the Lorentz gauge (see Ref. 12), where

$$\vec{B}_R = \frac{1}{\alpha_R}\,\partial\cdot\vec{A}_R , \tag{2.27}$$

and just as in the Abelian case, where the fields and currents are gauge-group scalars. The renormalized ETC

$$[A^a_{R0}(x),\ B^b_R(y)]\delta(x_0 - y_0) = i\delta_4(x-y)\delta^{ab} \tag{2.28}$$

is the same as the unrenormalized one (2.18d) and is therefore finite in perturbation theory in all gauges (Lorentz or Landau) for all gauge groups (Abelian or non-Abelian). However, in the Abelian models, one also has

$$K_{R\mu} = J_{R\mu} = \text{generating current} , \tag{2.29}$$

which is not valid in the non-Abelian models, as already noted in the unrenormalized theories.

Let us consider a scale-invariant limit of the above models (e.g., a fixed point of the renormalization group). In the Abelian models, where both $K_{R\mu} = J_{R\mu}$ and $\partial_\mu B_R$ generate the renormalized gauge transformations, one has

$$\dim K_{R\mu} = \dim\partial_\mu B_R = \dim J_{R\mu} = 3 \quad \text{(Abelian)} , \tag{2.30}$$

where "dim" is the scale-invariance dimension. In the non-Abelian models, one has only

$$\dim\vec{J}_{R\mu} = 3 \quad \text{(non-Abelian)} . \tag{2.31}$$

Also, in all models, (2.28) implies that

$$\dim\vec{A}_{R\mu} + \dim\vec{B}_{R\mu} = 3 \quad \text{(always)} , \tag{2.32}$$

and in the Lorentz gauges, one can conclude from (2.27) that

$$\dim\vec{B}_R = \dim\vec{A}_{R\mu} + 1 \quad \text{(Lorentz gauges)} , \tag{2.33}$$

so that

$$\dim\vec{A}_{R\mu} = 1, \quad \dim\vec{B}_{R\mu} = 2 \quad \text{(Lorentz gauges)} . \tag{2.34}$$

We are not able to derive (2.34) in the Landau gauge.

III. GREEN'S FUNCTIONS

We start with the B-A propagator, defined by[14]

$$E^{ab}_\mu(k) = \int d^4x e^{-ik\cdot x}\langle T[B^a(0)A^b_\mu(x)]\rangle . \tag{3.1}$$

It follows from (2.11) and (2.18d) that

$$k^\mu E^{ab}_\mu(k) = -\delta^{ab} , \tag{3.2}$$

or

$$E^{ab}_\mu(k) = -\frac{k_\mu}{k^2}\delta^{ab} . \tag{3.3}$$

Since the only equation of motion used is $\partial\cdot\vec{A} = 0$, which is true to all orders in g, Eq. (3.3) for E^{ab}_μ is exact. Now the renormalized B-A propagator

$$E^{ab}_{R\mu}(k) = \int d^4x e^{-ik\cdot x}\langle T[B^a_R(0)A^b_{R\mu}(x)]\rangle \tag{3.4}$$

is evidently

$$E^{ab}_{R\mu}(k) = (Z_B Z_3)^{1/2}E^{ab}_\mu(k)$$
$$= (Z_B Z_3)^{1/2}\left(-\frac{k_\mu}{k^2}\delta^{ab}\right) , \tag{3.5}$$

and must of course be finite. Thus the choice $Z_B = Z_3^{-1}$ is seen to be appropriate. As discussed in the last section, this is expected.

Next we consider the B-B propagator, defined by

$$D^{ab}(k) = \int d^4x e^{ik\cdot x}\langle T[B^a(x)B^b(0)]\rangle . \tag{3.6}$$

If we calculate $k^2 D^{ab}(k)$, there will be a T-product contribution involving $\langle T[B^a(x)B^b(0)]\rangle$, which vansihes to lowest order in g. The ETC contributions also vanish since B commutes both with itself and its own time derivative. So to lowest order in g

$$k^2 D^{(0)ab}(k) = 0 , \tag{3.7}$$

i.e., $D^{(0)ab}(k)$ is concentrated at $k^2 = 0$. Since $\delta(k^2)$ has the wrong asymptotic properties for a propagator, we see that $D^{(0)ab}(k) \propto \delta^4(k)$, i.e.,

$$D^{(0)ab}(x) = \text{constant} . \tag{3.8}$$

We shall see later that this state of affairs persists to all orders and that the constant actually vanishes.

The gauge field propagator is defined by

$$\Delta^{ab}_{\mu\nu}(k) = \int d^4x e^{ik\cdot x}\langle T[A^a_\mu(x)A^b_\nu(0)]\rangle . \tag{3.9}$$

As usual we consider

$$k^2\Delta^{ab}_{\mu\nu}(k) = -\int d^4x e^{ik\cdot x}\left(\frac{\partial}{\partial x_0}\{\delta(x_0)\langle[A^a_\mu(x),\ A^b_\nu(0)]\rangle\} + \delta(x_0)\langle[\partial_0 A^a_\mu(x),\ A^b_\nu(0)]\rangle + \langle T[\Box A^a_\mu(x) A^b_\nu(0)]\rangle\right) . \quad (3.10)$$

The ETC's can be evaluated from (2.18). The first ETC is zero, while the second is nonzero only for spacelike μ, ν, i.e.,

$$-\int d^4x e^{ik\cdot x}\delta(x_0)\langle[\partial_0 A^a_\mu(x),\ A^b_\nu(0)]\rangle = i(-g_{\mu\nu} + g_{\mu 0}g_{\nu 0})\delta^{ab} . \quad (3.11)$$

To evaluate the T product, we use (2.12) and (2.11) to obtain

$$\Box \vec{A}_\mu = -\partial_\mu \vec{B} - \vec{K}_\nu , \quad (3.12)$$

and find

$$-\int d^4x e^{ik\cdot x}\langle T[\Box A^a_\mu(x) A^b_\nu(0)]\rangle = \int d^4x e^{ik\cdot x}\{\partial_\mu\langle T[B^a(x) A^b_\nu(0)]\rangle - g_{\mu 0}\delta(x_0)\langle[B^a(x),\ A^b_\nu(0)]\rangle + \langle T[K_\mu(x) A^b_\nu(0)]\rangle\}$$

$$= i\,\frac{k_\mu k_\nu}{k^2}\,\delta^{ab} - i g_{\mu 0} g_{\nu 0}\delta^{ab} + \int d^4x e^{ik\cdot x}\langle T[K_\mu(x) A^b_\nu(0)]\rangle , \quad (3.13)$$

where we have used (3.3) for the B-A propagator and the fact that B has a nonzero ETC only with the zeroth component of A. Combining (3.11) and (3.13), we find that the $g_{\mu 0}g_{\nu 0}$ terms cancel, and we have

$$i\Delta^{ab}_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}\,\delta^{ab} + \frac{i}{k^2}\int d^4x e^{ik\cdot x}\langle T[K^a_\mu(x) A^b_\nu(0)]\rangle , \quad (3.14)$$

where the last term is of order g, so the first two terms represent the bare propagator.

$$i\Delta^{(0)ab}_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}\delta^{ab} . \quad (3.15)$$

It is interesting to see how the B field enters to ensure Lorentz invariance. The result (3.15) for the bare propagator is of course identical with the $\alpha \to 0$ limit of the bare propagator obtained in the usual second-order formulation. Since the bare vertices implied by (2.1) and (2.8) are obviously identical, this shows that all Green's functions without external B lines are to all orders identical with those obtained in the second-order formulation. This proves what has been expected all along, namely that (2.8) defines the $\alpha \to 0$ limit of (2.1).

The Feynman rules to be used in constructing the perturbation-theory expansion for the (unrenormalized[15]) Green's functions[16]

$$G^{(n,s,r,m)}_{\alpha_1\cdots\alpha_n}(q_1,\ldots,q_n;p_1,\ldots;l_1,\ldots;k_1,\ldots) = \text{Fourier transform of } \langle T[A_{\alpha_1}\cdots A_{\alpha_n}\psi\cdots\psi\bar\psi\cdots\bar\psi C_1\cdots C_1 C_2\cdots C_2 B\cdots B]\rangle , \quad (3.16)$$

where there are s factors each of ψ and $\bar\psi$, r factors each of C_1 and C_2 and m factors of B, can be read off from the Gell-Mann–Low expansion using as the interaction Lagrangian L_I the terms proportional to g and g^2 in (2.8) and using Wick's theorem with the propagators (3.3), (3.8), (3.15), together with the usual fermion and ghost propagator. Note that the B field does not occur in L_I, its effect having been exactly accounted for in the evaluation of (3.3), (3.8), and (3.15). In particular, the Green's functions $G^{(n,s,r,o)}_{\alpha_1\cdots\alpha_n}$ with no external B lines are to be evaluated exactly as in the Lorentz gauges except that the transverse bare gauge-field propagator (3.15) is to be used. Coupling to B fields never enters this evaluation and the resulting Green's functions are clearly the $\alpha \to 0$ limits of the Lorentz gauge Green's functions. It then follows from the transversality of the Landau gauge field propagator (3.15) that all of these Green's functions are transverse:

$$q_i^{\alpha_i} G^{(n,s,r,o)}_{\alpha_1\cdots\alpha_i\cdots\alpha_n} = 0, \quad i = 1,\ldots,n . \quad (3.17)$$

When B fields are present ($m > 0$), the Green's functions (3.16) are no longer transverse in the gauge-field momenta because of the presence of disconnected A-B propagators (3.3) which are not transverse. It is therefore convenient to define the amplitudes $\overline{G}(A\cdots A,\ B\cdots B$, fermions, ghosts) as those corresponding to all the diagrams for the full Green's functions $G(A\cdots A,\ B\cdots B$, fermions, ghosts) except those with disconnected A-B propagators $G(AB)$. The analytical definition is

$$G(A\cdots A, B\cdots B,\dots)=\overline{G}(A\cdots A, B\cdots B,\dots)+\sum G(AB)\delta_4\cdots G(AB)\delta_4\overline{G}(A\cdots A, B\cdots B,\dots)\ , \tag{3.18}$$

the sum being over all possible such factorizations (including permutations). The δ-function factors here express the equality of the A and B momenta in each $G(A_\alpha B)=E_\alpha(k)$, $k=P_A+P_B$. It then follows from the definition of the $\overline{G}$ amplitudes that they are transverse:

$$q_i^{\alpha_i}\overline{G}^{(n,s,r,m)}_{\alpha_1\cdots\alpha_i\cdots\alpha_n}=0,\quad i=1,\dots,n\ . \tag{3.19}$$

Note that (3.17) is a special case of (3.19) since $\overline{G}=G$ for $m=0$. Note also that the Green's functions (3.18) are the $\alpha\to 0$ limits of the Lorentz-gauge Green's functions $G(A\cdots A, \alpha^{-1}\partial\cdot A,\dots)$.

The transversality conditions (3.17) and (3.19) can be deduced directly from the representation (3.16). For $m=0$, (3.19) follows immediately from $\partial\cdot A=0$ and the field ETC's. For general m, (3.16) gives

$$q_i^{\alpha_i}G^{(n,s,r,m)}_{\alpha_1\cdots\alpha_i\cdots\alpha_n}(q_1,\dots,q_n;\dots;k_1,\dots,k_m)=\sum_{j=1}^{m}\delta^4(q_i+k_j)G^{(n-1,s,r,m-1)}_{\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n}(q_1,\dots,q_{i-1},q_{i+1},\dots,q_n;\dots;k_1,\dots,k_{j-1}k_{j+1},\dots,k_m) \tag{3.20}$$

using (2.11) and (2.18d), whereas (3.18) gives

$$\text{LHS}(3.20)=\text{RHS}(3.20)+q_i\cdot\overline{G}-\text{terms}\ , \tag{3.21}$$

using (3.2). It therefore follows (inductively) that (3.19) is valid.

Some further exact results can be deduced from the invariance of the vacuum under the Slavnov transformation (2.24). For example, the vanishing of the variation of $G(C_2\cdots C_2)$ under (2.24c) plus ghost number conservation leads immediately to the vanishing of all Green's functions of B fields alone:

$$G(B\cdots B)=0\ ; \tag{3.22}$$

in particular

$$G(B^aB^b)=D^{ab}(k)=0\ , \tag{3.23}$$

as we mentioned previously. More generally, consideration of the invariance of the Green's functions $G^{(n,m)}(A\cdots AC_2\cdots C_2)=0$ for $m>0$ under (2.24a) and (2.24c) using also ghost number conservation leads to the identities

$$\sum_{i=0}^{n}\binom{n}{i}\binom{m}{m-i}G(A\cdots A\mathfrak{D}C_1\cdots\mathfrak{D}C_1B\cdots BC_2\cdots C_2)=0,\quad m>0 \tag{3.24}$$

where there are $n-i$ factors of A, i factors of $\mathfrak{D}C_1$, $m-i$ factors of B, and i factors of C_2. For $n=0$, this gives back (3.22) and for $n=m=1$, this gives

$$G(A_\mu B)=-G(\mathfrak{D}_\mu C_1C_2)\ . \tag{3.25}$$

The renormalizations of the Green's functions considered above proceeds in the usual manner.[9–12] Recall first that in the Lorentz ($\alpha\neq 0$) gauges, the Green's functions of the renormalized fields

$$A_R=Z_3^{-1/2}A,\quad \psi_R=Z_2^{-1/2}\psi,\quad C_R=\tilde{Z}_3^{-1/2}C \tag{3.26}$$

are finite when expressed in terms of the renormalized parameters

$$g_R=(Z_3^{3/2}/Z_1)g,\quad \alpha_R=Z_3^{-1}\alpha\ , \tag{3.27}$$

and the renormalization-point mass μ. For example, the renormalized gauge-field Green's functions are

$$G_R^{(n)}(A_R\cdots A_R;g_R,\alpha_R;\mu)=(Z_3^{1/2})^{-n}G(A\cdots A;g,\alpha)\ . \tag{3.28}$$

The arbitrariness of the choice of μ is expressed by the renormalization-group equation[17]

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\delta\frac{\partial}{\partial\alpha_R}+n\gamma\right)G_R^{(n)}=0\ , \tag{3.29}$$

where

$$\beta(g_R,\alpha_R)=\mu\frac{\partial g_R}{\partial\mu}\bigg|_{g,\alpha}\ ,\qquad \delta(g_R,\alpha_R)=\mu\frac{\partial\alpha_R}{\partial\mu}\bigg|_{g,\alpha}\ ,\qquad \gamma(g_R,\alpha_R)=\tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_3\bigg|_{g,\alpha}\ , \tag{3.30}$$

with the partial derivatives evaluated at fixed g, α and fixed cutoff. The relation $\alpha_R=Z_3^{-1}\alpha$ gives the important identity

$$\delta(g_R,\alpha_R)=-\alpha_R\gamma(g_R,\alpha_R)\ . \tag{3.31}$$

In the Landau gauge, there is one less parameter ($\alpha=\alpha_R=0$) but one more field (B) to be renormalized:

$$B_R=Z_3^{1/2}B\ . \tag{3.32}$$

It is immediate that the Landau-gauge renormalized Green's functions and renormalization-group equations are the $\alpha \to 0$ limits of those, (3.28)–(3.30), in the Lorentz gauges, e.g.,

$$\left(\mu \frac{\partial}{\partial\mu} + \beta \frac{\partial}{\partial g_R} + n\gamma\right) G_R^{(n)} = 0 \,. \qquad (3.33)$$

Equation (3.31) has been used here to conclude that $\delta(g_R, 0) = 0$, a result which leads to an enormous simplification in the analysis of the renormalization-group equations. On the other hand, in the Landau gauge one must contend with the mixed Green's functions

$$G_R^{(n,m)}(A_R \cdots A_R, B_R \cdots B_R; g_R, \mu) = (Z_3^{1/2})^{-n}(Z_3^{-1/2})^{-m} G(A \cdots A, B \cdots B; g) \,, \qquad (3.34)$$

which satisfy

$$\left[\mu \frac{\partial}{\partial\mu} + \beta \frac{\partial}{\partial g_R} + (n-m)\gamma\right] G_R^{(n,m)} = 0 \,. \qquad (3.35)$$

Note, however, that no new functions are involved here.

In the asymptotically free models, one can exactly[18] calculate the ultraviolet behavior of the Green's functions, etc. The results are often considerably simpler than in the Lorentz gauges. We refer to Ref. 4 for a detailed discussion of these points.

IV. FUNCTIONAL FORMALISM

Functional integration gives in principle a closed solution to the field equations in quantum field theory. The relevant mathematics is, however, so undeveloped that the generating functional integral for a given theory is at best a formal representation of the exact solution. Functional integral techniques, however, have proved to be valuable tools in the development of NAGT's, especially in connection with the investigation of symmetries of the theory.[6,8–10] In spite of the rather formal nature of functional manipulations, our attitude is that many combinational problems can be investigated with much reduced effort with this formalism, and the proofs are often more transparent. In this section we study Landau-gauge NAGT in the framework of functional formalism.

In the first-order formulation, NAGT's in the Lorentz gauges are described in terms of the generating functional

$$\mathcal{W} = \int [d\vec{A}_\mu][d\vec{B}][d\vec{G}_{\mu\nu}] \cdots \exp\left\{ i \int d^4x[-\tfrac{1}{2}\vec{G}_{\mu\nu}\cdot(\partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu + g\vec{A}^\mu \times \vec{A}^\nu) + \tfrac{1}{4}\vec{G}_{\mu\nu}\cdot\vec{G}^{\mu\nu} - \vec{B}\cdot\partial\vec{A} + \tfrac{1}{2}\alpha\vec{B}^2 + \mathcal{L}_G + \mathcal{L}_F + \vec{J}_\mu\cdot\vec{A}^\mu + \vec{J}_B\cdot\vec{B} + \vec{J}_G^{\mu\nu}\cdot\vec{G}_{\mu\nu} + \cdots]\right\} \,, \qquad (4.1)$$

where $\vec{A}_\mu, \vec{G}_{\mu\nu}$, and $\vec{B}$ are independent dynamical variables, and the dots indicate corresponding entities for ghost and fermion fields. This formulation can be shown to be equivalent to the second-order one. For example, the terms involving $\vec{B}$ in (4.1) are

$$\mathcal{L}_B = -\vec{B}\cdot\partial\vec{A} + \frac{\alpha}{2}\vec{B}^2 = \frac{\alpha}{2}\left[\vec{B} - \frac{1}{\alpha}(\partial\cdot\vec{A})\right]^2 - \frac{1}{2\alpha}(\partial\cdot\vec{A})^2 \,. \qquad (4.2)$$

All the $\vec{B}$ dependence in the path integral defining the generating functional can be collected into

$$\int [d\vec{B}] e^{i\int dx \mathcal{L}_B} \sim (\text{const}) \times \exp\left[i \int dx \left(-\frac{1}{2\alpha}\right)(\partial\cdot\vec{A})^2\right] \,, \qquad (4.3)$$

which just produces the gauge-fixing term in the second-order Lagrangian.

In the $\alpha \to 0$ limit, (4.1) becomes the generating functional corresponding to the Lagrangian (2.8):

$$\mathcal{W} = \int [d\vec{A}_\mu][d\vec{B}][d\vec{G}_{\mu\nu}] \cdots \exp\left\{ i \int d^4x[-\tfrac{1}{2}\vec{G}_{\mu\nu}\cdot(\partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu + g\vec{A}^\mu \times \vec{A}^\nu) + \tfrac{1}{4}G_{\mu\nu}\cdot\vec{G}^{\mu\nu} - \vec{B}\cdot(\partial\cdot\vec{A}) + \mathcal{L}_G + \mathcal{L}_F + \vec{J}_\mu\cdot\vec{A}^\mu + \vec{J}_B\cdot\vec{B} + \vec{J}_G^{\mu\nu}\cdot\vec{G}_{\mu\nu} + \cdots]\right\} \,. \qquad (4.4)$$

Here the $\vec{B}$ dependence can again be integrated outto give

$$\int [d\vec{B}]\exp\left(i\int dx \mathcal{L}_B\right) \sim (\text{const}) \times \delta(\partial\cdot\vec{A}) \,, \qquad (4.5)$$

so that (4.4) becomes the same generating functional as that used as a starting point for the discussion of the Landau gauge in Ref. 6, where [Eq. (2.39) of Ref. 6]

$$\mathcal{W}=\int [d\vec{A}]\delta(\partial\cdot\vec{A})\exp\left[i\int dx(\mathcal{L}+\vec{J}_\mu\cdot\vec{A}^\mu)+\mathrm{tr}\ln(\Box^{-1}\mathfrak{D}\cdot\partial)\right] . \tag{4.6}$$

In Ref 6, the explicit form $\mathrm{tr}\ln(\Box^{-1}\mathfrak{D}\cdot\partial)$ was used instead of the more usual ghosts. The trace expression is of course what is obtained by carrying out the $[d\vec{C}_2][d\vec{C}_1]$ integration in our expression.

We now proceed to investigate the Green's functions in this first-order formalism with functional methods. We first consider the Lagrangian in zeroth order (in g):

$$\mathcal{L}^{(0)}=-\tfrac{1}{2}\vec{F}_{\mu\nu}\cdot(\partial^\mu\vec{A}^\nu-\partial^\nu\vec{A}^\mu)+\tfrac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} -\vec{B}\cdot(\partial\cdot\vec{A})+\vec{J}_\mu\cdot\vec{A}^\mu+\vec{J}_B\cdot\vec{B}+\cdots . \tag{4.7}$$

The generating functional, defined as usual, can be simplified by carrying out the $[d\vec{F}_{\mu\nu}]$ integrations to give

$$\mathcal{W}=\int [d\vec{A}_\mu][d\vec{B}]\cdots\exp\left(i\int d^4x\mathcal{L}'\right) , \tag{4.8}$$

where

$$\mathcal{L}'=-\tfrac{1}{4}(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu)^2-\vec{B}\cdot(\partial\cdot\vec{A}) +\vec{J}_\mu\cdot\vec{A}^\mu+\vec{J}_B\cdot\vec{B}+\cdots . \tag{4.9}$$

We next make the change of variables

$$\vec{A}_\mu(x)\rightarrow\vec{A}_\mu+\partial_\mu\frac{1}{\Box}\vec{J}_B(x) , \tag{4.10}$$

which induces the following changes in (4.9):

$$-\vec{B}\cdot(\partial\cdot\vec{A})\rightarrow-\vec{B}\cdot(\partial\cdot\vec{A})-\vec{J}_B\cdot\vec{B} , \tag{4.11}$$

$$\vec{J}_\mu\cdot\vec{A}^\mu\rightarrow\vec{J}_\mu\cdot\vec{A}^\mu+\vec{J}_\mu\cdot\frac{\partial^\mu}{\Box}\vec{J}_B(x) . \tag{4.12}$$

The extra term in (4.11) cancels the original source term to give

$$\mathcal{L}'=(\vec{B}-\text{independent terms})-\vec{B}\cdot(\partial\cdot\vec{A}) +\vec{J}_\mu\cdot\frac{\partial^\mu}{\Box}\vec{J}_B . \tag{4.13}$$

The transformation (4.10) is surely measure-preserving, and so $\mathcal{W}$ is unchanged with $\mathcal{L}\rightarrow\mathcal{L}'$. The $[d\vec{B}]$ integration is, however, now trivial, and we obtain

$$\mathcal{W}=\int[d\vec{A}_\mu]\cdots\delta(\partial\cdot\vec{A})\exp\left\{i\int d^4x\left[(\vec{B}-\text{independent})+\vec{J}_\mu\cdot\frac{\partial^\mu}{\Box}\vec{J}_B\right]\right\} . \tag{4.14}$$

The $\vec{B}$ field enters into $\mathcal{W}$ only in the form of a phase factor $\vec{J}_\mu\cdot(\partial^\mu/\Box)\vec{J}_B$, and so the generating functional of connected Green's functions is given by

$$Z=\ln\mathcal{W} =(\vec{B}-\text{independent terms})+i\int d^4x\vec{J}_\mu\cdot\partial^\mu\Box^{-1}\vec{J}_B . \tag{4.15}$$

The only connected Green's function to lowest order involving B is therefore

$$\langle A_\mu B\rangle_+^{(0)}\sim i\partial^\mu\Box^{-1} . \tag{4.16}$$

This has already been derived to all orders in Sec. III [Eq. (3.3)]. In particular, in lowest order $\langle BB\rangle_+$ vanishes:

$$\langle BB\rangle_+^{(0)}=0 . \tag{4.17}$$

This has also already been derived to all orders from Slavnov invariance.

The same result can be deduced by using the Slavnov invariance in the functional context. We consider $\mathcal{W}$ where the $[d\vec{G}_{\mu\nu}]$ dependence has been integrated out:

$$\mathcal{W}=\int [d\vec{A}_\mu][d\vec{B}][d\vec{C}_1][d\vec{C}_2]\exp\left\{i\int d^4x[-\tfrac{1}{4}(\partial^\mu\vec{A}^\nu-\partial^\nu\vec{A}^\mu+g\vec{A}^\mu\times\vec{A}^\nu)^2-\vec{B}\cdot(\partial\cdot\vec{A})-\partial_\mu\vec{C}_2\cdot\mathfrak{D}^\mu C_1 +\vec{J}_\mu\cdot\vec{A}+\vec{J}_B\cdot\vec{B}+\vec{J}_{C_1}\cdot\vec{C}_1+\vec{J}_{C_2}\cdot\vec{C}_2]\right\} . \tag{4.18}$$

The invariance of the Lagrangian in (4.18) under (2.24) implies the following Ward-Takahashi identity:

$$\left\{\vec{J}^\mu\cdot\mathfrak{D}_\mu\left[\frac{\delta}{\delta\vec{J}}\right]\frac{\delta}{\delta\vec{J}_{C_1}}+\vec{J}_{C_1}\cdot\left(-\frac{1}{2}g\frac{\delta}{\delta\vec{J}_{C_1}}\times\frac{\delta}{\delta\vec{J}_{C_1}}\right)+\vec{J}_{C_2}\cdot\frac{\delta}{\delta\vec{J}_B}\right\}\mathcal{W}=0 . \tag{4.19}$$

By taking $\delta^2/\delta J_{C_2}(z)\delta J_B(y)$, and then letting all J's vanish, we easily see that

$$0=\frac{\delta^2 \mathcal{W}}{\delta J_B(x)\delta J_B(y)}=\langle B(x)B(y)\rangle_+ \ . \tag{4.20}$$

Similarly, by taking $\partial_\nu^y(\delta^2/\delta \vec{J}_{C_2}(z)\delta \vec{J}_\nu(y))$, we obtain

$$\left\{\partial_y^\nu \mathfrak{D}_\nu\left[\frac{\delta}{\delta \vec{J}(y)}\right]\frac{\delta^2}{\delta \vec{J}_{C_1}(y)\delta \vec{J}_{C_2}(z)} + \partial_\nu \frac{\delta^2}{\delta \vec{J}_B(z)\delta \vec{J}_\nu(y)}\right\}\mathcal{W}=0 \ . \tag{4.21}$$

By considering the transformation

$$\vec{C}_2(x)\to \vec{C}_2(x)+\vec{r}(x) \tag{4.22}$$

(which generates the field equation for $\vec{C}_1$), we get

$$0=\int d^4x\, \vec{r}(x)\cdot\left[-\partial^\nu \mathfrak{D}_\nu \frac{\delta}{\delta \vec{J}_{C_1}(x)}+\vec{J}_{C_2}(x)\right]\mathcal{W} \ , \tag{4.23}$$

and hence (4.21) gives

$$\partial_\nu \frac{\delta^2 Z}{\delta \vec{J}_B(z)\delta \vec{J}_\nu(y)}+\delta^4(y-z)=0 \ . \tag{4.24}$$

This is precisely Eq. (3.2). Equation (4.24) is the equivalent of the usual statement on the nonrenormalization of the longitudinal part of the gauge-field propagator in the Lorentz gauges.

Taking functional derivatives with respect to $\vec{J}$, $\vec{J}_{C_2}$, and $\vec{J}_B$ gives the Slavnov[10] identity for the transverse part of the three-point vertex. If we consider the longitudinal part, we find

$$0=\left\{\partial\cdot\mathfrak{D}\left[\frac{\delta}{\delta \vec{J}}\right]\frac{\delta}{\delta \vec{J}_{C_1}(x)}\frac{\delta}{\delta \vec{J}_{C_2}(x)}\frac{\delta}{\delta \vec{J}_B(y)} + \partial_\mu^x \frac{\delta^3}{\delta \vec{J}_B(z)\delta \vec{J}_\mu(x)\delta \vec{J}_B(y)}\right\}\mathcal{W} \ . \tag{4.25}$$

Using (4.23) again, we readily obtain

$$0=\left\{\delta(x-z)\frac{\delta}{\delta \vec{J}_B(y)} + \partial_\mu^x \frac{\delta^3}{\delta \vec{J}_B(y)\delta \vec{J}_\mu(x)\delta \vec{J}_B(y)}\right\}\mathcal{W} \ . \tag{4.26}$$

This derivation of (4.26) is simpler than the direct approach, which would start with[12]

$$0=\partial^\mu[\langle \mathfrak{D}_\mu \vec{C}_1(x)\vec{C}_2(y)\vec{B}(z)\rangle_+ + \langle \vec{A}_\mu(x)\vec{B}(y)\vec{B}(z)\rangle_+] . \tag{4.27}$$

When one tries to pull the ∂_μ through the T product, one encounters the equal-time commutator $\delta(x^0-z^0)[\mathfrak{D}_\mu C_1(x),\ B(z)]$, which is seen to vanish only after some calculation.

As a final illustration of the usefulness of the functional formalism, let us write the renormalization-group equation (3.35) in functional form. It follows, for example, from the explicit representation (4.4) that

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\gamma\int d^4x\left(J_\mu\frac{\delta}{\delta J_\mu}-J_B\frac{\delta}{\delta J_B}\right)\right]\times Z(J;g_R,\mu)=0 \ . \tag{4.28}$$

V. VERTEX FUNCTIONS

If one were to try to calculate the gauge-field vertex functions by amputating $\langle AA\rangle_+$ propagators in the usual way in a gauge theory in the Landau gauge, one would immediately be faced with the difficulty that the transverse $\langle AA\rangle_+$ propagator, being a projector, does not have an inverse. In fact there is an arbitrary longitudinal ambiguity that can be added to the vertex function without changing the Green's functions at all. A related difficulty is encountered in the extraction of vertex functions from Green's functions involving B, since $\langle BB\rangle_+=0$. The resolution of these problems is the same: since $\langle BA\rangle_+\neq 0$, a B can propagate into an A, and so we must consider a propagator matrix involving A and B entries. This matrix will be seen to have an inverse and therefore lead to unique vertex functions.

The propagator matrix has the form

$$\underline{G}^{(2)}=\begin{bmatrix}\langle AA\rangle_+ & \langle BA\rangle_+\\ \langle AB\rangle_+ & \langle BB\rangle_+=0\end{bmatrix} \ . \tag{5.1}$$

More explicitly, the $\langle AA\rangle_+$ sector is given by the 4×4 matrix

$$\langle AA\rangle_+^{\mu\nu}=\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)d(k^2) \ , \tag{5.2}$$

the $\langle AB\rangle_+$ sector is a row vector

$$\langle AB\rangle_+^\nu=-k^\nu/k^2 \ , \tag{5.3}$$

and $\langle BA\rangle_+$ is a column vector

$$\langle BA\rangle_+^\mu=k^\mu/k^2 \ . \tag{5.4}$$

$\underline{G}^{(2)}$ is thus a 5×5 matrix, with B acting as a fifth component of A^μ. (The internal symmetry is irrelevant here and hence suppressed.)

The 4×4 matrix $\langle AA\rangle_+^{\mu\nu}$, being a projector, has no inverse, and indeed has a zero determinant. The problem associated with the nonexistence of $\langle AA\rangle_+^{-1}$ in the Landau gauge is automatically resolved by the need for the presence of the B field in the Landau gauge. The coupled propagator matrix (5.1) *is* invertible even though $\langle AA\rangle_+$ is not. Indeed, it has as determinant

$$\det\underline{G}^{(2)}=d(k^2)/k^2 \ . \tag{5.5}$$

The vertex functions can be most simply defined by functional methods. The generating functional

$\Gamma(S)$ of proper vertices is the Legendre transform

$$\Gamma = Z - J\cdot S\ , \tag{5.6}$$

$$\frac{\delta\Gamma}{\delta S} = -J\ , \tag{5.7}$$

$$\frac{\delta Z}{\delta J} = S\ , \tag{5.8}$$

where $J=\{J_i\}$ and $S=\{S_i\}$ denote 5 vectors with i running over A_μ and B. Functional differentiation of the completeness relation

$$\frac{\delta J_i(x)}{\delta J_j(y)} = \delta_{ij}\delta^4(x-y) \tag{5.9}$$

then gives the relation between the connected Green's functions

$$G_{i_1\cdots i_n} = \frac{\delta}{\delta J_{i_1}}\cdots\frac{\delta}{\delta J_{i_n}} Z(J)\bigg|_{J=0} \tag{5.10}$$

and the vertex functions

$$\Gamma_{i_1\cdots i_n} = \frac{\delta}{\delta S_{i_1}}\cdots\frac{\delta}{\delta S_{i_n}} \Gamma(S)\bigg|_{S=0}\ . \tag{5.11}$$

For example,

$$-\frac{\delta^2 Z}{\delta J_j(y)\delta J_k(z)}\ \frac{\delta^2\Gamma}{\delta S_k(z)\delta S_i(x)} = \delta_{ij}\delta^4(x-y) \tag{5.12}$$

gives the expected result that the two-point vertex function is the (five-dimensional) inverse of the two-point Green's function. The 5×5 vertex function matrix is thus

$$\underline{\Gamma}^{(2)} = [\underline{G}^{(2)}]^{-1} = \begin{bmatrix} d(k^2)^{-1}(g^{\mu\nu} - k^\mu k^\nu/k^2) & -k^\mu \\ k^\nu & 0 \end{bmatrix}\ . \tag{5.13}$$

We therefore find

$$\Gamma_{A_\mu A_\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{d(k^2)}\ , \tag{5.14}$$

$$\Gamma_{A_\mu B}(k) = k_\mu\ , \tag{5.15}$$

and

$$\Gamma_{BB}(k) = 0\ . \tag{5.16}$$

The three-point vertex $\Gamma^{(3)}$ is similarly obtained:

$$\Gamma^{(3)}_{nim} = \Gamma^{(2)}_{nj}\Gamma^{(2)}_{ml}G^{(3)}_{ljk}\Gamma^{(2)}_{ki}\ , \tag{5.17}$$

with $\Gamma^{(2)}$ given by (5.13). The higher vertex functions can now be obtained in the usual manner.

One immediate consequence of the above formalism is the vanishing of all vertex functions $\Gamma(B\cdots)$ with at least one B field except for $\Gamma(AB)$ [Eq. (5.15)]. This is because any such vertex function can be expressed as a sum of products involving the factors

$$\Gamma^{(2)}(BA_\mu)G_{\text{conn}}(A^\mu\cdots) = 0\ , \tag{5.18}$$

which vanish because the first factor is longitudinal [Eq. (5.15)] and the second (being connected) is transverse [Eq. (3.19)].

In the Lorentz gauges with $\alpha\neq 0$, the gauge-field vertex functions $\Gamma^{(\alpha)}$ can be defined from the Green's functions $G^{(\alpha)}$ simply by an amputation procedure since $\langle AA\rangle_+$ is no longer purely transverse. We now show that our formalism gives the same vertex functions as the usual Lorentz-gauge formalism upon taking the limit $\alpha\to 0$. Since Γ vanishes for any nonzero number of B's, it suffices to prove the equivalence for $\Gamma_{A\cdots A}$ with any number of A's.

We therefore consider the n-point vertex functions obtained by our procedure:

$$\Gamma_{A_{\mu_n}\cdots A_{\mu_1}} = G_{A_{\nu_n}\cdots A_{\nu_1}}\Gamma_{A_{\nu_n}A_{\mu_n}}\cdots\Gamma_{A_{\nu_1}A_{\mu_1}} + (G_{A_{\nu_n}\cdots A_{\nu_2}B}\Gamma_{A_{\nu_n}A_{\mu_n}}\cdots\Gamma_{A_{\nu_2}A_{\mu_2}}\Gamma_{BA_{\mu_1}} + \text{permutations}) + \cdots + G_{B\cdots B}\Gamma_{BA_{\mu_n}}\cdots\Gamma_{BA_{\mu_1}}\ , \tag{5.19}$$

where the G's here are all understood to be *connected*. Now the usual procedure for $\alpha\neq 0$ uses the inverse propagator

$$\Gamma^{(\alpha)\mu\nu}(k) = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^\nu}\right)d^{-1}(k^2) + \frac{1}{\alpha}k^\mu k^\nu\ , \tag{5.20}$$

which is the inverse of

$$G^{(\alpha)\mu\nu}(k) = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)d(k^2) + \alpha\frac{k^\mu k^\nu}{k^4}\ . \tag{5.21}$$

We need the superscript (α) to denote quantities in gauges with $\alpha\neq 0$. We previously showed that the formalism with the auxiliary field B is equivalent to the $\alpha\to 0$ limit of the Lorentz gauge theory. Hence it is the same $d(k^2)$ that appears in $G_{\mu\nu}$ and $G^{(\alpha)}_{\mu\nu}$ (although we need not make use of this fact). Thus the Lorentz gauge procedure gives

$$\Gamma^{(\alpha)}_{A_{\mu_n}\cdots A_{\mu_1}} = G^{(\alpha)}_{A_{\nu_n}\cdots A_{\nu_1}}\Gamma^{(\alpha)}_{\mathrm{tr},A_{\nu_1}A_{\mu_n}}\cdots\Gamma^{(\alpha)}_{\mathrm{tr},A_{\nu_1}A_{\mu_1}} + \left(G^{(\alpha)}_{A_{\nu_n}\cdots A_{\nu_1}}\Gamma^{(\alpha)}_{\mathrm{tr},A_{\nu_n}A_{\mu_n}}\cdots\Gamma^{(\alpha)}_{\mathrm{tr},A_{\nu_2}A_{\mu_2}}\Gamma^{(\alpha)}_{l,A_{\nu_1}A_{\mu_1}} + \text{permutations}\right) + \cdots + G^{(\alpha)}_{A_{\nu_n}\cdots A_{\nu_1}}\Gamma^{(\alpha)}_{l,A_{\nu_n}A_{\mu_n}}\cdots\Gamma^{(\alpha)}_{l,A_{\nu_1}A_{\mu_1}}\,, \tag{5.22}$$

where tr and l denote the transverse and longitudinal parts of $\Gamma^{(\alpha)}_{\mu\nu}$. The structures of (5.19) and (5.22) are in one-to-one correspondence with each other. Obviously, $\Gamma^{(\alpha)}_{\mathrm{tr},\mu\nu}=\Gamma_{\mu\nu}$, and so for our equivalence proof it suffices to show that multiplication by $\Gamma^{(\alpha)}_{l,\mu\nu}$ is equivalent to B-field amputation when $\alpha\to 0$.

We consider first

$$G^{(\alpha)}_{A_{\nu_n}\cdots A_{\nu_1}}\Gamma^{(\alpha)}_{l,A_{\nu_1}A_{\mu_1}} = \langle A_{\nu_n}\cdots A_{\nu_1}\rangle_+^{(\alpha)}\,\frac{1}{\alpha}k^{\nu_1}k^{\mu_1} = \left\langle A_{\nu_n}\cdots A_{\nu_2}\left(\frac{1}{\alpha}\partial^{\nu_1}A_{\nu_1}\right)\right\rangle_{+\mathrm{conn}} k_1^{\mu_1} \underset{\alpha\to 0}{\longrightarrow} G_{A_{\nu_n}\cdots A_{\nu_2}B}\,\Gamma_{BA_{\mu_1}}\,. \tag{5.23}$$

The second amputation gives

$$G^{(\alpha)}_{A_{\nu_n}\cdots A_{\nu_1}}\Gamma^{(\alpha)}_{l,A_{\nu_2}A_{\mu_2}}\Gamma^{(\alpha)}_{l,A_{\nu_1}A_{\mu_1}} = \left\langle A_{\nu_n}\cdots A_{\nu_3},\ \frac{1}{\alpha}\,\partial\cdot A,\ \frac{1}{\alpha}\,\partial\cdot A\right\rangle_{+\mathrm{conn}} k_2^{\mu_2}k_1^{\mu_1} + \text{equal-time commutators.} \tag{5.24}$$

These equal-time commutators are, however, easily seen to arise just from the *disconnected* parts of $G^{(\alpha)}_{A_{\nu_n}\cdots A_{\nu_1}}$, and so disappear when the connected part is used. In (5.24), the ETC is from the disconnected contribution

$$\langle A_{\nu_n}\cdots A_{\nu_3}\rangle\langle A_{\nu_2}A_{\nu_1}\rangle\ , \tag{5.25}$$

and thus we find equivalence term by term between (5.19) and (5.22) in the limit $\alpha\to 0$.

To conclude our study of vertex functions, we exhibit the renormalization-group equations satisfied by the renormalized vertices Γ_R. It follows from (3.35) and the structural relations (5.13), (5.17), etc., that these equations read

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}-(n-m)\gamma\right]\Gamma_R^{(n,m)}=0\ . \tag{5.26}$$

More simply, the functional form (4.28) together with the Legendre-transform relations (5.6)—(5.8) imply the equivalent statement

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R} -\gamma\int d^4x\left(S_\mu\frac{\delta}{\delta S_\mu}-S_B\frac{\delta}{\delta S_B}\right)\right]\Gamma_R(S;g_R\mu)=0\ . \tag{5.27}$$

VI. CONCLUSION

In the previous sections, various aspects of the Landau gauge formulation of NAGT's were discussed and compared with the conventional Lorentz gauge formulations. It was seen in particular how potential difficulties associated with the vanishing of the gauge parameter (so that $(1/\alpha)\partial\cdot\vec{A}$ is ill-defined in the field equations) and the transversality of the gauge field Green's functions (so that vertex functions cannot be defined by amputation) were overcome by the introduction of the $\vec{B}$ field.

In the Lagrangian formalism [e.g., Eq. (2.8)], the Landau gauge involves one less parameter ($\alpha=0$) but one additional field $\vec{B}$. There is also the new field equation $\partial\cdot\vec{A}=0$, which is responsible for much of the simplicity of the Landau gauge. The $\vec{B}$ field provides a canonical momentum conjugate to $\vec{A}_0$, but satisfies a field equation (2.14) which is not independent of the other field equations. The $\vec{B}$ field is of course not "physical,"[19] just as $\partial\cdot\vec{A}$ in the Lorentz gauges, and it has unusual properties, such as a vanishing propagator (3.23). Also, its presence in Green's functions leads to disconnected nontransverse parts proportional to the A-B propagator (3.3), as illustrated in Eq. (3.18).

For Green's functions not containing B fields, the B field can be completely ignored provided the gauge-field propagator is taken to be the transverse expression (3.15). This exactly incorporates the effect of the A-B coupling and the perturbation theory (in g) expansion is about the bare theory which includes this coupling. We thus obtain the expected Feynman rules for these Green's functions and find that they are transverse in the vector indices and are the $\alpha\to 0$ limits of the Lorentz-gauge Green's functions.

Renormalization proceeds as usual, according to (3.26), (3.27), and (3.32). The renormalization-group equations (3.33) are *much* simpler than those (3.29) in the Lorentz gauges. This is basically because the Landau-gauge Green's functions depend on only one coupling constant g_R instead of the two

(g_R and α_R) which are present in the Lorentz gauges. Correspondingly, the solutions to the renormalization group equations involve only one "effective" coupling constant $\bar{g}(g)$ instead of the two $[\bar{g}(g,\alpha),\ \bar{\alpha}(g,\alpha)]$ which enter in the Lorentz gauges. This leads to an enormous simplification in the analysis of the consequences of renormalization invariance. We refer to Ref. 4 for more details and some illustrations of this.

Because $\langle T(A_\mu A_\nu)\rangle$ is transverse, gauge-field vertex functions cannot be defined in the usual amputational manner. Once the coupling to the B field is taken into account, however, unique vertex functions can be defined and employed in the usual way, e.g., for renormalization and renormalization group purposes. In the gauge-field subsector, the vertex functions were seen to be the $\alpha \to 0$ limits of the Lorentz-gauge vertex functions.

We have thus shown how all of the field-theoretic formalism used in the study of NAGT's in the Lorentz gauges can, *mutatis mutandis,* also be used in the Landau gauge. This provides a field-theoretical basis for the simplicity, both foundational and computational, of the Landau gauge.

Note added in proof. Landau gauge formalisms for non-Abelian gauge theories have also been discussed by N. Nakanishi [Phys. Rev. D 5, 1324 (1972)] and W. Kummer [Acta Physica Austriaca, Suppl. XV, 423(1976)]. We thank these authors for informing us of their work.

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*Research supported in part by the National Science Foundation under Grant No. PHY74-22218A01.

†Research supported in part by the National Science Foundation under Grant No. GP-43662X.

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[13]This is shown for the Lorentz gauges in Refs. 9–12.

[14]In this paper, we use the following different notations for Green s functions: $\langle 0|T(A\cdots A)|0\rangle = \langle T[A\cdots A]\rangle = G(A\cdots A) = \langle A\cdots A\rangle_+ \equiv G_{A\cdots A}$.

[15]We may assume that a regularization is present which renders the unrenormalized Green's functions finite.

[16]Fermion-number conservation implies an equal number of ψ's and $\bar{\psi}$'s and ghost-number conservation implies an equal number of C_1's and C_2's.

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[18]Provided that the perturbation expansion is an asymptotic series.

[19]The question of what are the physical observables in the unbroken gauge theories under discussion is at present unanswered.