Continuum limit of lattice gauge theories in the context of renormalized perturbation theory*

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It is shown that in spite of the modifications introduced by Wilson and Polyakov, the gauge theory on a lattice in the Abelian case in the limit of zero lattice spacing has the same renormalized S matrix as quantum electrodynamics, to all orders in the renormalized coupling constant. Apparently nonrenormalizable vertices contained in the lattice Lagrangian contribute to mass, wave-function, and coupling-constant renormalizations, but do not contribute to the "finite parts" as a result of being multiplied by additional powers of lattice spacing. It is crucial for this renormalizability that the lattice theory respects local gauge symmetry and discrete symmetries and has the correct "classical continuum limit." The fact that in a renormalizable field theory divergences are contained in the first few terms of the Taylor series expansion of the Green's functions about the external momenta, and that these divergences are mild, play an important role in our proof. Umklapp processes characteristic of the lattice regularization do not have any observable consequences in the continuum limit. Thus Wilson's lattice action is well suited for nonperturbative considerations of gauge theories.

I. INTRODUCTION

The gauge theory on a lattice, first formulated by Wilson and Polyakov,[1] has been exhaustively studied[2-7] in order to discover tractable methods of calculation for the color-gluon model, especially the hadron spectrum. The theory has a natural ultraviolet cutoff, the inverse lattice spacing. It can serve as a starting point for renormalization-group calculations[6,8,9] of quantum chromodynamics. In fact, even preliminary considerations such as an expansion in the inverse bare coupling constant have provided valuable insights into quantum chromodynamics. Quark confinement occurs for any nonzero lattice spacing, to any finite order in the inverse bare coupling constant. It can be seen how it is possible for asymptotic freedom and quark confinement to coexist.[2] Hamiltonian perturbation theory in conjunction with Padé summation techniques gives[4] quite good fits to the hadron spectrum. Crude renormalization-group calculations[6] also indicate significant possibilities.

The cutoff is not covariant. Also, the action is not just a straightforward discretization (i.e., replacement of continuous space-time by a lattice and derivatives by differences) of the continuum action. For example, in the Abelian case the following changes have been made[2]:

(1) There are $\bar{\psi}\psi$-multiphoton vertices with an arbitrarily large number of photons to maintain local gauge invariance.

(2) The gauge-field part of the action is suitably modified to have the *ad hoc* requirement of periodicity (the period is $2\pi/ea$ in the Abelian case[10]). This apparently technical requirement makes the strong-coupling expansion possible.

(3) The conventional γ_μ of the fermion vertex is replaced by $(1 \pm \gamma_I^E)$. This modification seems necessary, as otherwise electrons with momenta π/a would have very low energies and behave like a new species of particles[2] in low-energy experiments.

In spite of these modifications, when certain parameters are held fixed and the limit of zero lattice spacing is taken, we recover[1] the continuum action if we further assume that the lattice degrees of freedom pass over to smoothly varying fields. This is the classical continuum limit in contrast to the quantum (or the "statistical") continuum limit defined by Wilson.[9] This means that the classical theory of these lattice actions has the correct continuum limit. In classical theory, we are interested in initial configurations that have a smooth continuum limit. Then, because the action has a correct continuum limit in the above sense, so do the Euler-Lagrange equations, and hence the field configuration at any later instant will have a continuum limit corresponding to time evolution of the initial configuration in the continuum theory.

The continuum limit of the corresponding quantum theory is more delicate. If we use the lattice action to define the path integral for the continuum theory as with the time-slicing definition of Feynman,[11] we see that the fluctuations of the order of inverse lattice spacing (in case of Bose fields) contribute significantly[2] to the quantum amplitudes (Appendix A). Is it possible to choose the dependence of the parameters in the lattice action on the lattice spacing such that in the limit of zero lattice spacing we recover the consequences of the continuum theory?

We propose to check this in the Abelian case within the canons of renormalized perturbation theory. The multiphoton vertices give divergent contributions even when a $\to 0$, but their effects can be absorbed in the renormalization of the parameters of the lattice action and do not have any observable consequences.

Roth[12] has considered the continuum limit of U(1) and SU(N) lattice gauge theories in $1+1$ dimensions. His method is to sum the strong-coupling expansion. The continuum theory is made a potential theory by choosing the Coulomb gauge and is made discrete for comparison. The two theories give identical results when the bare coupling constant is fixed and the limit of zero lattice spacing is taken. This is to be expected as the theories are superrenormalizable. In our perturbative treatment equivalence in this case is straightforward.

This paper is organized as follows: In Sec. II, we develop a systematic perturbation expansion in e. This requires a proper choice of gauge and a nonlinear transformation on the "photon" field to make the range of its quantum fluctuations $(-\infty, +\infty)$. We analyze a few low-order diagrams to demonstrate how the renormalized Green's functions becomes identical to that of quantum electrodynamics. In Sec. III equivalence is exhibited to all orders in the renormalized coupling constant. In Sec. IV we discuss the implications of our analysis, in particular, the limitations of our perturbative proof.

In Appendix A we demonstrate effects of finite range of quantum fluctuations and write the full effective action relevant for perturbation expansion.

In Appendix B we discuss the nature of regularization provided by a lattice, and in particular, the umklapp conservation of momentum. In Appendix C power-counting arguments are briefly justified. In Appendix D, we derive the Ward-Takahashi identities relevant for our analysis.

II. CHOICE OF GAUGE, PROPAGATORS, AND PERTURBATION THEORY

We limit our discussion to the Abelian case. Wilson's action on a Euclidean space-time lattice is[2]

$$S=\tfrac{1}{2}a^3\sum_{ni}[\bar{\psi}_n(1-\gamma_i^E)e^{-ieaA_{ni}}\psi_{n+i} + \bar{\psi}_{n+i}(1+\gamma_i^E)e^{+ieaA_{ni}}\psi_n] - (ma^4+4a^3)\sum_n \bar{\psi}_n\psi_n + \frac{1}{2e^2}\sum_{nij}(e^{ieaf_{nij}}-1)\,. \tag{2.1}$$

Here n is a four-vector with integer components (in units of the lattice spacing) representing the sites of the lattice. i denotes a unit vector along the ith axis. γ_i^E are the Dirac matrices for the Euclidean space-time with the following algebra: $\{\gamma_i^E,\gamma_j^E\}=2\delta_{ij}$, $(\gamma_i^E)^\dagger=\gamma_i^E$. ψ_n is a 4-component Dirac spinor associated with site n. A_{ni} associated with the "link" joining sites n and $n+i$ is the gauge field. $f_{nij}=A_{ni}+A_{n+i,j}-A_{nj}-A_{n+j,i}$ is the analog of the field tensor.

Use of a Euclidean lattice does not present any conceptual problems. The Lorentzian Green's functions can be obtained from the Euclidean ones by an analytic continuation.[13]

We have explicitly exhibited dependence on a in the action as this is more relevant to see the continuum limit in perturbation theory. This action is invariant under the local gauge transformations

$$\psi_n \to e^{-i\theta_n}\psi_n\,,\quad \psi_n^\dagger \to \psi_n^\dagger e^{+i\theta_n}\,,$$
$$A_{n,i}\to A_{n,i}-\frac{1}{ea}(\theta_{n+i}-\theta_n)\,. \tag{2.2}$$

The action is also periodic in the A_{ni}'s with a periodicity $2\pi/ea$. This enables us to consider the range of quantum fluctuations of A_{ni} to be $(-\pi/ea, +\pi/ea)$. We therefore get

$$\langle m\rangle = \frac{\prod_{ni}\int_{-\pi/ea}^{+\pi/ea}dA_{ni}\prod_n\int d\psi_n\, d\bar{\psi}_n e^S m}{\prod_{ni}\int_{-\pi/ea}^{+\pi/ea}dA_{ni}\prod_n\int d\psi_n\, d\bar{\psi}_n e^S}\,, \tag{2.3}$$

where m is any gauge-invariant combination of operators and $\langle m\rangle$ represents the vacuum expectation value of the "T product" of this combination, where ordering is with respect to the n_4 coordinate. Integration over the fermion fields must be regarded in the sense of integration over anticommuting c numbers.[1,14] When $a\to 0$, the range becomes $(-\infty,+\infty)$ and hence naively speaking the finite range does not matter in the continuum limit.

When $a\to 0$, if e and m are fixed in (2.1) and we assume A_{ni}, ψ_n, and $\bar{\psi}_n$ pass over to continuous functions, we recover the QED action.[1] In particular,

$$\frac{1}{2a}(\bar{\psi}_n e^{-ieaA_{ni}}\psi_{n+1}+\bar{\psi}_{n+1}e^{+ieaA_{ni}}\psi_n)-\frac{1}{a}\bar{\psi}_n\psi_n \to 0\,,$$

so that replacing γ_i^E by $(1\pm\gamma_i^E)$ has no new effects.

Fields and parameters in (2.1) must be regarded as unrenormalized quantities. To develop a perturbation expansion in powers of e, starting from (2.3), we have to choose a gauge using the technique of Faddeev and Popov.[15,16] The matrix of quadratic form in A_{ni} in the action (2.1) is not invertible even in the discrete case.

Because the range of A_{ni} is finite the Faddeev-Popov procedure can be carried through only if we choose a gauge that is periodic in A_{ni} with a periodicity $2\pi/ea$ (or a submultiple of it). Consider the identity

$$\Delta(\{A_{ni}\})\prod_{ni}\int_{-\pi}^{+\pi} d\theta_n\, \delta\left(\sin\left[\sum_i ea(A_{ni}^{\theta} - A_{n-i,i}^{\theta})\right] - c_n\right) = 1 \tag{2.4}$$

where the c_n's are arbitrary parameters in the range $(-1,+1)$ and $\Delta(\{A_{ni}\})$ is the Jacobian factor coming from integrating the δ function:

$$\Delta(\{A_{ni}\}) = \prod_{ni} \cos\left(ea\sum_i (A_{ni}^{\theta^0} - A_{n-i,i}^{\theta^0})\right) \times (\text{constant})\,.$$

Here θ_n^0 is the value of θ_n where the δ-function constraint is satisfied. As usual[17] it follows that Δ is insensitive to the gauge transformations of A_{ni}. Therefore, it is independent of $\{\theta_n^0\}$ and hence of the c_n's. Thus integrating (2.4) over the c_n's with a weight factor $\exp(-c_n^2/2\alpha e^2)$ we may write

$$(\text{constant}) \times \Delta(\{A_{ni}\})\prod_n \int_{-\pi}^{+\pi} d\theta_n \times \exp\left(-\frac{1}{2\alpha e^2}\sin^2\left[ea\sum_i (A_{ni}^{\theta} - A_{n-i,i}^{\theta})\right]\right) = 1\,,$$

with Δ factoring out.

Substituting for the factor 1 from this identity in both the numerator and denominator of (2.3) we may make a change of variables $A_{ni} \to A_{ni}^{-\theta}$. Since the entire integrand including the gauge choice and the Jacobian factor are periodic in A_{ni}, we may move back the range of integration over A_{ni} to $(-\pi/ea, +\pi/ea)$, extracting a group volume factor $\prod_n \int d\theta_n$. We thus get

$$\langle m\rangle = Z^{-1}\prod_{ni}\int_{-\pi/ea}^{+\pi/ea} dA_{ni}\prod_n \int d\psi_n d\bar{\psi}_n\, m \exp\left[S - \frac{1}{2\alpha e^2}\sum_n \sin^2\left(ea\sum_i (A_{ni} - A_{n-i,i})\right) + \sum_n \ln\cos\left(\sum_i ea(A_{ni} - A_{n-i,i})\right)\right] \tag{2.5}$$

where Z is the integral on the right-hand side without the factor m.

Equality of (2.5) and (2.3) is mathematically exact and meaningful if we work with a finite lattice and periodic boundary conditions.

Now that a gauge is chosen, we may evaluate the Green's functions for non-gauge-invariant combinations of operators also; residues at the poles of such Green's functions provide the appropriate S matrix elements, which will be gauge invariant.

If we retain only the Fermi fields in the exponent, we can apply Wick's theorem to products of Fermi fields with a propagator[2]

$$S(p) = \frac{1}{m + (1/a)\sum_i (Q_i - iR_i\gamma_i^E)} \tag{2.6}$$

where

$$Q_i = 2\sin^2(\tfrac{1}{2}p_i a)\,,\quad R_i = \sin(p_i a)\,.$$

As $a \to 0$, we recover the continuum propagator

$$S(p) \to \frac{1}{m - i\sum_i p_i\gamma_i^E}\,.$$

Because of the finite range of integration over the A_{ni}'s, we do not recover the simple "Wick's theorem" for T products of the A_{ni}'s, even if we retain in the exponent only terms quadratic in A_{ni} (Appendix A). We stretch the range of integration to $(-\infty, +\infty)$ via a nonlinear transformation,

$$\tfrac{1}{2}eaB_{ni} = \tan(\tfrac{1}{2}ea)A_{ni} \tag{2.7}$$

which is an identity transformation when $ea = 0$. In fact, this procedure is necessary if we want to develop a consistent perturbation series in the coupling constant. e in (2.7) is the bare charge. We can now apply Wick's theorem with a propagator[18]

$$D_{ij}(k) = \frac{\delta_{ij}}{\sum_i (4/a^2)\sin^2(\frac{1}{2}k_i a)} \tag{2.8}$$

when $\alpha = 1$.

The Jacobian of the transformation (2.7)

$$\prod_{ni}\frac{1}{1 + (\frac{1}{2}ea)^2 B_{ni}^2} \tag{2.9}$$

is equivalent to a term $-\sum_{ni}\ln[1 + (\frac{1}{2}ea)^2 B_{ni}^2]$ in the effective action. This is similar to the ghost part[16] of the action in non-Abelian gauge theories. To $O(e^2)$ this shows a "mass" equal to $e/(a\sqrt{2})$ for the photon. Its function is to precisely cancel a singular mass appearing in $O(e^2)$ photon self-energy, thereby leaving the renormalized photon mass zero. This will be explicitly demonstrated later.

With transformation (2.7), calculating $\langle A_{n_1 i_1}\cdots A_{n_r i_r}\rangle$ amounts to evaluating a more complicated T product,

$$\langle [B_{n_1 i_1} - \tfrac{1}{3}(ea)^2 B_{n_1 i_1}^{\;3} + \cdots] \times [B_{n_2 i_2} - \tfrac{1}{3}(ea)^2 B_{n_2 i_2}^{\;3} + \cdots]\cdots\rangle\,.$$

At the poles of the Green's functions the nonlinear terms only contribute to a wave-function renormalization so that the renormalized S matrix can be calculated[17] by just considering $\langle B_{n_1 i_1}\cdots B_{n_r i_r}\rangle$.

We illustrate the nature of regularization provided by the lattice in Appendix B. A characteristic feature of lattice regularization is the umklapp conservation[19] of wave vectors. We demonstrate that

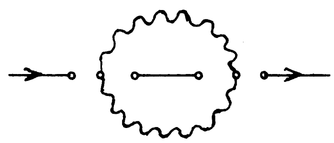

FIG. 1. An anomalous contribution to the self-energy of the electron. The gaps occur because the interaction involves a flip to the neighboring site. A_{ni} is supposed to be located at the midpoint of the sites n and $n+i$.

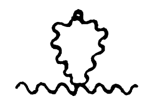

FIG. 2. Order-e^2 anomalous contribution to the photon self-energy.

the umklapp processes do not have any observable consequences in the continuum limit.

Looking at the action (Eq. (2.1), the effective action in terms of B_{ni} is exhibited in Appendix A [Eq. (A5)]), we observe that there are multiphoton vertices, apparently nonrenormalizable, not present in the continuum theory. These lead to anomalous diagrams (see Figs. 1–4 for examples). These new vertices occur with extra powers of a. But since there is a possibility that the integrals are also singular in a, we cannot immediately conclude that the new diagrams do not contribute when $a \to 0$. Indeed, if we consider an anomalous contribution of $O(e^4)$ to the electron self-energy (Fig. 1),

$$\Sigma^{(4)}_{an}(p) \sim e^4 a^2 \int_{-\pi/a}^{+\pi/a} \int_{-\pi/a}^{+\pi/a} \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{1}{\sum_i (4/a^2)\sin^2(\frac{1}{2}k_{1i}a)} \frac{1}{\sum_i (4/a^2)\sin^2(\frac{1}{2}k_{2i}a)}$$
$$\times \sum_i \{\cos([p-\tfrac{1}{2}(k_1+k_2)]_i a) + i\gamma_i^E \sin([p-\tfrac{1}{2}(k_1+k_2)]_i a)\}$$
$$\times \frac{1}{m+(1/a)\sum_i (Q_i - iR_i\gamma_i^E)}\bigg|_{p-k_1-k_2}$$
$$\times \{\cos([p-\tfrac{1}{2}(k_1+k_2)]_i a) + i\gamma_i^E \sin([p-\tfrac{1}{2}(k_1+k_2)]_i a)\}. \tag{2.10}$$

We note an additional power of a^2 from the two anomalous vertices, whereas when $a=0$ the integral has a superficial degree of divergence three leading to at most[20] a $1/a^3$ singularity when $a=0$. Thus a $1/a$ singularity is left, precisely as for the electron self-energy in a renormalizable theory. We justify such power-counting arguments in Appendix C. Note the strong momentum dependence in the vertex factors, which appears because of flip from one site to the next, occurring in the interaction term:

$$a^3 \sum_{ni} [\bar{\psi}_n (1-\gamma_i^E)\psi_{n+i} + \bar{\psi}_{n+i}(1+\gamma_i^E)\psi_n]\, \tfrac{1}{2}(iea)^2 A_{ni}{}^2 .$$

The momentum which appears at the vertex is the average of incoming and outgoing fermion momenta.

It is easy to see that first two terms in the Taylor series expansion about $p=0$ have the structure $\delta m(a) + z_2(a)\sum_i p_i\gamma_i^E$: Thus, for example, the linear term in p coming from the electron propagator is

$$\int_{-\pi/a}^{+\pi/a} \int_{-\pi/a}^{+\pi/a} \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{1}{\sum_i \sin^2(\frac{1}{2}k_{1i}a)} \frac{1}{\sum_i \sin^2(\frac{1}{2}k_{2i}a)}$$
$$\times \sum_i (E_i' + i\gamma_i^E O_i') \sum_n (E_n + i\gamma_i^E O_n') \sum_m (+O_m p_m - i\gamma_m^E E_m'' p_m) \sum_p (E_p' - i\gamma_p^E O_p')(E_i' + i\gamma_i^E O_i'),$$

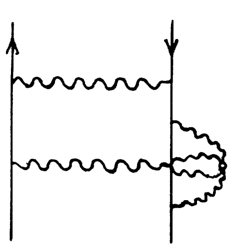

FIG. 3. A superficially convergent diagram with an internal anomalous renormalization part.

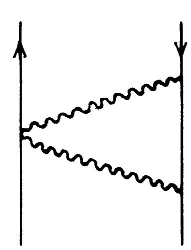

FIG. 4. An anomalous contribution to the electron-positron scattering amplitude.

where O_i, O_i' are odd and E_i, E_i', E_i'' are even in (k_1+k_2). Also O_i goes over to O_j under $(k_1+k_2)_i \to (k_1+k_2)_j$ and likewise for E_i. Then the symmetric way in which different components of internal momenta enter shows that only $\sum_i \gamma_i^E p_i$ terms survive, other terms becoming zero on integration because of the oddness of the integrand.

Now consider the third term in the Taylor series expansion[21] of $\Sigma_{an}^{(4)}(p)$ about $p=0$, obtained by differentiating twice with respect to the p_i's and setting all p_i's to zero. Differentiation of propagators will increase the powers of internal momenta in the denominator, in the limit $a=0$, so that with a^2 outside, $\Sigma(p)$ has the form $a^2 O(1/a)$ and vanishes in the limit $a\to 0$. When the vertices are differentiated, we notice a form $a^4 O(1/a^3)$ which again vanishes. This argument is valid for all higher terms in the Taylor series expansion. Thus we see that anomalous diagrams do not lead to "finite parts" when $a\to 0$.

Note that in the case of the normal self-energy diagram there are no explicit powers of a outside the integral. The integral is linearly divergent when $a=0$. For the third term in the Taylor series expansion, about $p=0$, the situation is now different. Differentiating the vertices, we get an a^2 outside, and the integral remains linearly divergent if a is set to zero. We thus get $a^2 O(1/a)$ which becomes zero when $a\to 0$. But if we differentiate propagators, there are no powers of a coming out, and the integral is finite when $a=0$. Indeed, we recover exactly the corresponding expression of the continuum theory. Thus we see that momentum-dependent vertices do not lead to finite parts and the finite part is exactly as in the continuum theory.

Now consider the $O(e^2)$ contribution to the photon self-energy, due to the anomalous 4-photon vertex (see Fig. 2). Relevant vertices are [see Eq. (A5) of Appendix A]

$$\frac{e^2a^4}{2^3 3!}\sum_{nij}(B_{ni}+B_{n+i,j}-B_{nj}-B_{n+j,i})^4 + \frac{e^2a^4}{2^3 3}\sum_{nij}(B_{ni}+B_{n+i,j}-B_{nj}-B_{n+j,i}) \times ({B_{ni}}^3+{B_{n+i,j}}^3-{B_{nj}}^3-{B_{n+j,i}}^3) + \frac{1}{\alpha}\frac{e^2a^4}{3!}\sum_n\left(\sum_i(B_{ni}-B_{n-i,i})\right)^4 + \frac{1}{\alpha}\frac{e^2a^4}{2^2 3}\sum_n\left(\sum_i(B_{ni}-B_{n-i,i})\right)\left(\sum_i({B_{ni}}^3-{B_{n-i,i}}^3)\right). \tag{2.11}$$

Moreover, there are counterterms

$$-Z_3^{(2)}(a)\frac{a^2}{4}\sum_{nij}(B_{ni}+B_{n+i,j}-B_{nj}-B_{n+j,i})^2 - [Z_3^{(2)}(a)-\alpha^{(2)}(a)]\frac{a^2}{2}\sum_n\left(\sum_i(B_{ni}-B_{n-i,i})\right)^2 \tag{2.12}$$

coming from rescalings of B_{ni} and α, respectively, and counterterms

$$-\frac{(ea)^2}{2!}\sum_n\left(\sum_i(B_{ni}-B_{n-i,i})\right)^2-(\tfrac{1}{2}ea)^2\sum_{ni}{B_{ni}}^2 \tag{2.13}$$

coming from the ghost part of the action.

Vertices (2.11) give an amplitude[18]

$$\Pi_{IJ}^{(2)\,\text{loop}}(p)=\left[\delta_{IJ}\sum_j\sin^2(\tfrac12 p_j a)-\left(1-\frac{1}{\alpha}\right)\sin(\tfrac12 p_I a)\sin(\tfrac12 p_J a)\right]\frac{2Ae^2}{a^2}\left(\frac{3}{2}+\frac{1}{2\alpha}\right)\frac{e^2}{4a^2} +\sin(\tfrac12 p_I a)\sin(\tfrac12 p_J a)\frac{2^3}{\alpha a^2}e^2+\delta_{IJ}\sum_j\sin^2(\tfrac12 p_j a)\frac{e^2}{a^2}-\sin(\tfrac12 p_I a)\sin(\tfrac12 p_J a)\frac{e^2}{a^2}, \tag{2.14}$$

where

$$\frac{A}{a^2}=\int_{-\pi/a}^{+\pi/a}\frac{d^4k}{(2\pi)^4}\frac{1}{\sum_i(4/a^2)\sin^2(\frac12 k_i a)}$$

and we have used the fact that

$$\int_{-\pi/a}^{+\pi/a}\frac{d^4k}{(2\pi)^4}\frac{\sin^2(\frac12 k_j a)}{\sum_i(4/a^2)\sin^2(\frac12 k_i a)}$$

is independent of j and is $1/16a^2$.

Note that the singular photon mass from the loop [Eq. (2.14)] is canceled by the ghost contribution Eq. (2.13). Choosing

$$Z_3^{(2)}=\frac{I(a)}{2a^2}e^2+\frac{1}{4a^2}e^2,$$

$$\alpha^{(2)}=-\frac{11}{4a^2}e^2,$$

$\Pi_{IJ}^{(2)\,\text{loop}}$ can be completely canceled in the limit $a\to 0$.

III. PROOF OF RENORMALIZABILITY TO ALL ORDERS

Following the lines indicated in low-order calculations in Sec. II, we prove renormalizability to all orders below:

A. Normal diagrams contribute the same finite part as in the continuum case

By "normal diagrams" we mean those that have counterparts in the continuum theory, i.e., all the vertices involve one photon and two electron lines. The vertex factor and the propagators are different, however. By finite part we mean the part that is left when subtractions are made exactly as for the corresponding Green's functions in the continuum theory.

Amplitudes for normal diagrams reduce to those of the continuum case, when we formally set a to zero. This is because all propagators go over to the continuum ones, and the range of integration $(-\pi/a, +\pi/a)$ becomes $(-\infty, +\infty)$. Normal vertices also go over to the continuum ones, since we know that we recover the continuum action when $a \to 0$. In particular, the $\bar{\psi}\psi A_i$ vertex

$$a^4 \frac{ie}{2} \sum_{ni} [\bar{\psi}_n (1-\gamma_i^E) \psi_{n+i} A_{ni} - \bar{\psi}_{n+i}(1+\gamma_i^E)\psi_n A_{ni}]$$

gives

$$\frac{ie}{2}[(1-\gamma_i^E) e^{-ip_i a} - (1+\gamma_i^E) e^{+ip_i a}],$$

where p is the average of the incoming and outgoing fermion momenta.[18] As $a \to 0$ we recover $-ie\gamma_i^E$ and the unit Dirac matrix does not contribute.

Later we consider the structure of the terms singular in a and show that they can be removed by a renormalization of parameters. Therefore, the finite part is exactly as in the continuum theory.

B. Anomalous diagrams give a vanishing contribution to the finite parts

As there are "nonrenormalizable" vertices, we encounter integrals of increasing singularities in a. However, powers of a associated with anomalous vertices occur outside the integral. Counting powers of a, from both sources, we see that the superficial singularity in a of any diagram is $D_a = 4 - \frac{3}{2}F - B$ where B and F are the numbers of external photon and electron lines, respectively.[22] This is exactly as in a renormalizable theory. The singularity in a does not increase with the order of the diagram and only the self-energy and $\bar{\psi}\psi A_i$ vertex diagrams have superficial singularities.[23] Thus new diagrams (examples, Figs. 1–3) not present in the continuum theory contribute to the divergence structure.

Such primitively divergent anomalous diagrams have a general form[20]

$$a^n \times (\text{an integral with at most a } 1/a^{n+D_a} \text{ singularity}),$$

where[24] $n > 0$. If first $D_a + 1$ terms in the Taylor series expansion about external momenta are removed by counterterms, the integral takes the form $a^n O(1/a^{n-1})$ (Appendix C) and vanishes when $a \to 0$ (provided that all internal renormalization parts have been already rendered finite by similar subtractions). We consider the γ-matrix structure and Lorentz structure of the singular terms later.

Now anomalous diagrams with $D_a < 0$ can be handled. If anomalous vertices are contributing to renormalization parts (example, Fig. 3), these become zero with suitable subtractions and hence the entire diagram. If anomalous vertices are not contributing to renormalization parts (example, Fig. 4), the diagram becomes zero when $a = 0$, because of excess factors of a outside the integral (provided normal renormalization parts, if any, have been rendered finite by subtractions). It will be shown below that there are no singular counterterms resulting from rescalings for diagrams with $D_a < 0$.

C. Structure of the singular terms

(a) The fermion self-energy $\Sigma(p)$ has a divergence structure $\delta m(a) + z_2(a) \sum_i p_i \gamma_i^E$. $\Sigma(p)$ is invariant under $p_I \to -p_I$, $\gamma_I^E \to -\gamma_I^E$ for any particular index I. This follows from invariance of the action under inversion of the Ith axis.[25] This invariance gives the structure $\delta m(a) + \sum_i z_i(a) p_i \gamma_i^E$ for the first two terms of the Taylor series expansion of $\Sigma(p)$ about $p = 0$, which are the only terms singular in a. The symmetric way in which all axes are treated makes the z_i's all the same.

(b) The $\bar{\psi}\psi A_I$ vertex has a divergence structure $Z_1(a)\gamma_I^E$. The discrete symmetry[25] mentioned above implies that $\Gamma_I(p,k)$ is odd under $p_I \to -p_I$, $k_I \to -k_I$, $\gamma_I^E \to -\gamma_I^E$ and even under these reversals for other indices. This immediately leads to the divergence structure $Z_1 \gamma_I^E$.

(c) Divergent parts of the photon self-energy can be removed by rescalings of the A_{ni} and the gauge-fixing parameter α. We may rewrite the Ward-Takahashi identity (D4) (Appendix D) in the form

$$\sum_j 2i \sin(\tfrac{1}{2} p_j a) \tilde{D}_{ji}^{-1}(p) = D^{-1}(p) \times (\text{nonpole terms})_i, \tag{3.1}$$

where $\tilde{D}_{ji}(p)$ is the full photon propagator, and the nonpole terms come from one particle irreducible parts of the Green's functions

$$\left\langle \sin\left[2ea \sum_i (B_{ni} - B_{n-i,i})\right], B_{0j} \right\rangle$$

and

$$\left\langle \tan\left[ea \sum_i (B_{ni} - B_{n-i,i})\right], B_{0j} \right\rangle.$$

This means that when $\tilde{D}_{ji}^{-1}(p)$ is "contracted" with $\sin(\frac{1}{2} p_j a)$, we must be able to remove a factor

$$D^{-1}(p) = \sum_i \frac{4}{a^2} \sin^2(\tfrac{1}{2} p_i a).$$

This implies that the photon remains massless to all orders. (A nonzero mass would lead to a term linear in p on the left-hand side.) Moreover, $\tilde{D}_{ji}^{-1}(p)$ must have a structure,

$$\tilde{D}_{ji}^{-1}(p) = A_{ji}(p) \sum_k \frac{4}{a^2} \sin^2(\tfrac{1}{2} p_k a) + \frac{2}{a} \sin(\tfrac{1}{2} p_j a) B_i(p).$$

Since $\tilde{D}_{ji}^{-1}(p)$ is invariant under $p \to -p$ and $i \leftrightarrow j$, $B_i(p)$ must have the form $(2/a)C(p)\sin(\frac{1}{2}p_i a)$, where $C(p)$ has no dependence on index i. Moreover, $\frac{1}{2}[A_{ij}(p) + A_{ij}(-p)]$ must be a symmetric matrix. The symmetric way in which different space-time axes are treated shows that if we choose $K = (k,k,k,k)$, $\frac{1}{2}[A_{ij}(K) + A_{ij}(-K)]$ can only be a linear combination $D\delta_{ij} + E$, where the second term is the same for all j and i. But a term like E leads to $\sum_j 2i \sin(\frac{1}{2}p_j a)$ on the left-hand side of (3.1), and such terms independent of index i are absent on the right-hand side.

In contrast to the conventional treatment of quantum electrodynamics, it cannot be shown that $C = -D$. Indeed, in our order-e^2 calculation in Sec. II, this did not happen. But in addition to a rescaling of the photon field, we can make a renormalization of the gauge-fixing parameter α. This gives counterterms of the form

$$-D\delta_{ij} \sum_m \frac{4}{a^2} \sin^2(\tfrac{1}{2} p_m a) - C \frac{4}{a^2} \sin(\tfrac{1}{2} p_i a) \sin(\tfrac{1}{2} p_j a). \tag{3.2}$$

A_{ij} (as also C) is made finite (in the limit $a \to 0$) by one subtraction. Therefore $A_{ij}(p) - A_{ij}(k)$ and $A_{ij}(p) - A_{ij}(-k)$ are finite, and so also is $A_{ij}(p) - \frac{1}{2}[A_{ij}(k) + A_{ij}(-k)]$.

Since we have proven that $\frac{1}{2}[A_{ij}(k) + A_{ij}(-k)]$ is diagonal for $K = (k,k,k,k)$, the counterterm (3.2) will make the photon self-energy finite in the limit $a \to 0$. We have to make subtractions off the $p = 0$ point to avoid infrared divergences.

(d) The photon-photon scattering amplitude has no superficial singularities in a. Equation (D8) of Appendix D,

$$\lim_{k\to 0} \Big(\sum_i \sin(\tfrac{1}{2} k_i a) \Pi_{ijkm}(k, k_1, k_2, k_3) \langle 1 + (\tfrac{1}{2} ea)^2 B_{ni}{}^2 \rangle = \sum_{j_1} \sin(\tfrac{1}{2} k_{j_1} a) V_{j_1;km}(k + k_1, k_2, k_3) \times \tilde{D}_{j_1 j}{}^{-1}(k_1) + \text{cyclic} \Big),$$

where $V(k + k_1, k_2, k_3)$ has no poles in its arguments, shows that the term independent of external momenta is indeed absent in Π_{ijkm}. Hence the photon-photon scattering amplitude does not have logarthmic singularities in a as superficial power counting suggests.

(e) Additive and mutliplicative renormalization of parameters do not generate nonvanishing counterterms from the anomalous vertices: To perform renormalization, we make the substitutions

$$\psi_n = Z_2{}^{1/2} \psi_n^R, \quad A_{ni} = Z_3{}^{1/2} A_{ni}^R,$$
$$m = m_R + \frac{\delta m}{Z_2}, \quad e = e_R Z_1 / (Z_2 Z_3{}^{1/2}),$$

where Z_1, Z_2, Z_3, e, m are to be regarded as series in e_R, the renormalized coupling constant, with coefficients as functions of a and m_R. This gives the counterterm for electron self-energy:

$$(Z_2 - 1)\Big(\sum_i \frac{1}{2a} [(1 - \gamma_i^E) e^{-ip_i a} + (1 + \gamma_i^E) e^{+ip_i a}] - (m_R + 4/a) \Big) - \delta m .$$

Since Z_2 has at the most a logarithmic singularity in a, the part that survives as $a \to 0$ is

$$(Z_2 - 1)\Big(\sum_i (+ip_i \gamma_i^E) - m_R \Big) - \delta m ,$$

which is precisely what is needed to cancel the divergent part. Similar results follow for the photon self-energy and vertex correction counterterms.

For anomalous vertices, the counterterms obtained by this procedure become zero when $a = 0$. Thus, for example, the anomalous vertex $ae_0{}^2 \bar{\psi}\psi A^2$ gives

$$aZ_1(Z_1/Z_2) e_R{}^2 \bar{\psi}_R \psi_R A_R{}^2$$

and since $Z_1{}^2/Z_2$ is a power series in e_R with powers of $\ln(am_R)$ as coefficients, $aZ_1{}^2/Z_2 \to 0$ as $a \to 0$.

IV. DISCUSSION

We have shown that the only effect of the anomalous vertices with corresponding powers of the lattice spacing multiplying them is to contribute to the unobservable wave-function, coupling-constant, and mass renormalizations. The fact that the classical lattice theory has the correct continuum limit was relevant in giving the correct finite part. The feature of renormalizable theories that the divergences are limited to the first few terms of the Taylor expansion of the Green's functions about the external momenta together with the local gauge invariance, the symmetric manner in which all the space-time axes were treated, discrete symmetries like parity, charge-conjugation invariance, all have played a crucial role in giving divergent terms having the same Lorentz and γ-matrix structure as in the continuum theory. Since the divergences were logarithmic, it was possible to remove all the divergent parts by a rescaling of the few parameters available. Apart from the features involved here, we could have chosen any lattice action, any gauge, and any nonlinear transformation without changing the continuum limit.

Our proof was in the context of renormalized perturbation theory. If the renormalized perturbation series were convergent, we could have concluded that within the domain of convergence the lattice action in the continuum limit is equivalent to QED even in a nonperturbative context. Even in this case, since at a critical point the dependence on the coupling constant is nonanalytic, in principle the lattice action could give a different theory in the continuum limit, for strong couplings.

But indications are that the renormalized perturbation series in QED is an asymptotic series.[26] Therefore it is mathematically possible that the continuum limit of the lattice theory is altogether different from quantum electrodynamics as defined through a Lorentz-covariant regularization.

However, quantum electrodynamics as we know it today is defined by renormalized perturbation theory. Our faith in this prescription comes from the excellent agreement with experiments.[27] In principle it is possible that QED by itself is mathematically meaningless, its successes with experiments being due to a coincidence that the renormalized perturbation theory in lower orders is a valid approximation to the complete theory. Taken in this light, our proof is of the same status as our belief that renormalized perturbation theory reflects the characteristics of the exact theory.

Divergences are not just a feature of the perturbation theory.[28,8,9] Thus for any nonperturbative considerations, whether it be constractive field theory[28] or the renormalization group,[8,9] we must first define the theory as a singular limit of theories corresponding to a sequence of actions with a cutoff. In this context, a lattice cutoff has certain natural advantages.[8,28] Since by faithfully following the prescription of renormalized perturbation theory we have shown that the lattice action of Wilson and Polyakov has the same consequences as QED with a Lorentz-invariant cutoff, it may be used for investigating questions such as the existence of the infinite cutoff limit of QED.

We expect our analysis to go through in the non-Abelian case also. Hence the lattice action may be used for renormalization-group calculations of quantum chromodynamics.

Our proof involved an expansion of the effective action (A5) in powers of e. Such an expansion is absolutely convergent only for $\frac{1}{2}ea|B_{ni}|<1$. But B_{ni} ranges over $(-\infty,+\infty)$. However, if we sum all diagrams of a given order in $\hbar$ i.e., diagrams with a specified number of loops), we should not expect any trouble on this account.

The case of an anisotropic lattice, which is relevant for the Hamiltonian formulation[4] and for the transverse lattice formulation,[5] will be treated elsewhere.[29]

The anomalous vertices of our action are anologous to the irrelevant variables in the renormalization-group formalism.[8,9] Our analysis has shown that the cutoff dependence of the bare charge to reach the required continuum limit can be altered by adding vertices of higher dimensions with corresponding powers of the cutoff. The freedom this provides us will be analyzed elsewhere.

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APPENDIX A: EFFECTIVE ACTION

Consider a free scalar field on a lattice. The r-point Green's function is given by

$$G_{n_1\cdots n_r}=Z^{-1}\prod_n \int d\phi_n \exp\left(-\tfrac{1}{2}a^2(8+m^2a^2)\sum_n \phi_n{}^2 + \tfrac{1}{2}a^2\sum_{ni}\phi_n\phi_{n+i}\right)\phi_{n_1}\cdots\phi_{n_r}. \quad \text{(A1)}$$

Fluctuations of $O(1/a)$ for each ϕ_n give a contribution of $O(1)$ to the action and hence contribute significantly to the integral. Hence[2] it is difficult to make a connection between continuum limits of the classical theory where $\phi_q-\phi_{q-i}$ is of order a (because of continuity of the classical fields) and the corresponding limit in the quantum theory.

To evaluate $G_{n_1\cdots n_r}$ we regard the term

$$\tfrac{1}{2}a^2\sum_{ni}\phi_n\phi_{n+i}$$

as an interaction (flip from one site to the next) and calculate

$$\prod_m \int d\phi_m \exp\left(-\tfrac{1}{2}a^2(8+m^2a^2)\sum_n \phi_n{}^2\right)\phi_{m_1}\cdots\phi_{m_k} \quad \text{(A2)}$$

using

$$\prod_m \int d\phi_m \exp\left(-\tfrac{1}{2}a^2(8+m^2a^2)\sum_n \phi_n^{\,2}+\sum_n j_n\phi_n\right) = \exp\left(\tfrac{1}{2}\sum_{qp}\frac{j_q\,\delta_{qp}\,j_p}{a^2(8+m^2a^2)}\right).$$

This last identity is valid only if the range of integration is $(-\infty,+\infty)$ for each ϕ_m. Only then, evaluating (A2) is equivalent to applying Wick's theorem for the product $\phi_{m_1}\cdots\phi_{m_k}$ with $\phi_n^{\bullet}\phi_m^{\bullet} = \delta_{nm}/a^2(8+m^2a^2)$. In this case

$$\begin{aligned} G_{n_1\cdots n_r} &= Z^{-1}\left\langle \sum_{N=0}^{\infty}\frac{a^{2N}}{N!}\left(\sum_{ni}\phi_n\phi_{n+i}\right)^N \phi_{n_1}\cdots\phi_{n_r}\right\rangle \\ &= Z^{-1}\left\langle \sum_{N=0}^{\infty}\frac{a^{2N}}{N!}\left(\sum_{ni}\phi_n\phi_{n+i}\right)^N \sum_{i=1}^{r-1}\delta_{n_r n_i}\phi_{n_1}\cdots\phi_{n_{i-1}}\phi_{n_{i+1}}\cdots\phi_{n_{r-1}}\right\rangle \\ &\quad + Z^{-1}\left\langle \sum_{N=1}^{\infty}\frac{a^{2N}}{N!}\left(\sum_{ni}\phi_n\phi_{n+i}\right)^{N-1} N\phi_{n_1}\cdots\phi_{n_{r-1}}\sum_i(\phi_{n_r+1}+\phi_{n_r-1})\right\rangle . \end{aligned}$$

Hence, we get a recursion relation,

$$G_{n_1\cdots n_r} = \sum_{i=1}^{r-1}\delta_{n_r n_i}G_{n_1\cdots n_{i-1},n_{i+1}\cdots n_{r-1}} + a^2\sum_{\pm i} G_{n_1\cdots n_{r-1},n_r+i}\,.$$

This can be solved by considering the Fourier coefficients[2] and is given by applying Wick's theorem to the product $\phi_{n_1}\cdots\phi_{n_r}$ with a propagator in momentum space,

$$D(p) = \frac{1}{m^2+\sum_i(4/a^2)\sin^2(\frac{1}{2}p_i a)}\,. \tag{A3}$$

If the range of integration is finite, say $(-\pi/ea, +\pi/ea)$, the above result is no longer valid. We may evaluate (A1) in this case, as a power series in ea, by making a nonlinear transformation,

$$\tfrac{1}{2}ea\,\theta_n = \tan(\tfrac{1}{2}ea)\phi_n\,. \tag{A4}$$

Under this transformation, we get nonquadratic terms in θ_n, in the exponent. We may now make an expansion in ea, using the propagator (A3). Thus, a finite range of integration effectively introduces interactions. (If there are interactions already, it modifies them.)

Applying the transformation $\frac{1}{2}eaB_{ni} = \tan(\frac{1}{2}ea)A_{ni}$ for our theory, we get an effective action

$$\begin{aligned} S_{\rm eff} =& -\frac{1}{e^2}\sum_{P(n)}\left[1+\left(\frac{1-I_2^{P(n)}+I_4^{P(n)}}{I_1^{P(n)}-I_3^{P(n)}}\right)^2\right]^{-1} - \frac{2}{\alpha e^2}\sum_{F(n)}\frac{X_{F(n)}{}^2}{(1+X_{F(n)}{}^2)^2} + \sum_{F(n)}\ln\left(\frac{1-X_{F(n)}{}^2}{1+X_{F(n)}{}^2}\right) \\ & - \sum_{L(n)}\ln\left[(1+iI_1^{L(n)})(1-iI_1^{L(n)})\right] - (ma^4+4a^3)\sum_{S(n)}\bar\psi_n\psi_n \\ & + \tfrac{1}{2}a^3\sum_{L(n)}\left(\bar\psi_n(1-\gamma_i^E)\frac{1-iI_1^{L(n)}}{1+iI_1^{L(n)}}\psi_{n+i} + \bar\psi_{n+i}(1+\gamma_i^E)\frac{1+iI_1^{L(n)}}{1-iI_1^{L(n)}}\psi_n\right), \end{aligned} \tag{A5}$$

where

$$X_n = \frac{I_1^{F(n)}-I_3^{F(n)}+I_5^{F(n)}-I_7^{F(n)}}{I_0^{F(n)}-I_2^{F(n)}+I_4^{F(n)}-I_6^{F(n)}+I_8^{F(n)}}\,.$$

Here I_k^N $(k\le N)$ for a set $(x_1,\ldots,x_N)$ stands for an algebraic multinomial of degree k,

$$I_k^N = \sum_{(i_1,\ldots,i_k)}^{N} x_{i_1},\ldots,x_{i_k}$$

where no two indices of the set $(i_1,\ldots,i_k)$ are identical. $I_k^{L(n)}, I_k^{P(n)}, I_k^{F(n)}$ stand for such a multinomial formed out of $\frac{1}{2}ea\,B_{ni}$'s with ni's corresponding to a "link" [Fig. 5(a)], the sides of an oriented "placquette" [Fig. 5(b)] with vertex at site n, and the sides of an oriented frame [Fig. 5(c)], respectively.

In deriving this form of $S_{\rm eff}$ we have used the trigonometric identity

$$\tan^{-1}X_1+\cdots+\tan^{-1}X_n = \tan^{-1}\frac{I_1^N-I_3^N+I_5^N-\cdots}{I_0^N-I_2^N+I_4^N-\cdots}\,,$$

where $I_0^N=1$ and the highest term I_N^N is in the numerator or denominator according to whether N is odd or even, respectively.

Observing that I_k^N is the coefficient of x^k in

$$f_N = (1+ixx_1)(1+ixx_2)\cdots(1+ixx_N)\,,$$

we can rewrite S_{eff} in the form

$$S_{\text{eff}} = -\frac{1}{e^2}\sum_{P(n)}\sin^2\theta_{P(n)} - \frac{1}{2\alpha e^2}\sum_{F(n)}\sin^2 2\theta_{F(n)} + \sum_{F(n)}\ln\cos 2\theta_{F(n)} - \sum_{L(n)}\ln|f_{L(n)}|^2$$
$$-(ma^4+4a^3)\sum_n\bar\psi_n\psi_n + \tfrac{1}{2}a^3\sum_{L(n)}[\bar\psi_n(1-\gamma_i^E)e^{-2i\theta_{L(n)}}\psi_{n+i} + \bar\psi_{n+i}(1+\gamma_i^E)e^{2i\theta_{L(n)}}\psi_n]\,, \tag{A6}$$

where θ is $\arg f$. $f_{L(n)}$, $f_{P(n)}$, and $f_{F(n)}$ are the functions f_N associated with the link, placquette, and frame, respectively, at the site n.

It is interesting to observe that the B_{ni} transforms under local gauge transofrmations as

$$B_{ni} \to \frac{B_{ni} - (2/ea)\tan\frac{1}{2}(\theta_{n+1}-\theta_n)}{1+\frac{1}{2}eaB_{ni}\tan\frac{1}{2}(\theta_{n+1}-\theta_n)}$$

which is a projective transformation.

APPENDIX B: NATURE OF REGULARIZATION PROVIDED BY A LATTICE

In this appendix we demonstrate the umklapp conservation of momentum provided by a lattice and show that it does not lead to any observable effects. We also exhibit the nature of regularization provided by a lattice.

We consider the decay process "1" → "2" + "3" + "4", with "1" at rest. With a simple $\lambda\chi\varphi^3$ vertex, we get in the lowest order, an amplitude

$$m \sim \sum_n \lambda a^4 \exp[i(k_1-k_2-k_3-k_4)na]\,. \tag{B1}$$

The sum vanishes unless $p_i = (k_1-k_2-k_3-k_4)_i$ is an integral multiple of $2\pi/a$ for each i. It is the periodic δ function $(2\pi)^4\delta^4(p\mid \text{mod } 2\pi/a)$ which has the property that its integral is one whenever the range of integration covers the point where p_i is an integral multiple of $2\pi/a$ for each i. All observables in a lattice theory have this periodictiy. States with momenta p_i and $p_i+2N_i\pi/a$ are identical: the phase factor $e^{ip\cdot x}$ assumes identical values at the lattice sites. Therefore to label the states, it is sufficient to restrict each p_i to an interval of size $2\pi/a$, say $(-\pi/a, +\pi/a)$. An enumeration of the number of states in a finite lattice and the available number of degrees of freedom justifies this.

If now the χ field is sufficiently massive, decay processes wherein all the wave vectors are in the forward direction are possible. [For example, "1" at rest and "2", "3", "4" each carrying a momentum $2\pi/(3a)$.] But such processes can occur only if at least one particle has a momentum of the order π/a. In the limit $a\to 0$, this means "1" must be infinitely massive. Hence such processes are irrelevant.

But virtual particles in loops can have arbitrarily large momenta. We consider the order-λ^2 self-energy in $\sum_n \lambda\phi_n{}^4$ theory (Fig. 6):

$$\Sigma(p) \sim \sum_n a^4\exp(ip\cdot na)D_{n0}{}^3 = \int_{-\pi/a}^{+\pi/a}\cdots\int_{-\pi/a}^{+\pi/a}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\frac{1}{m^2+(4/a^2)\sum_i\sin^2(\frac{1}{2}k_{1i}a)}$$
$$\times\frac{1}{m^2+(4/a^2)\sum_i\sin^2(\frac{1}{2}k_{2i}a)}\;\frac{1}{m^2+(4/a^2)\sum_i\sin^2(\frac{1}{2}k_{3i}a)}$$
$$\times(2\pi)^4\delta^4(p-k_1-k_2-k_3\mid \text{mod } 2\pi/a)\,. \tag{B2}$$

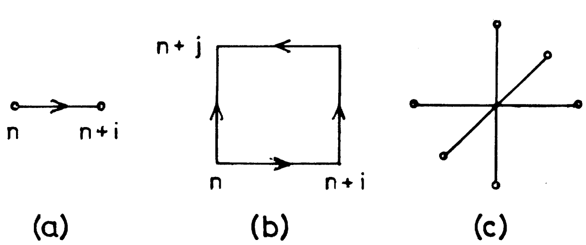

FIG. 5. (a) Link, (b) plaquette, and (c) frame. The fourth axis is not indicated in (c).

If p, k_1, and k_2 are such that $(p-k_1-k_2)_i > \pi/a$, the value of k_{3i} that contributes to the integral is $(p-k_1-k_2)_i - 2\pi/a$ (an umklapp process[19]). This is true generally. For any given configuration of the other momenta, there is a unique value of the

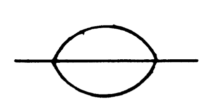

FIG. 6. The lowest-order self-energy diagram in ϕ^4 theory.

momentum we are integrating out, in the range $(-\pi/a, +\pi/a)$. But as the integrand is periodic in k_{3i}, the effect is just that of an ordinary δ function:

$$\Sigma(p) \sim \int_{-\pi/a}^{+\pi/a}\int_{-\pi/a}^{+\pi/a} \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} D(k_1)D(k_2)D(p-k_1-k_2)\,. \tag{B3}$$

This formally goes over to the divergent integral encountered in the continuum theory when $a \to 0$.

The contribution of the "normal" processes to $\Sigma(p)$ is

$$\Sigma_{\text{normal}}(p) \sim \int_{-\pi/a}^{+\pi/a}\int_{-\pi/a}^{+\pi/a} \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} \times D(k_1)D(k_2)D(p-k_1-k_2-k_3) \times \prod_i \theta(\pi/a-(p-k_1-k_2)_i) \times \theta((p-k_1-k_2)_i+\pi/a)$$

which also goes over to the divergent integral of the continuum theory when a is set equal to zero. This does not mean that the contribution of the Umklapp processes vanishes when $a \to 0$, as will be shown below. The reason is that $\lim_{a\to 0}\Sigma(p)$ is divergent and $\infty = \infty + \infty$ is possible.

It is seen from Fig. 7 that the area of the k_{1i}-k_{2i} space which contributes to the umklapp processes is $2\times\frac{1}{2}(\pi/a)^2$ for each i. (We have set $p=0$.) Using the lower bound, $D(k) > (m^2+16/a^2)^{-1}$,

$$\Sigma_{\text{umklapp}}(0) > \text{const}\times\left[\left(\frac{\pi}{a}\right)^2\right]^4\left(\frac{1}{m^2+16/a^2}\right)^3 \sim \frac{1}{a^2} \tag{B5}$$

which blows up as $a \to 0$.

On the other hand, if the integral were convergent when $a=0$, since the total and the normal contributions have the same limit, the umklapp contribution should vanish in the limit $a \to 0$. This can also be justified using simple bounds for the integrals. We will do this for the case of two internal-momentum integrations, and the proof for the general case will be a straightforward extension. The umklapp process can occur only if at least one component, say $k_{2(4)}$ of either momenta, is greater than $\frac{1}{2}\pi/a$. We now integrate over all variables except $k_{2(4)}$. Because we have assumed that the integral converges in the limit $a \to 0$, we must get

$$m_{\text{total}} = \int_{-\pi/a}^{+\pi/a} dk_{2(4)} f(k_{2(4)}; a)\,, \tag{B6}$$

where f falls off faster than $1/k_{2(4)}$ when $a=0$. Therefore, the umklapp contribution

$$m_{\text{umklapp}} > \int_{\pi/2a}^{\pi/a} dk_{2(4)} f(k_{2(4)}; a) \tag{B7}$$

goes to zero as $a \to 0$.

Thus the umklapp processes do not contribute to the finite parts. On the other hand, in Sec. III we analyzed the structure of the terms singular in a using expressions like (B3) which include both the normal and the umklapp contributions. Thus the umklapp contributions are completely absorbed in the divergent counterterms.

There is nothing scared about the particular range $(-\pi/a, +\pi/a)$ we have chosen, except that it is symmetric about the origin. Any range of width $2\pi/a$ would give the same observable effects. The terminology "normal" and "umklapp" depend on the choice of this range. But for every choice, we have the same number of umklapp processes and we get the same contribution from them.

We observe that the propagator (in the scalar case) is positive definite and has the bounds

$$\frac{1}{m^2} \geq D(k) \geq \frac{1}{m^2+16/a^2}\,. \tag{B8}$$

The range of integration is also finite and therefore the integrals [e.g., (B3)] are finite. However, if $m^2=0$ the propagators are bounded on one side only, so that the lattice does not provide an infrared regularization.

To compare with the amplitudes of the continuum theory, we make the transformation

$$K_i = \frac{2}{a}\tan(\tfrac{1}{2}ak_i) \tag{B9}$$

which is an identity transformation when $a=0$, so that we may continue to regard K_i as the momentum. Now,

$$dk_i = (1+\tfrac{1}{4}a^2K_i^2)^{-1}dK_i$$

and

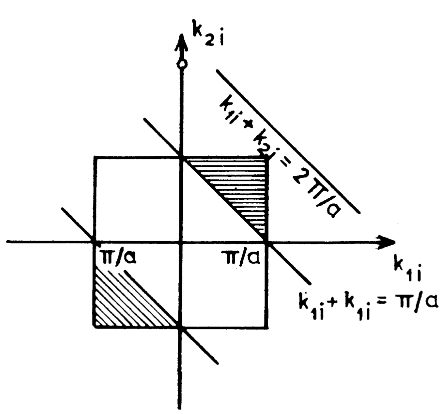

FIG. 7. The Umklapp processes correspond to the shaded region in this k_1-k_2 space.

$$\frac{1}{m^2+(4/a^2)\sum_i \sin^2(\frac{1}{2}k_i a)} = \frac{1}{m^2+\sum_i [1+(\frac{1}{2}K_i a)^2]^{-1}K_i^2} = \frac{1}{m^2+K^2-\frac{1}{4}a^2\sum_i K_i^4+\cdots}\,. \tag{B10}$$

The range of integration becomes $(-\infty, +\infty)$. Thus we get integrals very similar to the corresponding continuum integrals. However, the propagator is worse behaved at high momenta than it is in the continuum theory. It approaches a constant $(m^2 + 16/a^2)^{-1}$ instead of falling off like $1/k^2$. However, the Jacobian factor $(1+\frac{1}{4}a^2K_i^2)^{-1}$ makes every loop integration finite.

APPENDIX C: JUSTIFICATION OF THE POWER-COUNTING ARGUMENTS

We will briefly justify the power-counting arguments we have extensively used. A detailed treatment is given in Ref. 29.

We first remark that the leading singularities in a if any, can be obtained by replacing the propagator denominators by the corresponding relativistic expressions.[29] To this modified expression we apply a straightforward extension[29] of Weinberg's theorem,[30] which we state below:

Definition. A real function $f(P)$ (where $P\in R^n$) belongs to the class A_n if to every subspace $S \subset R^n$ there exists an integer $\alpha(S)$ (power) and $\beta(S)$ (logarithmic power) such that for any choice of $m\le n$ independent vectors $\vec{M}_1, \ldots, \vec{M}_m$,

$$|f(\vec{M}_1\eta_1\cdots\eta_m+\vec{M}_2\eta_2\cdots\eta_m+\cdots+\vec{M}_m\eta_m+\vec{C})| \le M\eta_1^{\alpha(\vec{M}_1)}(\ln\eta_1)^{\beta(\vec{M}_1)}\eta_2^{\alpha(\{M_1,M_2\})}\cdots\eta_m^{\alpha(\{M_1,\ldots,M_m\})}(\ln\eta_m)^{\beta(\{M_1,\ldots,M_m\})} \tag{C1}$$

whenever $\vec{C}\in W$, $\eta_1>b_1$, $\eta_2>b_2, \ldots, \eta_m>b_m$, where $b_1, \ldots, b_m$ depend on $\vec{M}_1, \ldots, \vec{M}_m$ and on $\vec{C}$, but not on $\eta_1, \ldots, \eta_m$.

If we collect together the n internal momenta in (C1) into a $4n$-dimensional vector P, then the integrands of our modified expressions belong[30] to the class A_{4n}.

We are interested in the dominant singularity in a of

$$f_n=\int_{-\pi/a}^{+\pi/a} dy_1\cdots\int_{-\pi/a}^{+\pi/a} dy_n f(\vec{M}_1 y_1+\cdots+\vec{M}_n y_n)\,. \tag{C2}$$

Theorem. The dominant singularity of f_n in a is given by the (superficial) degree of divergence of (superficially) the most divergent subintegration.

The theorem can be proven[29] by following exactly the technique developed by Weinberg.[30] Only the intervals $J^{\pm}(\eta)$ (in Weinberg's[30] notation) are now bounded [because the range of integration is $(-\pi/a, +\pi/a)$ instead of $(-\infty, +\infty)$].

It is immediately clear from the theorem that any diagram involving the anomalous vertices does not contribute to the finite parts. When we have made suitable subtractions from all the renormalization parts, the superficial degree of divergence of any subdiagram is less than the power of a outside the integral. Also when subtractions are made on the internal renormalization parts, the most singular subintegration is the overall integral, so that our estimates of the singularities of the counterterms is also correct.

APPENDIX D: WARD-TAKAHASHI IDENTITIES

Consider the generating functional in the presence of external sources,

$$Z(j,\bar\eta,\eta)=Z^{-1}\prod_{ni}\int_{-\pi/ea}^{+\pi/ea} dA_{ni}\int d\psi_n d\bar\psi_n \left\{\exp\left[S-\frac{1}{2\alpha e^2}\sum_n \sin^2\left(\sum_i ea(A_{ni}-A_{n-i,i})\right)\right] + \sum_n \ln\cos\left(ea\sum_i (A_{ni}-A_{n-i,i})\right) + a^4\sum_{ni} j_{ni}\frac{2}{ea}\tan(\tfrac{1}{2}ea)A_{ni}+a^4\sum_n \bar\psi_n\eta_n+a^4\sum_n \bar\eta_n\psi_n\right\}. \tag{D1}$$

We have considered j_{ni} as the source of $B_{ni}=(2/ea)\tan(\frac{1}{2}ea)A_{ni}$, because we are interested in the Green's functions involving B_{ni}'s.

Ward-Takahashi identities are derived in the standard fashion[16,31] by making a change of variables corresponding to an infinitesimal gauge transformation:

$$-ea\left\langle\sum_j\left[\tan\left(ea\sum_i (A_{n-j,i}-A_{n-i-j,i})\right)+\tan\left(ea\sum_i (A_{n+j,i}-A_{n-i+j,i})\right)\right]-8\tan\left(ea\sum_i (A_{ni}-A_{n-i,i})\right)\right\rangle_J + \frac{1}{ea}a^4\sum_i [j_{n-i,i}\langle\sec^2(\tfrac{1}{2}eaA_{n-i,i})\rangle_J - j_{ni}\langle\sec^2(\tfrac{1}{2}eaA_{ni})\rangle_J]+ia^4\bar\eta_n\langle\psi_n\rangle_J - ia^4\langle\bar\psi_n\rangle_J\eta_n=0\,, \tag{D2}$$

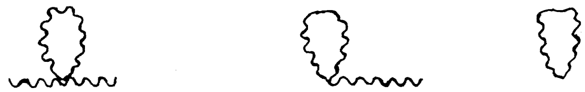

FIG. 8. Contributing diagrams in order e^2 to the Ward-Takahashi identity for the photon propagator.

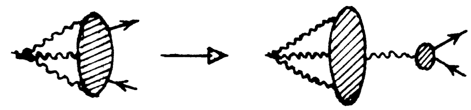

FIG. 9. The one-photon pole dominates in the limit of zero momentum transfer, in the Ward-Takahashi identity for the $\bar{\psi}\psi A_i$ vertex.

where $\langle\cdots\rangle_J$ means the corresponding Green's function in the presence of external sources $j, \bar{\eta}, \eta$.

Differentiating once with respect to B_{0j} and setting $J=0$, we get

$$\langle O_n B_{0j}\rangle + \frac{1}{ea}(\delta_{n-j,0} - \delta_{n,0})\left\langle \frac{1}{1+(\frac{1}{2}ea)^2 B_{0k}{}^2}\right\rangle = 0\,, \tag{D3}$$

where O_n stands for sum of the operators occurring in the first term of (D2).

This equation looks simpler if we consider the Fourier coefficients,

$$D^{-1}(p)\langle O(p), B_{0j}\rangle e^{ip_j a/2} = \frac{2i}{ea}\sin(\tfrac{1}{2}p_j a)\left\langle \frac{1}{1+(\frac{1}{2}ea)^2 B_{0k}{}^2}\right\rangle\,, \tag{D4}$$

where

$$O(p) = a^4 \sum_n e^{-ip\cdot na} \times \left[\frac{1}{2\alpha e^2 a^2}\sin\left(2ea\sum_i (A_{ni} - A_{n-i,i})\right) + \frac{1}{a^2}\tan\left(ea\sum_i (A_{ni} - A_{n-i,i})\right)\right]\,, \tag{D5}$$

where $D^{-1}(p)$ is the denominator of free photon propagator. This identity is easily verified[18] for our free photon propagator:

$$D^{-1}(p)\frac{1}{2\alpha e^2 a^2}2(ea)2i\sin(\tfrac{1}{2}p_j a) \times \frac{1}{\sum_i (4/a^2)\sin^2(\frac{1}{2}p_i a)} = 2i\sin(\tfrac{1}{2}p_j a)$$

which is true when $\alpha = 1$.

Identity (D4) is complicated because of our complicated choice of gauge. Some lower-order diagrams which enter in this identity are shown in Fig. 8. We have verified the identity to $O(e^2)$.

For the $\bar{\psi}\psi A_1$ vertex, we get the identity

$$D^{-1}(k)\langle O(k)\psi_\alpha(p+k)\bar{\psi}_\beta(p)\rangle = i[S_F(p+k) - S_F(p)]\,. \tag{D6}$$

In the limit $k \to 0$, the leading contributions comes from the one-phonon pole of the Green's function on the left-hand side (see Fig. 9, for an example). Using (D4) and considering terms linear in k, we get

$$\lim_{k\to 0}\left[\sum_i \frac{2i}{ea}\sin(\tfrac{1}{2}k_i a)\langle 1+(\tfrac{1}{2}ea)^2 B_{0i}{}^2\rangle[-ie\Gamma_i(p,k)] - i[S_F{}^{-1}(p+k) - S_F{}^{-1}(p)] = 0\right]\,.$$

Thus we no longer have $Z_1 = Z_2$; rather,

$$Z_1\langle 1+(\tfrac{1}{2}ea)^2 B_{0i}{}^2\rangle = Z_2\,.$$

For the photon-photon scattering amplitude, we get the identity

$$D^{-1}(k)\langle O(k)B_{i_1}(k_1)B_{i_2}(k_2)B_{i_3}(k_3)\rangle = 2i\sin(\tfrac{1}{2}k_{i_3}a)\sum_{j_1 j_2} V_{i_3;j_1 j_2}(k_3+k;k_1,k_2) \times \tilde{D}_{j_1 i_1}(k_1)\tilde{D}_{j_2 i_2}(k_2) + (\text{cyclic})\,, \tag{D8}$$

where $V_{i_1;i_2 i_3}(k_1;k_2,k_3)$ is the one-particle irreducible part of the vertex

$$\langle[1+(\tfrac{1}{2}ea)^2 B_{n_1 i_1}{}^2]B_{n_2 i_2}B_{n_3 i_3}\rangle\,.$$

*This work is based on a thesis submitted to the Department of Physics, Indian Institute of Technology, Kanpur, in partial fulfillment of the requirements for a Doctor's degree.

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[18]For the photon field it is necessary to define the Fourier transform by

$$A_i(p) = e^{-ik_i a/2}\sum_n e^{-ik\cdot na}A_{ni}$$

to get real expressions for the propagator and other Green's functions. The presence of $e^{-ik_i a/2}$ is natural if we regard A_{ni} to be located at the midpoint of sites n and $n+i$.

[19]See, e.g., C. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1966), p. 189.

[20]Henceforth our estimate of singularities in a must be regarded as possibly overestimates.

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[22]The superficial degree of divergence of the momentum integrals is $D=4-\frac{3}{2}F-B-\sum_d n_d d$ where n_d is the number of vertices with canonical dimension d (exclusive of the powers of ea). See Ref. 21. But a vertex of dimension d is associated with a^d. Hence the superficial singularity in a is $D_a=4-\frac{3}{2}F-B$.

[23]The photon-photon scattering amplitude has no superficial divergence, as in the continuum theory, as a consequence of gauge invariance (as shown below). Charge-conjugation invariance, which is valid for our lattice-action if we normal order the fermion bilinears, makes the amplitudes with just an odd number of external photons vanish.

[24]An exception is the anomalous 4-photon vertex of $O(e^4)$. Thus for example, in $e^2a^4[\sum_i(B_{ni}-B_{n-i,i})]^4$, the factor a^4 gets absorbed in momentum conservation at the vertex so that there are no extra powers of a outside. But, in fact, extra powers of a are hidden in the vertex factors, as this corresponds to a derivative interaction, and our argument of counting powers of a goes through. This is not true of the vertex $e^4a^4\sum_{ni}B_{ni}{}^4$ coming from the nonlinear transformation of A_{ni} to B_{ni}. But this serves precisely to cancel out a gauge-noninvariant finite piece coming from loop contributions to the photon-photon scattering of $O(e^4)$. This can be seen from the relevant Ward-Takahashi identity (as shown below).

[25]The action (2.1) is invariant under $n\to n^I$ (where n^I has the same components as n, except that the Ith component has an opposite sign), $\gamma_I^E\to-\gamma_I^E$, $\psi_n \to\psi_{n^I}$, $A_{nI}\to A_{n^I,-I}=-A_{n^I-I,I}$, and $A_{nJ}=A_{n^I,J}$ for $J\neq I$. This means that all the momentum-space Green's functions are unchanged under $p_I\to-p_I$ for both internal and external momenta and $\gamma_I^E\to-\gamma_I^E$ for any particular I, provided we also include a factor -1 for each external photon of polarization I. The internal momenta can be reversed back by a change of variables. Since B_{nI} is expressible as a series involving only odd powers of A_{nI}, our conclusions are valid for the action (A5) also.

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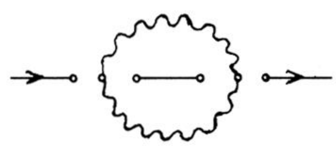

FIG. 1. An anomalous contribution to the self-energy of the electron. The gaps occur because the interaction involves a flip to the neighboring site. A_{ni} is supposed to be located at the midpoint of the sites n and $n+i$.

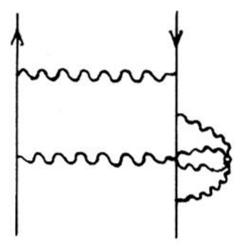

FIG. 3. A superficially convergent diagram with an internal anomalous renormalization part.

FIG. 2. Order-e^2 anomalous contribution to the photon self-energy.

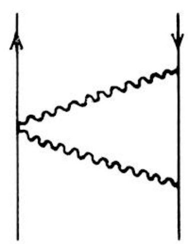

FIG. 4. An anomalous contribution to the electron-positron scattering amplitude.

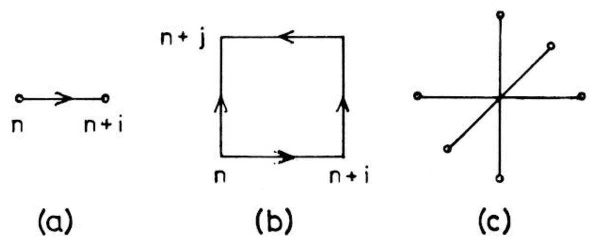

FIG. 5. (a) Link, (b) plaquette, and (c) frame. The fourth axis is not indicated in (c).

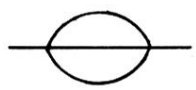

FIG. 6. The lowest-order self-energy diagram in ϕ^4 theory.

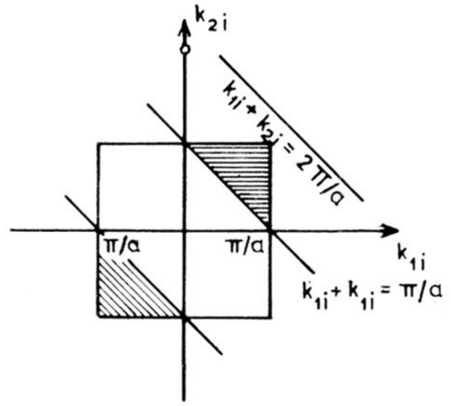

FIG. 7. The Umklapp processes correspond to the shaded region in this k_1-k_2 space.

FIG. 8. Contributing diagrams in order e^2 to the Ward-Takahashi identity for the photon propagator.

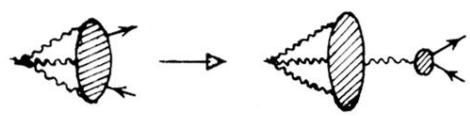

FIG. 9. The one-photon pole dominates in the limit of zero momentum transfer, in the Ward-Takahashi identity for the $\overline{\psi}\psi A_i$ vertex.