Structure and renormalizability of massive Yang-Mills field theories

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We observe that the theory of the Yang-Mills field with bare mass is included in the framework of spontaneously broken gauge theories, except that the Higgs field is realized nonlinearly. With the nonlinear symmetry associated with the Higgs field taken into account, the SU(N) massive Yang-Mills theory and the associated $SU(N)_L \times SU(N)_R$ nonlinear σ model are shown to be renormalizable and asymptotically free theories in two dimensions.

I. INTRODUCTION

Extensive studies of non-Abelian gauge theories[1] in recent years, both theoretical and phenomenological, originate in the proof of their renormalizability[2] in 1971. Renormalizable massless Yang-Mills theories[1] give rise, via the Higgs-Kibble mechanism,[3] to renormalizable theories of non-Abelian massive vector mesons. Such spontaneously broken gauge theories,[1,2,4] however, exclude the theory of the Yang-Mills field with bare mass, conventionally defined by the Lagrangian

$$\mathcal{L}[A] = -\tfrac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c)^2 + \tfrac{1}{2} m^2 (A_\mu^a)^2 . \tag{1.1}$$

In this form the massive vector field (Proca field[5]) A_μ^a is no longer regarded as a gauge field (since the mass term is not gauge invariant) and the renormalizability problem is difficult to investigate.

There have been various attempts[6-10] to explore the renormalizability of the massive Yang-Mills field. Most approaches are, in essence, based upon the non-Abelian generalization of the Stueckelberg formalism for neutral vector fields.[5] Within the generalized Stueckelberg formalism the gauge-field character[8] of the massive Yang-Mills field manifests itself at the cost of introducing unphysical scalar fields. It becomes possible to quantize the massive Yang-Mills field in arbitrary gauges by use of the path-integral quantization. In four space-time dimensions power counting in covariant gauges indicates the nonrenormalizability of this field theory, which has been verified by explicit one-loop calculations.[6,10] Recently the renormalizability of the two-dimensional massive Yang-Mills theory was studied; the conclusion was yet indecisive.[9,10,11]

It has often been observed[7-10] that in the generalized Stueckelberg formalism the most divergent part of the massive Yang-Mills theory is lumped in the form of nonlinear chiral Lagrangians. This remarkable chiral structure is neither accidental nor approximate. In this paper we shall point out that the massive Yang-Mills theory is expressed as a spontaneously broken gauge theory in which the Higgs field is introduced according to the nonlinear realization of chiral symmetry.[12] Viewed as a chiral gauge theory, massive Yang-Mills theory exhibits a richer symmetry structure than in its conventional form. In particular, the combined use of the gauge symmetry and nonlinear chiral symmetry enables one to discuss the renormalization of massive Yang-Mills theory. We shall prove the renormalizability of SU(N) massive Yang-Mills theory and, as a by-product, that of the $SU(N)_L \times SU(N)_R$ nonlinear σ model in two dimensions. The latter by-product should be regarded as a generalization of the case of the O(N) nonlinear σ model,[13,14] the renormalizability of which was recently studied in connection with the problem of phase transitions[13-15] in $2+\epsilon$ dimensions.

Interest in the nonlinear σ model stems from analogous nonlinear structures of lattice gauge theories, as formulated by Wilson,[16] and aims at understanding the phase transitions leading to quark confinement in gauge theories. In particular, the O(N) nonlinear σ model, which in ordinary perturbation theory realizes a spontaneously broken [O($N-1$)-symmetric] phase, is known[13-15] to possess, in the nonperturbative large-N limit, a fully O(N)-symmetric phase realized linearly with the occurrence of a bound state; such a symmetry restoration mechanism leads to color confinement in the transverse-lattice gauge theory of Bardeen and Pearson.[17]

In Sec. II we begin with a brief review of the nonlinear realization of chiral $SU(N)_L \times SU(N)_R$ symmetry and set up our notation. In Sec. III we couple the $SU(N)_L$ gauge field to the Higgs field which is realized nonlinearly according to the regular representation of the $SU(N)_L \times SU(N)_R$ symmetry. A suitable choice of the gauge condition shows that this chiral gauge theory is identical with the massive Yang-Mills theory. The content of this section can be extended to other gauge groups in a straightforward manner.

In Sec. IV we derive the Ward-Takahashi (WT) identities[2,18] associated with the local gauge symmetry and nonlinear chiral symmetry of our massive gauge theory. In Sec. V we prove, by intensive use of the WT identities, the renormalizability of two-dimensional SU(N) massive Yang-Mills theory and the associated $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ nonlinear σ model. An important observation here is that the ultraviolet structure of massive Yang-Mills theory is essentially determined by its nonlinear σ-model component.

In Sec. VI we present an explicit calculation of the one-loop renormalization counterterm for SU(2) massive Yang-Mills theory, mainly to verify the argument in the foregoing sections.

In Sec. VII we discuss the short-distance behavior of our SU(N) models. Like the O(N) nonlinear σ model,[13-15] both SU(N) massive Yang-Mills theory and the associated nonlinear σ model are asymptotically free in two dimensions[19] and have a nontrivial ultraviolet-stable fixed point in $2+\epsilon$ dimensions with $\epsilon>0$ infinitesimal.

In Sec. VIII we present a summary of our result. We speculate on the phase properties of massive Yang-Mills theories and the related SU(N) nonlinear σ model in two and higher dimensions.

Throughout this paper we shall employ dimensional regularization[20] to control ultraviolet divergences, preserving the WT identities in arbitrary dimensions. Special care is taken to separate infrared and ultraviolet divergences; we shall refer to the infrared regularization procedure in Sec. V.

II. NONLINEAR σ MODEL

In this section we review the $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ nonlinear σ model which is used to describe the Higgs field in the succeeding sections.

Let us consider an $(N\times N)$-matrix field M transforming according to the $(N,\bar N)$ representation of $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$,[12] i.e.,

$$M \to M' = U_L M U_R^\dagger \,, \tag{2.1}$$

where the transformation matrices U_L and U_R are SU(N) matrices. The Hermitian conjugate $M^\dagger$ of M transforms according to the $(\bar N, N)$ representation,

$$M^\dagger \to M^{\dagger\prime} = U_R M^\dagger U_L^\dagger \,. \tag{2.2}$$

The nonlinear σ model based on the regular representation of $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ is defined by the Lagrangian

$$\mathcal{L} = \tfrac{1}{4}\mathrm{Tr}[(\partial_\mu M^\dagger)(\partial^\mu M)] \,, \tag{2.3}$$

with the field M subjected to the constraint

$$\begin{aligned} &M^\dagger M = M M^\dagger = F^2 1 \,, \\ &\det(M/F) = 1 \,, \end{aligned} \tag{2.4}$$

where F is a real constant. In the presence of this constraint the Lagrangian (2.3) is the only chiral-invariant Lagrangian that involves at most two space-time derivatives.

The constraint (2.4) means that the matrix M/F is an SU(N) matrix. Any SU(N) matrices are parametrized in terms of (N^2-1) real parameters. The parametrization we consider is provided by

$$M = (\sigma^a + i\pi^a)\lambda^a \equiv \hat\sigma + i\hat\pi \,, \tag{2.5}$$

defined in terms of real fields $\pi^a = (\pi^0, \underline{\pi}^b)$ and $\sigma^a = (\sigma^0, \underline{\sigma}^b)$ $(a=0,\ldots,N^2-1)$, where we regard (N^2-1) fields $\underline{\pi}^b$ as independent fields while treating π^0 and σ^a as dependent fields. The $N\times N$ matrices λ^a $(a=0,\ldots,N^2-1)$ are the SU(N) generalization[21] of the SU(3) λ matrices: λ^k $(k=1,\ldots,N^2-1)$ are Hermitian traceless matrices and $\lambda^0 = (2/N)^{1/2}1$, normalized so that

$$\mathrm{Tr}(\lambda^a\lambda^b) = 2\delta^{ab} \,. \tag{2.6}$$

The product $\lambda^a\lambda^b$ is written as

$$\lambda^a\lambda^b = (d^{abc} + i f^{abc})\lambda^c \,, \tag{2.7}$$

where the real coefficients d^{abc} and f^{abc} are totally symmetric and antisymmetric, respectively, in particular,

$$d^{ab0} = (2/N)^{1/2}\delta^{ab} \text{ and } f^{ab0} = 0 \,. \tag{2.8}$$

In what follows we shall make extensive use of the following U(N) notation: A nonunderlined vector A^a $(a=0,\ldots,N^2-1)$ always stands for an N^2-component vector, and in particular its (N^2-1) components A^b $(b=1,\ldots,N^2-1)$ are denoted by an underlined vector $\underline{A}^b$, i.e., $A^a = (A^0, \underline{A})$. We shall, however, often denote an (N^2-1)-component quantity $\underline{A}^a$ simply by A^a, regarding A^a as $A^a = (0, \underline{A})$. As in (2.5), the $N\times N$ matrix $A^a\lambda^a$ will be denoted by $\hat A$,

$$\hat A = A^a\lambda^a = (2/N)^{1/2}A^0 1 + \underline{\hat A} \,. \tag{2.9}$$

Similarly, we write the $N^2\times N^2$ matrices $d^{bca}A^a$ and $f^{bac}A^a$ as

$$\begin{aligned} &(\tilde A)^{bc} = d^{bca}A^a \,, \\ &(A\,\times)^{bc} = f^{bac}A^a \,. \end{aligned} \tag{2.10}$$

Vector and matrix indices will frequently be suppressed; e.g., A^aA^a, A^aB^a, $C^{ab}B^b$, and $A^aC^{ab}B^b$, are denoted simply by A^2, $A\cdot B$, CB, and $A\cdot CB$, respectively. Appendix A lists some useful formulas.

We express the dependent fields $\pi^0[\underline{\pi}]$ and $\sigma^a[\underline{\pi}]$ in a power series in $\underline{\pi}$ so that $M[\underline{\pi}] = F + i\underline{\hat\pi} + O(\underline{\pi}^2/F)$, i.e., $(2/N)^{1/2}\sigma^0 \to F$ and $(\pi^0, \underline{\sigma}) \to 0$ as $\underline{\pi} \to 0$. The

SU(N) matrix field $M[\underline{\pi}]/F$ defined in this way is determined uniquely, as demonstrated by explicit construction in Appendix B.

In terms of (π^a, σ^a) the field transformation property (2.1)

$$M[\underline{\pi}] \to M' = M[\underline{\pi}'] = U_L M[\underline{\pi}] U_R^\dagger \tag{2.11}$$

reads in infinitesimal form, i.e., $\pi'^a = \pi^a + \delta\pi^a$, $\sigma'^a = \sigma^a + \delta\sigma^a$, $U_L = 1 + i\frac{1}{2}\delta\underline{\hat{\epsilon}}_L$, and $U_R = 1 + i\frac{1}{2}\delta\underline{\hat{\epsilon}}_R$, etc.,

$$\begin{aligned} \delta\pi &= \tfrac{1}{2}[(\tilde{\sigma} + \pi\times)\delta\epsilon_L + (-\tilde{\sigma} + \pi\times)\delta\epsilon_R], \\ \delta\sigma &= \tfrac{1}{2}[(-\tilde{\pi} + \sigma\times)\delta\epsilon_L + (\tilde{\pi} + \sigma\times)\delta\epsilon_R], \end{aligned} \tag{2.12}$$

where $\delta\epsilon_L^a = (0, \delta\underline{\epsilon}_L)$, $\delta\epsilon_R^a = (0, \delta\underline{\epsilon}_R)$, and $\pi^a = (\pi^0, \underline{\pi})$, $\sigma^a = (\sigma^0, \underline{\sigma})$. If we rewrite (2.12) in the form

$$\begin{aligned} \delta\pi &= \pi \times (\delta\epsilon_L + \delta\epsilon_R)/2 + \tilde{\sigma}(\delta\epsilon_L - \delta\epsilon_R)/2, \\ \delta\sigma &= \sigma\times(\delta\epsilon_L + \delta\epsilon_R)/2 - \tilde{\pi}(\delta\epsilon_L - \delta\epsilon_R)/2, \end{aligned} \tag{2.13}$$

it is clear that the fields $\underline{\pi}$ and $\underline{\sigma}$ belong to the regular representation of $\mathrm{SU}(N)_{L+R}$ while π^0 and σ^0 are $\mathrm{SU}(N)_{L+R}$ singlets. [Referring to the combinations $\frac{1}{2}(\delta\epsilon_L + \delta\epsilon_R)$ and $\frac{1}{2}(\delta\epsilon_L - \delta\epsilon_R)$ we call the corresponding two SU(N) symmetries $\mathrm{SU}(N)_{L+R}$ and $\mathrm{SU}(N)_{L-R}$, respectively.] The chiral Lagrangian (2.3) may also be written as

$$\mathcal{L} = \tfrac{1}{2}[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2]. \tag{2.14}$$

We note that the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ version of the nonlinear σ model is renormalizable and asymptotically free in two dimensions, as a particular case of the O(N) nonlinear σ model previously studied.[13,14] We shall, in later sections, extend the proof of the renormalizability of the nonlinear σ model to the $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ case, by an argument which emphasizes the symmetry structure of nonlinear realizations.

III. MASSIVE YANG-MILLS FIELD

Let us introduce the $\mathrm{SU}(N)_L$ Yang-Mills field $\underline{A}_\mu^a$ $(a = 1, \ldots, N^2-1)$ into the $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ nonlinear σ model (2.14). The gauge-invariant Lagrangian is given by

$$\begin{aligned} \mathcal{L}_0[\underline{A}, \underline{\pi}] = &-\tfrac{1}{4}F_{\mu\nu}^a[A]^2 \\ &+ \tfrac{1}{4}\mathrm{Tr}[M^\dagger \mathfrak{D}_\mu^\dagger[A]\mathfrak{D}^\mu[A]M], \end{aligned} \tag{3.1}$$

with $F_{\mu\nu}^a[A] = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$, where we have denoted $\underline{A}_\mu$ simply by $A_\mu^a = (0, \underline{A}_\mu)$. Here g is the coupling constant, and the covariant derivative $\mathfrak{D}_\mu[A]$ is defined as

$$\mathfrak{D}_\mu[A] = \partial_\mu - i\tfrac{1}{2}gA_\mu^a\lambda^a = \partial_\mu - i\tfrac{1}{2}g\hat{A}_\mu,$$

while its Hermitian conjugate $\mathfrak{D}_\mu^\dagger[A]$ acts on the left, i.e., $B\mathfrak{D}_\mu^\dagger[A] = B(\overleftarrow{\partial}_\mu + i\frac{1}{2}g\hat{A}_\mu)$. In the form of (3.1) it is evident that $\mathcal{L}_0[\underline{A}, \underline{\pi}]$ has the local $\mathrm{SU}(N)_L$ symmetry as well as the global $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ symmetry. Expressed in terms of (π^a, σ^a), $\mathcal{L}_0[\underline{A}, \underline{\pi}]$ is given by

$$\begin{aligned} \mathcal{L}_0[\underline{A}, \underline{\pi}] = &-\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu + gA_\mu \times A_\nu)^2 + \tfrac{1}{2}m^2 A_\mu{}^2 \\ &+ \tfrac{1}{2}[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2] \\ &+ \tfrac{1}{2}gA_\mu \cdot [(\tilde{\pi} + \sigma\times)\partial^\mu\sigma + (-\tilde{\sigma} + \pi\times)\partial^\mu\pi], \end{aligned} \tag{3.2}$$

with the vector-meson mass given by

$$m = \tfrac{1}{2}gF. \tag{3.3}$$

To quantize this system one has to impose a gauge condition. If we adopt the gauge condition $\underline{\pi}^a = 0$ $(a = 1, \ldots, N^2-1)$, $\mathcal{L}_0[\underline{A}, \underline{\pi}]$ takes the form

$$\begin{aligned} \mathcal{L}_0[\underline{A}, \underline{\pi} = 0] = &-\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu + gA_\mu \times A_\nu)^2 \\ &+ \tfrac{1}{2}m^2 A_\mu{}^2, \end{aligned} \tag{3.4}$$

which is precisely the Lagrangian for the massive Yang-Mills field. The pure gauge-field nature of the massive Yang-Mills field is not explicit in this conventional form while it is manifest in the gauge-invariant form of (3.1) or (3.2). In the latter the vector-meson mass $m = gF/2$ comes from the nonlinearity of the realization

$$M^\dagger M = (2/N)(\sigma^2 + \pi^2) = F^2 = 4m^2/g^2,$$

or, equivalently, from the nonvanishing vacuum expectation value $\langle M^\dagger M\rangle_0 = F^2$. Correspondingly, in perturbation theory based on power-series expansions in g or $1/F = (g/2m)$, the field $(2/N)^{1/2}\sigma^0 = F + i\tilde{\underline{\pi}} + O(\underline{\pi}^2/F)$ develops a nonvanishing vacuum expectation value so that the global $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ symmetry of $\mathcal{L}_0[\underline{A}, \underline{\pi}]$ is spontaneously broken down to global $\mathrm{SU}(N)_{L+R}$ symmetry. As a consequence of the spontaneous breakdown of chiral $\mathrm{SU}(N)_{L-R}$ symmetry, (N^2-1) massless Goldstone bosons π^a appear in perturbation theory.

In the (so-called unitary) gauge $\underline{\pi}^a = 0$, only the physical vector fields $\underline{A}_\mu^a$ remain and the equivalence of the present chiral gauge theory and massive Yang-Mills theory is made manifest. This gauge however, is unsuited for the discussion of renormalizability because of the vector-meson propagator of the Proca[5] form $(-g^{\mu\nu} + p^\mu p^\nu/m^2)/(p^2 - m^2)$. For this very purpose, we may employ the standard gauge-fixing procedure[1,21] to quantize the theory in a gauge characterized by $F^a[\underline{A}] = -\partial^\mu \underline{A}_\mu^a$. In this [Lorentz- and global $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$-].covariant gauge the Feynman rules are derived from the functional

$$W[\underline{J},\underline{K}] = \int [d\underline{A}][\mathfrak{D}\underline{\pi}][d\underline{C}][d\underline{\overline{C}}] \times \exp\left\{ i \int dx (\mathcal{L}_{\rm eff}[A,\pi,C,\overline{C}] + \underline{J}^{\mu}\cdot\underline{A}_{\mu} + \underline{K}\cdot\underline{\pi})\right\} \qquad (3.5)$$

with

$$\mathcal{L}_{\rm eff}[A,\pi,C,\overline{C}] = \mathcal{L}_0[\underline{A},\underline{\pi}] - \frac{1}{2\alpha}(\partial^{\mu}\underline{A}_{\mu})^2 - \underline{\overline{C}}\cdot\partial^{\mu}\nabla_{\mu}[\underline{A}]\underline{C}\,, \qquad (3.6)$$

$$\nabla_{\mu}^{ab}[\underline{A}] = \delta^{ab}\partial_{\mu} + g f^{acb}\underline{A}_{\mu}^{c} = \partial_{\mu} + g(\underline{A}_{\mu}\times)\,, \qquad (3.7)$$

where $C^a = (0, \underline{C})$ and $\overline{C}^a = (0, \underline{\overline{C}})$ are the Faddeev-Popov ghost fields[22] obeying Fermi statistics. The functional measure $[\mathfrak{D}\underline{\pi}]$ stands for the invariant measure (the Haar measure[23]) on $SU(N)_L \times SU(N)_R$ and is given by

$$[\mathfrak{D}\underline{\pi}] = [d\underline{\pi}]w[\underline{\pi}]\,,$$
$$1/w[\underline{\pi}] = \prod_x \det[(2/N)^{1/2}\sigma^0 1 + (\underline{\tilde{\sigma}} - \underline{\pi}\times)]_x\,, \qquad (3.8)$$

where the matrix $[\cdots]$ in the last brackets is an $(N^2-1)\times(N^2-1)$ matrix. Accordingly perturbation theory is constructed on the basis of the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\rm eff}[A,\pi,C,\overline{C}] + i\delta^{(n)}(0)\ln[(2/N)^{1/2}\sigma^0 + (\underline{\tilde{\sigma}} - \underline{\pi}\times)]\,, \qquad (3.9)$$

where n is the dimension of space-time. The last term proportional to $\delta^{(n)}(0)$ serves to cancel[24] the leading divergences (quadratic in two dimensions) in the $\underline{\pi}^k$ $(k \geq 2)$ proper vertices; practically one may entirely ignore this term by using dimensional regularization in which $\delta^{(n)}(0)$ can be set equal to zero. In the present covariant gauge $\mathcal{L}_{\rm eff}[A,\pi,C,\overline{C}]$ preserves the global $SU(N)_L \times SU(N)_R$ symmetry of Eq. (3.1) and Eq. (3.2); the fields $\underline{A}_{\mu}$, $\underline{C}$, and $\underline{\overline{C}}$ belong to the regular representation of $SU(N)_L$ (i.e., $\underline{A}_{\mu} \to \underline{A}'_{\mu} = U_L \underline{A}_{\mu} U_L^{\dagger}$, etc.) and are $SU(N)_R$ singlets. In addition, $\mathcal{L}_{\rm eff}[A,\pi,C,\overline{C}]$ is invariant under the local $SU(N)_L$ transformations

$$\delta\pi = (\tilde{\sigma} + \pi\times)\delta\epsilon/2\,,$$
$$\delta\sigma = (-\tilde{\pi} + \sigma\times)\delta\epsilon/2\,,$$
$$\delta A_{\mu} = g^{-1}\nabla_{\mu}[A]\delta\epsilon\,, \qquad (3.10)$$
$$\delta C = C\times\delta\epsilon/2\,,$$
$$\delta\overline{C}^a = -\alpha^{-1}F^a[A]\delta\xi\,,$$

where the group parameter $\delta\epsilon^a = (0, \delta\underline{\epsilon})$ is defined as $\delta\underline{\epsilon}^a(x) = \underline{C}^a(x)\delta\xi$ in terms of an infinitesimal anticommuting number $\delta\xi$ independent of the space-time coordinate x_{μ}; $(\delta\xi, \underline{C}, \underline{\overline{C}})$ anticommute among themselves. This gauge transformation is known as the Becchi-Rouet-Stora (BRS) transformation.[18]

In the present covariant gauge the most ultraviolet-divergent portion of the theory is contained in the nonlinear σ-model sector rather than the pure Yang-Mills sector. Power-counting and, in fact, an explicit one-loop calculation[10] show the nonrenormalizability of massive Yang-Mills theory in four dimensions. (Although ordinary spontaneously broken gauge theories are renormalizable in four dimensions, the present chiral gauge theory fails to inherit this feature because of the nonlinearity of the σ-model sector.) However, massive Yang-Mills theory and the associated $SU(N)_L \times SU(N)_R$ nonlinear σ model become renormalizable in two dimensions; we shall show this in the following two sections.

With the present parametrization (2.5) of the M field the theory turns out to be multiplicatively renormalizable. This multiplicative nature of renormalization, however, is not common to arbitrary parametrizations. With those parametrizations where renormalizations are not multiplicative, the renormalization procedure is more complicated and the renormalization constants may be field dependent, as observed in the case of the $O(N)$ nonlinear σ model in Ref. 13.

With the exponential parametrization

$$M = F\exp(i\Theta^a\lambda^a/F)\,, \qquad (3.11)$$

where Θ^a $(a = 1,\ldots,N^2-1)$ are real fields, the gauge-invariant Lagrangian (3.1) is written as

$$\mathcal{L}_0(A,\Theta) = -\tfrac{1}{4}F_{\mu\nu}[A]^2 + \tfrac{1}{2}[mA_{\mu}^a - E^{ab}(\Theta)\partial_{\mu}\Theta^b]^2\,,$$
$$E^{ab}(\Theta) = \{[\exp(i\overline{\Theta}) - 1]/(i\overline{\Theta})\}^{ab} = \sum_{n=1}^{\infty}[(i\overline{\Theta})^n/(n+1)!]^{ab}\,, \qquad (3.12)$$
$$\overline{\Theta}^{ab} \equiv if^{acb}\Theta^c/F\,.$$

The Lagrangian of this form was previously derived from a different viewpoint[8] and used to discuss the renormalization problem.[10] With this exponential parametrization, however, the WT identities associated with the nonlinear chiral symmetry is not as simple as the one to be derived in the next section.

IV. THE WARD-TAKAHASHI IDENTITIES

Investigation into the renormalizability of gauge theories relies on the Ward-Takahashi (WT) identities which express the symmetry property of the Lagrangian in terms of relations for the Green's functions.

To derive the WT identities let us consider the generating functional of the Green's functions

$$W[J,K,\bar{\eta},\eta,H,\rho,G,\Sigma,P]=\int[d\underline{A}][\mathfrak{D}\underline{\pi}][d\underline{C}][d\underline{\bar{C}}]\exp\left\{i\int dx(\mathcal{L}_{\text{eff}}[A,\pi,C,\bar{C}]+\mathcal{L}_{\text{source}})\right\}, \tag{4.1}$$

$$\mathcal{L}_{\text{source}}=J_\mu\cdot A^\mu+K\cdot\pi+\bar{\eta}\cdot C+\bar{C}\cdot\eta+H\cdot\sigma+\rho^\mu\cdot\nabla_\mu[A]C+\tfrac{1}{2}gG\cdot(C\times C)+\tfrac{1}{2}g\Sigma\cdot(-\bar{\pi}+\sigma\times)C+\tfrac{1}{2}gP\cdot(\bar{\sigma}+\pi\times)C\,, \tag{4.2}$$

where $J^a_\mu=(0,\underline{J}_\mu)$, $K^a=(K^0,\underline{K})$, $H^a=(H^0,\underline{H})$, and $G^a=(0,\underline{G})$ are commuting source functions while $\bar{\eta}^a=(0,\underline{\bar{\eta}})$, $\eta^a=(0,\underline{\eta})$, $\rho^a_\mu=(0,\underline{\rho}_\mu)$, $\Sigma^a=(\Sigma^0,\underline{\Sigma})$, and $P^a=(P^0,\underline{P})$ are anticommuting source functions. We have, following Zinn-Justin,[18] introduced the external sources K^0, H^a, ρ^a_μ, G^a, Σ^a, and P^a for the composite operators $\pi^0[\underline{\pi}]$, $\sigma^a[\underline{\pi}]$, $\nabla^\mu[A]C$, etc. to simplify the WT identities. Let us note that under infinitesimal global $SU(N)_R$ transformations (2.12) the source term $\mathcal{L}_{\text{source}}$ changes such that $\mathcal{L}_{\text{source}}\to\mathcal{L}_{\text{source}}+\delta_R\mathcal{L}_{\text{source}}$,

$$\delta_R\mathcal{L}_{\text{source}}=\tfrac{1}{2}[K\cdot(-\bar{\sigma}+\pi\times)+H\cdot(\bar{\pi}+\sigma\times)]\delta\epsilon_R+\tfrac{1}{4}g\delta\epsilon_R\cdot[(\bar{\Sigma}+P\times)(\bar{\sigma}+\pi\times)C+(-\bar{P}+\Sigma\times)(-\bar{\pi}+\sigma\times)C] \tag{4.3}$$

(see Appendix C). Accordingly, we make in (4.1) a change of field variables corresponding to this global $SU(N)_R$ transformation, noting the invariance of the integration measure. Then the result is the WT identity associated with the $SU(N)_R$ symmetry:

$$\int dx\left[(-\bar{\sigma}-\pi\times)K+(\bar{\pi}-\sigma\times)H+(\bar{\Sigma}+P\times)\frac{\delta}{i\delta P}+(-\bar{P}+\Sigma\times)\frac{\delta}{i\delta\Sigma}\right]^b W[J,\ldots]=0\,, \tag{4.4}$$

where $b=1,\ldots,N^2-1$ (i.e., $b\neq 0$), and the replacement

$$\underline{\pi}^a\to\delta/i\delta\underline{K}^a,\quad \pi^0[\underline{\pi}]\to\pi^0[\delta/i\delta\underline{K}],\quad \sigma^a[\underline{\pi}]\to\sigma^a[\delta/i\delta\underline{K}] \tag{4.5}$$

is understood within the square brackets.

Next let us derive the WT identities associated with the local $SU(N)_L$ symmetry. It is important to note that under the BRS transformation (3.10) the composite operators $\nabla_\mu[A]C$, $(C\times C)$, $(-\bar{\pi}+\sigma\times)C$, and $(\bar{\sigma}+\pi\times)C$ remain invariant, as is easily verified. On performing the BRS transformation on the integration variables A_μ and π in (4.1), we are left with the desired WT identity:

$$\int dx\left(J_\mu\cdot\frac{\delta}{\delta\rho_\mu}+K\cdot\frac{\delta}{\delta P}+H\cdot\frac{\delta}{\delta\Sigma}+\bar{\eta}\cdot\frac{\delta}{\delta G}-\frac{1}{\alpha}\eta\cdot\partial_\mu\frac{\delta}{\delta J_\mu}\right)W=0\,. \tag{4.6}$$

Similarly, a change of variables $\underline{\bar{C}}^a\to\underline{\bar{C}}^a+\delta\underline{\bar{C}}^a$ in (4.1) leads to the equation of motion for the ghost field C^a:

$$\left(\partial_\mu\frac{\delta}{\delta\rho^a_\mu}-i\eta^a\right)W=0\,. \tag{4.7}$$

Our next task is to rewrite (4.4), (4.6), and (4.7) in terms of the generating functional Γ of proper (i.e., one-particle irreducible) vertices defined by the Legendre transform[25]

$$\Gamma[\underline{A},\underline{\pi},\underline{C},\underline{\bar{C}};K^0,H^a,\underline{\rho},\underline{G},\Sigma^a,P^a]=Z[\underline{J},\underline{K},\ldots]-\int dx(\underline{J}^\mu\cdot\underline{A}_\mu+\underline{K}\cdot\underline{\pi}+\underline{\bar{\eta}}\cdot\underline{C}+\underline{\bar{C}}\cdot\underline{\eta})\,, \tag{4.8}$$

with

$$Z[J,K,\bar{\eta},\eta,H,\rho,G,\Sigma,P]=-i\ln W[J,K,\bar{\eta},\eta,H,\ldots]\,, \tag{4.9}$$

where $A^b_\mu=(0,\underline{A}_\mu)$, $\pi^a=(0,\underline{\pi})$, $C^a=(0,\underline{C})$, and $\bar{C}^a=(0,\underline{\bar{C}})$ now stand for the new variables defined by

$$A^a_\mu=\frac{\delta Z}{\delta J^{\mu a}},\quad \pi^a=\frac{\delta Z}{\delta K^a},\quad C^a=\frac{\delta Z}{\delta\bar{\eta}^a},\quad \bar{C}^a=-\frac{\delta Z}{\delta\eta^a}. \tag{4.10}$$

For later convenience we have duplicated the same notation $(\underline{A}_\mu,\underline{\pi},\underline{C},\underline{\bar{C}})$ in (4.10) as was used for the bare fields $(\underline{A}_\mu,\underline{\pi},\underline{C},\underline{\bar{C}})$ earlier. It is necessary to distinguish them for the time being, but this notation will turn out to be useful in Sec. V where no distinction is necessary. Note that the Legendre transformation is made on the independent fields A_μ,π,C, and $\bar{C}$ only. It follows from (4.8) and (4.10) that

$$J^a_\mu=-\frac{\delta\Gamma}{\delta A^a_\mu},\quad K^a=-\frac{\delta\Gamma}{\delta\pi^a},\quad \bar{\eta}^a=+\frac{\delta\Gamma}{\delta C^a},\quad \eta^a=-\frac{\delta\Gamma}{\delta\bar{C}^a}, \tag{4.11}$$

$$\frac{\delta Z}{\delta K^0}=\frac{\delta\Gamma}{\delta K^0},\quad \frac{\delta Z}{\delta H^a}=\frac{\delta\Gamma}{\delta H^a},\quad \text{etc.}$$

In terms of $\Gamma[A,\pi,\ldots]$, the WT identities (4.4), (4.6), and (4.7) are rewritten as

$$\int dx\left[\left(\frac{\delta\tilde{\Gamma}}{\delta H}+\pi\times\right)\frac{\delta\Gamma}{\delta\pi}+\left(\tilde{\pi}-\frac{\delta\Gamma}{\delta H}\times\right)H+(\tilde{\Sigma}+P\times)\frac{\delta\Gamma}{\delta P}+(-\tilde{P}+\Sigma\times)\frac{\delta\Gamma}{\delta\Sigma}\right]^a=0\quad(a=1,\ldots,N^2-1)\,, \tag{4.12}$$

$$\int dx\left[\frac{\delta\Gamma}{\delta A_\mu}\cdot\frac{\delta\Gamma}{\delta\rho^\mu}+\frac{\delta\Gamma}{\delta\pi}\cdot\frac{\delta\Gamma}{\delta P}-H\cdot\frac{\delta\Gamma}{\delta\Sigma}-\frac{\delta\Gamma}{\delta C}\cdot\frac{\delta\Gamma}{\delta G}-\frac{1}{\alpha}\frac{\delta\Gamma}{\delta\bar{C}}\cdot(\partial_\mu A^\mu)\right]=0\,, \tag{4.13}$$

$$\partial_\mu\delta\Gamma/\delta\rho_\mu^a+\delta\Gamma/\delta\bar{C}^a=0\,. \tag{4.14}$$

In (4.12) and (4.13)

$$\delta\Gamma/\delta\pi^0 \text{ stands for } -K^0 \text{ and } \pi^0 \text{ stands for } \delta\Gamma/\delta K^0. \tag{4.15}$$

[Such replacement is simply a matter of notational convenience, making the relevant expressions compact by use of the U(N) notation.]

As a counterpart of the $SU(N)_R$ WT identity (4.12) we can also derive the WT identity associated with global $SU(N)_L$ symmetry. Equivalently, we may simply use the WT identity associated with $SU(N)_{L+R}$,

$$\int dx\left(\underline{A}_\mu\times\frac{\delta}{\delta\underline{A}_\mu}+\underline{\pi}\times\frac{\delta}{\delta\underline{\pi}}+\underline{H}\times\frac{\delta}{\delta\underline{H}}+\underline{C}\times\frac{\delta}{\delta\underline{C}}+\cdots+\underline{P}\times\frac{\delta}{\delta\underline{P}}+\underline{\Sigma}\times\frac{\delta}{\delta\underline{\Sigma}}\right)\Gamma=0\,, \tag{4.16}$$

which means that $\underline{A}_\mu$, $\underline{\pi}$, $\underline{H}$, $\underline{C}$, $\underline{\bar{C}}$, $\underline{\rho}_\mu$, $\underline{G}$, $\underline{P}$, and $\underline{\Sigma}$ belong to the $SU(N)_{L+R}$ regular representation.

V. RENORMALIZATION

The procedure for the discussion of renormalizability of gauge theories is well known.[18] For completeness we outline the procedure due to Zinn-Justin. First by the renormalization transformation $\underline{A}^\mu=\sqrt{Z_A}\underline{A}_r^\mu$, $\underline{\pi}=\sqrt{Z_1}\underline{\pi}_r$, $\pi^0=\sqrt{Z_2}\pi_r^0$, $K^0=(Z_2)^{-1/2}K_r^0,\ldots$, $F=Z_F F_r$ $m=Z_m m_r,\ldots$, we define the renormalized quantities $\underline{A}_r^\mu$, $\underline{\pi}_r^\mu$, etc.; these renormalization constants are to be determined successively in the loopwise expansion of

$$\Gamma[\underline{A},\underline{\pi},\underline{C},\underline{\bar{C}},\ldots]=\sum_{l=0}^{\infty}\Gamma^{(l)}[\underline{A}_r,\underline{\pi}_r,\underline{C}_r,\underline{\bar{C}}_r,\ldots]\,.$$

At the zero-loop (tree) level, Γ is given by the bare Lagrangian $\mathcal{L}_{\rm eff}+\mathcal{L}_{\rm source}$ with every quantity replaced by the corresponding renormalized one,

$$\begin{aligned}\Gamma^{(0)}&=\int dx\,\mathcal{L}^{(0)}\\&=\int dx(\mathcal{L}_{\rm eff}[\underline{A}_r,\underline{\pi}_r,\ldots]+\mathcal{L}_{\rm source}[K_r^0,\ldots])\,.\end{aligned}$$

If we calculate the one-loop correction $\Gamma^{(1)}$ on the basis of this tree-level Lagrangian, we shall find the ultraviolet divergences. The one-loop renormalization counterterm $\Delta\mathcal{L}^{(1)}$ is then given by the divergent part of $\Gamma^{(1)}$ so that $\int dx\,\Delta\mathcal{L}^{(1)}=-\Gamma^{(1)}(\text{div})$. An important consequence of the WT identities for Γ, (4.12)–(4.14), is that the one-loop renormalized action $\int dx(\mathcal{L}^{(0)}+\Delta\mathcal{L}^{(1)})$ itself satisfies the WT identities up to the one-loop approximation; furthermore, it follows by induction that the complete renormalized action $\int dx\,\mathcal{L}_r=\int dx(\mathcal{L}^{(0)}+\Delta\mathcal{L})$ must obey the WT identities to each order in perturbation theory. The possible structure of the counterterm $\Delta\mathcal{L}$ is constrained by power-counting and is further restricted by the WT identities. The proof of renormalizability is completed if one can show that the renormalized Lagrangian $\mathcal{L}_r$ determined in this way has the same structure as the initial bare Lagrangian.

The above renormalization procedure is concerned with the removal of ultraviolet divergences alone. However, since the $\underline{\pi}$ fields are massless in perturbation theory, we must take special care to separate ultraviolet divergences from infrared ones in the course of renormalizations. A possible way of avoiding the infrared problem is to carry out renormalizations somewhere off the mass shell. An alternative trick is to introduce a soft symmetry-breaking term which gives rise to a mass term for the $\underline{\pi}$ fields, as employed in Refs. 13 and 14. In practice, the external source $H^0(x)$ can be used as an infrared regulator since, if $H^0(x)$ is kept to be a constant $H^0(0)$ in the course of perturbative calculations, the source term

$$H^0(0)\sigma^0(x)=(N/2)^{1/2}H^0(0)[F-(\underline{\pi}^2/NF)+\cdots]$$

generates a mass term for the $\underline{\pi}$ fields.

In two dimensions the fields A_μ^a, π^a, σ^a, C^a, and $\bar{C}^a$ are dimensionless in units of mass, and the coupling constants $1/F$ and $g=2m/F$ have dimension equal to zero and one, respectively. The external source ρ_μ^a has dimension one while $K^0, H^a, gG^a, g\Sigma^a, gP^a$ have dimension two. Power counting tells us that the counterterm $\Delta\mathcal{L}$ consists of local ploynomials (of fields and sources) of dimension two. The general form of $\Delta\mathcal{L}$ is considerably restricted by the following observation:

In the Lagrangian $\mathcal{L}_{\rm eff}$ (3.6) any interaction vertices that involve the A_μ field carry at least one power of the dimensional constant g. As a result of this, the counterterm $\Delta\mathcal{L}$ does not contain more than two A_μ fields; in particular, A_μ and g need no divergent wave-function and coupling-constant renormalization, respectively. (The gauge parameter α also requires no divergent renormalization.) In addition, the ghost fields are coupled to A_μ through the $g(\partial^\mu \overline{C})\cdot(A_\mu \times C)$ vertex which is proportional to the momentum of the outgoing ghost field $\overline{C}$, and they have no direct coupling to (σ^a, π^a). Consequently, $\Delta\mathcal{L}$ does not contain any terms that are proportional to $\underline{\overline{C}}^a$, $\underline{\rho}^a_\mu$, and/or $\underline{G}^a$. Thus the power-counting argument alone can restrict the renormalized Lagrangian $\mathcal{L}_r[A_r, \pi_r, \ldots]$ $=\mathcal{L}^{(0)}+\Delta\mathcal{L}$ to the form

$$\mathcal{L}_r=\mathcal{L}^*[\underline{A},\underline{\pi},\underline{C},\underline{\overline{C}},\underline{\rho},\underline{G}]+g\Phi[P,\Sigma;\underline{\pi},\underline{C}] +K^0\mathfrak{p}^0[\underline{\pi}]+H^a\mathfrak{s}^a[\underline{\pi}]\,, \tag{5.1}$$

$$\begin{aligned}\mathcal{L}^*=\mathcal{L}^{**}[\underline{A},\underline{\pi}]&-\tfrac{1}{4}F^a_{\mu\nu}[\underline{A}]^2 -\frac{1}{2\alpha}(\partial^\mu\underline{A}_\mu)^2-\underline{\overline{C}}\cdot\partial^\mu\nabla_\mu[\underline{A}]\underline{C}\\ &+\underline{\rho}^\mu\cdot\nabla_\mu[\underline{A}]\underline{C}+\tfrac{1}{2}g\underline{G}\cdot(\underline{C}\times\underline{C})\,.\end{aligned} \tag{5.2}$$

Here the functional $\Phi[P,\Sigma;\underline{\pi},\underline{C}]$ is linear in C and linear in either P or Σ, and $\mathfrak{p}^0[\underline{\pi}]$ and $\mathfrak{s}^a[\underline{\pi}]$ $=(\mathfrak{s}^0[\underline{\pi}],\underline{\mathfrak{s}}[\underline{\pi}])$ are functionals of $\underline{\pi}$ alone. On dimensional grounds, Φ, $\mathfrak{p}^0$, and $\mathfrak{s}^a$ contain the dimensionless coupling constant $1/F$ but neither g nor a space-time derivative.

Our next task is to determine the further structures of $\mathcal{L}^{**}[\underline{A},\underline{\pi}]$, $\Phi[P,\Sigma;\underline{\pi},\underline{C}]$, $\mathfrak{p}^0[\underline{\pi}]$, and $\mathfrak{s}^a[\underline{\pi}]$ by means of the WT identities. Now let us first substitute the action $\int dx\,\mathcal{L}_r$ into the $SU(N)_R$ WT identity (4.12) and isolate the coefficients of K^0, H^a, etc. The result is the following set of equations:

$$-\left(\frac{2}{N}\right)^{1/2}\underline{\mathfrak{s}}^k+\frac{\delta\mathfrak{p}^0}{\delta\underline{\pi}^l}\left[\left(\frac{2}{N}\right)^{1/2}\mathfrak{s}^0+\tilde{\underline{\mathfrak{s}}}-\underline{\pi}\times\right]^{lk}=0\,, \tag{5.3}$$

$$\left(\frac{2}{N}\right)^{1/2}\underline{\pi}^k+\frac{\delta\mathfrak{s}^0}{\delta\underline{\pi}^l}\left[\left(\frac{2}{N}\right)^{1/2}\mathfrak{s}^0+\tilde{\underline{\mathfrak{s}}}-\underline{\pi}\times\right]^{lk}=0\,, \tag{5.4}$$

$$\left[\left(\frac{2}{N}\right)^{1/2}\mathfrak{p}^0+\tilde{\underline{\pi}}+\underline{\mathfrak{s}}\times\right]^{kl} +\frac{\delta\mathfrak{s}^k}{\delta\underline{\pi}^m}\left[\left(\frac{2}{N}\right)^{1/2}\mathfrak{s}^0+\tilde{\underline{\mathfrak{s}}}-\underline{\pi}\times\right]^{ml}=0\,, \tag{5.5}$$

$$\int dx\left\{(\tilde{\Sigma}+P\times)\frac{\delta\Phi}{\delta P}+(-\tilde{P}+\Sigma\times)\frac{\delta\Phi}{\delta\Sigma} +\left[\left(\frac{2}{N}\right)^{1/2}\mathfrak{s}^0+\tilde{\underline{\mathfrak{s}}}+\underline{\pi}\times\right]\frac{\delta\Phi}{\delta\underline{\pi}}\right\}^k=0\,, \tag{5.6}$$

$$\int dx\left[\left(\frac{2}{N}\right)^{1/2}\mathfrak{s}^0+\tilde{\underline{\mathfrak{s}}}+\underline{\pi}\times\right]\frac{\delta\mathcal{L}^{**}[\underline{A},\underline{\pi}]}{\delta\underline{\pi}}=0\,, \tag{5.7}$$

where k, l, and m run from 1 to N^2-1. The first three sets of equations (5.3)—(5.5) serve to determine $\mathfrak{p}^0[\underline{\pi}]$ and $\mathfrak{s}^a[\underline{\pi}]$. If we make identification $\mathfrak{p}^0[\underline{\pi}]\to\pi^0[\underline{\pi}]$ and $\mathfrak{s}^a[\underline{\pi}]\to\sigma^a[\underline{\pi}]$ in these equations, they imply nothing but the $SU(N)_R$ transformation property (2.12) of π^a and σ^a; furthermore, if we take into account the $SU(N)_{L+R}$ symmetry (4.16) they represent the $SU(N)_L$ transformation property in (2.12) as well. Therefore, if we introduce the $N\times N$ matrix $\mathfrak{M}[\underline{\pi}]=\mathfrak{s}^a[\underline{\pi}]\lambda^a+i(\mathfrak{p}^0[\underline{\pi}]\lambda^0+\tilde{\underline{\pi}})$, it transforms according to the $(N,\overline{N})$ representation of $SU(N)_L\times SU(N)_R$, $\mathfrak{M}[\underline{\pi}]\to\mathfrak{M}'=\mathfrak{M}[\underline{\pi}']=U_L\,\mathfrak{M}[\underline{\pi}]U_R^\dagger$, where the transformed field $\underline{\pi}'$ is defined by (2.12) with (π^0,σ^a) replaced by $(\mathfrak{p}^0,\mathfrak{s}^a)$. We next note that the Hermitian matrix field $\mathfrak{M}[\underline{\pi}]\mathfrak{M}[\underline{\pi}]^\dagger$ is invariant under the $SU(N)_R$ transformation

$$\delta_R(\mathfrak{M}\mathfrak{M}^\dagger)=\tfrac{1}{2}\delta\underline{\epsilon}^a_R\cdot\left[\left(\frac{2}{N}\right)^{1/2}\mathfrak{s}^0+\tilde{\underline{\mathfrak{s}}}-\underline{\pi}\times\right]^{ab} \times(\delta/\delta\underline{\pi}^b)(\mathfrak{M}\mathfrak{M}^\dagger)=0\,,$$

which means that the $\mathfrak{M}[\underline{\pi}]\mathfrak{M}[\underline{\pi}]^\dagger$ is a constant Hermitian matrix independent of $\underline{\pi}$. (Recall that at the tree level $\mathfrak{s}^0[\underline{\pi}]\to\sigma^0[\underline{\pi}]=(N/2)^{1/2}[F+O(\underline{\pi}^2)]$ and therefore $\delta_R(\mathfrak{M}\mathfrak{M}^\dagger)=0$ implies $\delta(\mathfrak{M}\mathfrak{M}^\dagger)/\delta\underline{\pi}^a=0$.) Analogously, the Hermitian matrix $\mathfrak{M}[\underline{\pi}]^\dagger\mathfrak{M}[\underline{\pi}]$ turns out to be a constant matrix. It is equally important to note that the determinant $\det\mathfrak{M}[\underline{\pi}]$ also is independent of $\underline{\pi}$ since it is invariant under $SU(N)_L\times SU(N)_R$ transformations. With these observations in mind let us turn to (5.3)-(5.5). We should look for those expressions for $\mathfrak{p}^0[\underline{\pi}]$ and $\mathfrak{s}^a[\underline{\pi}]$ which are a power series in $\underline{\pi}$ and which are normalized so that $\mathfrak{s}^0[\underline{\pi}=0]=(N/2)^{1/2}F'$ and $\pi^0[0]$ $=\underline{\mathfrak{s}}[0]=0$, where F' is a real constant (recall in this connection that the counterterm consists of polynomials of fields and sources so that the above power-series nature is preserved in perturbation theory). With the normalization condition $\mathfrak{M}[\underline{\pi}=0]$ $=F'$, the matrix $\mathfrak{M}[\underline{\pi}]/F'$ becomes an $SU(N)$ matrix, i.e., $\mathfrak{M}[\underline{\pi}]\mathfrak{M}^\dagger[\underline{\pi}]=\mathfrak{M}^\dagger\mathfrak{M}=(F')^2$ and $\det(\mathfrak{M}[\underline{\pi}]/F')=1$. In Sec. II and Appendix B we have noted that the power-series parametrization of an $SU(N)$ matrix normalized in this way is uniquely determined. Therefore F', $\mathfrak{p}^0[\underline{\pi}]$, and $\mathfrak{s}^a[\underline{\pi}]$ can be set equal to F, $\pi^0[\underline{\pi}]$, and $\sigma^a[\underline{\pi}]$ defined in Sec. II, respectively.

Let us next look at the WT identity (5.6), which implies that $\Phi[P,\Sigma;\underline{\pi},\underline{C}]$ is a scalar under the $SU(N)_R$ transformation

$$\begin{aligned}\delta P^a&=-\tfrac{1}{2}\delta\epsilon^b_R(\tilde{\Sigma}+P\times)^{ba}\,,\\ \delta\Sigma^a&=-\tfrac{1}{2}\delta\epsilon^b_R(-\tilde{P}+\Sigma\times)^{ba}\,,\\ \delta\underline{\pi}^c&=-\tfrac{1}{2}\delta\epsilon^b_R[(2/N)^{1/2}\sigma^0+\tilde{\underline{\sigma}}+\underline{\pi}\times]^{bc}\,,\end{aligned}$$

or in the $U(N)$ notation

$$\delta\pi^a = -\tfrac{1}{2}\delta\epsilon_R^b \cdot (\vec{\sigma} + \pi\times)^{ba}, \tag{5.8}$$

where $\delta\epsilon_R^b = (0, \delta\underline{\epsilon}_R)$, and (b,c) run from 1 to N^2-1 while a runs from 0 to N^2-1. Analogously, it follows from the WT identity associated with the global $SU(N)_L$ symmetry that Φ is also invariant under the corresponding $SU(N)_L$ transformation on P, Σ, π, and C. Possible forms of Φ are determined from the following symmetry argument. Let us write

$$\begin{aligned} M &= \hat{\sigma} + i\hat{\pi}\,, \\ N &= \hat{\Sigma} + i\hat{P} = (\Sigma^a + iP^a)\lambda^a\,, \\ Q &= \hat{C} = \underline{\hat{C}}\,. \end{aligned} \tag{5.9}$$

The $(N\times N)$-matrix fields M and N transform according to the $(N,\bar{N})$ representation of $SU(N)_L \times SU(N)_R$, i.e., $M \to M' = U_L M U_R^\dagger$, etc., while Q transforms according to the $SU(N)_L$ regular representation $Q \to Q' = U_L Q U_L^\dagger$ and is an $SU(N)_R$ singlet. In general, $SU(N)_L \times SU(N)_R$ invariants are constructed from appropriate products of M, $M^\dagger, N, N^\dagger$, and Q by taking traces or determinants. We know that Φ is linear in Q and in either N or $N^\dagger$. Note that Q, $NM^\dagger$, and $MN^\dagger$, which are $SU(N)_R$ singlets, tranform according to the $SU(N)_L$ regular representation. Accordingly, both $\mathrm{Tr}(NM^\dagger Q)$ and $\mathrm{Tr}(MN^\dagger Q)$ are $SU(N)_L \times SU(N)_R$ invariants. Moreover, in view of the nonlinearity of the realization $M^\dagger M = F^2$ and $\det(M/F) = 1$, one may readily be convinced that these are the only invariants that meet our requirements for Φ. They lead to the following invariants:

$$\begin{aligned} (\mathrm{I}) &= \Sigma\cdot(-\vec{\pi} + \sigma\times)C + P\cdot(\vec{\sigma} + \pi\times)C\,, \\ (\mathrm{II}) &= \Sigma\cdot(\vec{\sigma} + \pi\times)C + P\cdot(\vec{\pi} - \sigma\times)C\,, \end{aligned} \tag{5.10}$$

in terms of which Φ is written as

$$\Phi[P, \Sigma; \underline{\pi}, \underline{C}] = a(\mathrm{I}) + b(\mathrm{II})\,. \tag{5.11}$$

A further constraint on Φ follows from the WT identity (4.13). Let us substitute (5.1) into (4.13) and pick up the coefficient of K^0:

$$\frac{\delta\Phi}{\delta P^0} - \frac{\delta\pi^0}{\delta\underline{\pi}}\cdot\frac{\delta\Phi}{\delta\underline{P}} = 0\,. \tag{5.12}$$

At the same time we obtain the WT identity

$$\int dx \left[\frac{\delta\Phi}{\delta\underline{P}}\cdot\frac{\delta\Phi}{\delta\underline{\pi}} - \frac{1}{2}\frac{\delta\Phi}{\delta\underline{C}}\cdot(\underline{C}\times\underline{C})\right] = 0\,. \tag{5.13}$$

Substitution of (5.11) into (5.12) and (5.13) shows that

$$b = 0 \text{ and } a = \tfrac{1}{2}\,. \tag{5.14}$$

In terms of $\mathcal{L}^{**}[\underline{A}, \underline{\pi}]$, the WT identity (4.13) now reads

$$\nabla_\mu^{ab}[\underline{A}]\frac{\delta\mathcal{L}^{**}}{\delta\underline{A}_\mu^b} - \frac{g}{2}(\vec{\sigma} - \pi\times)^{ab}\frac{\delta\mathcal{L}^{**}}{\delta\underline{\pi}^b} = 0\,, \tag{5.15}$$

where a and b run from 1 to N^2-1. From this identity and (5.7) it follows that $\mathcal{L}^{**}[\underline{A}, \underline{\pi}]$ has the local $SU(N)_L$ symmetry as well as the global $SU(N)_L \times SU(N)_R$ symmetry. This symmetry property and power-counting are sufficient to determine possible forms of $\mathcal{L}^{**}[\underline{A}, \underline{\pi}]$.

Let us first consider the case of the pure nonlinear σ model; we turn off the coupling to the gauge field $g_r \to 0$. According to power-counting $\mathcal{L}^{**}[\underline{A}=0, \underline{\pi}]$ can involve at most two space-time derivatives; this determines $\mathcal{L}^{**}[\underline{A}=0, \underline{\pi}]$ uniquely:

$$\mathcal{L}^{**}[\underline{A}=0, \underline{\pi}] = \tfrac{1}{4}\beta\,\mathrm{Tr}[(\partial_\mu M^\dagger)(\partial^\mu M)]\,, \tag{5.16}$$

where β is a constant which, at the tree level, is equal to one. If we make the scale transformation $\beta^{1/2}(M, \sigma^a, \pi^a, F) \to (M, \sigma^a, \pi^a, F)$ and $\beta^{-1/2}(K^0, H^a) \to (K^0, H^a)$ in (5.1) and (5.16), β can effectively be set equal to one. Consequently we have learned that the renormalized Lagrangian $\mathcal{L}_r[g=0, \underline{\pi}]$ in (5.1) has the same structure as our initial bare Lagrangian in (4.1); this shows the renormalizability of the $SU(N)_L \times SU(N)_R$ nonlinear σ model.

Let us next include couplings to the gauge field. Since $\mathcal{L}^{**}[\underline{A}, \underline{\pi}]$ can contain at most two A_μ fields, its most general form is given by

$$\begin{aligned} \mathcal{L}^{**}[\underline{A}, \underline{\pi}] = {} & \tfrac{1}{4}\mathrm{Tr}[M^\dagger \mathfrak{D}_\mu^\dagger[\underline{A}]\mathfrak{D}^\mu[\underline{A}]M] \\ & + \tfrac{1}{4}\gamma\,\mathrm{Tr}[(\partial_\mu M^\dagger)(\partial^\mu M)]\,, \end{aligned} \tag{5.17}$$

where γ is a dimensionless constant. We have fixed the overall normalization of the first term by an appropriate rescaling of M, F, etc. At the tree level γ vanishes, and the term proportional to γ, if any, is a part of higher-loop renormalization counterterms. Now recall the softness of the gauge-field coupling, as characterized by the dimentional constant g. In particular, on dimensional grounds the inclusion of the gauge-field coupling does not affect the divergent part of wave-function renormalization of the $\underline{\pi}$ fields. Therefore, comparing (5.16) and (5.17) we conclude that γ is an effect of finite renormalization (i.e., finite rescaling of $\underline{\pi}$, F, etc.). Since finite renormalization is inessential in the proof of renormalizability we may simply set $\gamma = 0$. Thus we have again learned that the renormalized Lagrangian $\mathcal{L}_r$ coincides with our bare Lagrangian in (4.1). This completes the proof of the renormalizability of the two-dimensional $SU(N)$ nonlinear σ model and its associated massive Yang-Mills theory.

The wave-function renormalization of the $\underline{\pi}$ field and the renormalization of the coupling constant $1/F$ are two basic divergent renormalizations we have to carry out; accordingly we introduce the renormalization transformation

$$\underline{\pi} = Z^{1/2}\underline{\pi}_r \text{ and } F^2 = XZF_r^{\,2}\,. \tag{5.18}$$

The mass renormalization of the vector field A_μ is not independent:

$$m = \tfrac{1}{2} gF = (XZ)^{1/2} m_r \,,$$

where $m_r = \frac{1}{2} g F_r$ is a renormalized mass, which in general is different from the physical vector-meson mass. [In (5.18) we have assumed the $\mathrm{SU}(N)_{L+R}$ symmetry. As is well known,[1] once we know the renormalizability in the $\mathrm{SU}(N)_{L+R}$ symmetric case, the theory remains renormalizable even when the global symmetry is spontaneously broken down to smaller symmetries.]

The foregoing proof tells us that the composite operators $\pi^0[\underline{\pi}]$ and $\sigma^a[\underline{\pi}]$ are multiplicatively renormalized. Although the $\mathrm{SU}(N)_{L+R}$ singlet fields $\pi^0[\underline{\pi}]$ and $\sigma^0[\underline{\pi}]$ appear to mix under renormalization, the mixing is avoided owing to the discrete symmetry $\underline{\pi}^a \to -\underline{\pi}^a$ of the chiral Lagrangian (2.14). (Recall that $\pi^0[-\underline{\pi}] = -\pi^0[\underline{\pi}]$ while $\sigma^a[-\underline{\pi}] = \sigma^a[\underline{\pi}]$; see Appendix B.) In perturbation theory the pure nonlinear σ model preserves this discrete symmetry. It is broken in the presence of the $\mathrm{SU}(N)_L$ gauge field but still governs the divergent part of the renormalization of the above composite operators. Accordingly, $\sigma^0[\underline{\pi}]$, for example, is made finite by an appropriate rescaling $\sigma_r^0 = (Y_0 Z)^{-1/2} \times \sigma^0[\underline{\pi}]$; the external source for this finite operator is given by $H_r^0 = (Y_0 Z)^{1/2} H^0$. The renormalization constant Y_0 can be set equal to one, as we shall see below. We differentiate (4.12) once with respect to $\underline{\pi}_r^b(x)$ and let all external sources vanish except for $H_r^0(x)$ which we set equal to a constant $H_r^0(0)$. The result is the WT identity

$$\langle \sigma_r^0 \rangle \, \Gamma^{ab}(p^2 = 0) + (H_r^0 / Y_0) \delta^{ab} = 0 \,, \tag{5.19}$$

where $\langle \sigma_r^0 \rangle_0 \equiv \delta\Gamma / \delta H_r^0(x)$ is the vacuum expectation value of $\sigma_r^0(x)$ and $\Gamma^{ab}(p^2)$ is the Fourier transform of the renormalized inverse $\underline{\pi}_r$ propagator $\delta^2\Gamma / \delta\underline{\pi}_r^a(x)\delta\underline{\pi}_r^b(y)$. Since $\langle \sigma_r^0 \rangle_0$, H_r^0, and $\Gamma^{ab}(p^2)$ are finite Y_0 must be finite; consequently we can simply choose $Y_0 = 1$. Proceeding with the WT identities in the same fashion, one can further show that $(\pi_r^0, \sigma_r^a) = Z^{-1/2}(\pi^0, \sigma^a)$ are finite operators. Thus the renormalization transformation we should use is $\mathrm{U}(N)_L \times \mathrm{U}(N)_R$ symmetric. The composite operators $(-\bar{\pi} + \sigma \times) C$ and $(\bar{\sigma} + \pi \times) C$ are multiplicatively renormalized but they in general mix under renormalization.

Perturbation theory is developed on the basis of the effective Lagrangian (3.6) with the source term $H_r^0 \sigma_r^0[\underline{\pi}]$ added as an infrared regulator:

$$\mathcal{L}_{\mathrm{eff}}[A, \pi, C, \bar{C}] + H_r^0 \sigma_r^0[\underline{\pi}] = \mathcal{L}_{\mathrm{free}}[\underline{A}, \underline{\pi}_r, \underline{C}_r, \underline{\bar{C}}_r] + \mathcal{L}_{\mathrm{int}} \,, \tag{5.20}$$

$$\mathcal{L}_{\mathrm{free}} = -\tfrac{1}{4}(\partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu)^2 + \tfrac{1}{2} m_r^2 (\underline{A}_\mu)^2 - \frac{1}{2\alpha}(\partial^\mu \underline{A}_\mu)^2$$

$$- m_r \underline{A}^\mu \cdot \partial_\mu \underline{\pi}_r + \tfrac{1}{2}(\partial_\mu \underline{\pi}_r)^2 - \tfrac{1}{2}\kappa^2 \underline{\pi}_r^2 - \underline{\bar{C}} \partial^2 \underline{C} \,,$$

where $\kappa^2 = (2/N)^{1/2} H_r^0 / F_r$ and we have chosen H_r^0 to be a constant. The free propagators derived from this free Lagrangian are listed in Fig. 1. The interaction Lagrangian $\mathcal{L}_{\mathrm{int}}$, when expanded in powers of $\underline{A}$ and $\underline{\pi}$, contains an infinite number of interaction vertices.

VI. ONE-LOOP CALCULATION

In this section we calculate the one-loop counterterm in the SU(2) case, to verify the argument of Sec. V. Expressions for SU(2) are obtained by setting $\sigma^0 \to \sigma$ and $(\pi^0, \underline{\sigma}) \to 0$ everywhere in the foregoing sections. In particular, the matrix field $M[\underline{\pi}]$ is given by $M[\underline{\pi}] = \sigma + i\underline{\pi} \cdot \underline{\tau}$ with σ and $\underline{\pi}^a$ ($a = 1, 2, 3$) subjected to a constraint $\sigma^2 + \underline{\pi}^2 = F^2$ or $\sigma[\underline{\pi}] = (F^2 - \underline{\pi}^2)^{1/2}$.

As is well known,[26] the steepest-descent ap-

$$\langle T^* A_\mu^a(p) A_\nu^b(p) \rangle = \frac{i}{p^2 - m^2}\left[-g_{\mu\nu} + \{(1-\alpha)(\kappa^2 - p^2) - \alpha m^2\} D(p^2) p_\mu p_\nu \right] \delta^{ab}$$

$$\langle T^* A_\mu^a(p) \pi^b(-p) \rangle = \alpha m p_\mu D(p^2) \delta^{ab}$$

$$\langle T^* \pi^a(p) \pi^b(-p) \rangle = i(\alpha m^2 - p^2) D(p^2) \delta^{ab}$$

$$\langle T^* C^a(p) \bar{C}^b(-p) \rangle = i/p^2$$

$$D(p^2) = 1/[p^2(\kappa^2 - p^2) - \alpha\kappa^2 m^2]$$

FIG. 1. Free propagators. All the fields and parameters are renormalized quantities; the subscript r is suppressed. The parameter κ^2 is an infrared cutoff.

proximation to the evaluation of the functional integral (4.1) gives the effective action Γ up to the one-loop approximation. [In what follows we regard (4.1) as written in terms of renormalized quantities, although the subscript r is suppressed. In addition, we set $\rho_\mu = G = P = \Sigma = 0$ for simplicity.]

We first shift the integration variables:

$$\begin{aligned} \underline{A}_\mu &= \underline{A}_\mu^{\rm st} + \underline{B}_\mu , \\ \underline{\pi} &= \underline{\pi}^{\rm st} + \underline{\chi} , \\ \underline{C} &= \underline{C}^{\rm st} + \underline{\psi} , \\ \underline{\overline{C}} &= \underline{\overline{C}}^{\rm st} + \underline{\overline{\psi}} , \end{aligned} \tag{6.1}$$

where $\underline{A}_\mu^{\rm st}$, $\underline{\pi}^{\rm st}$, $\underline{C}^{\rm st}$, and $\underline{\overline{C}}^{\rm st}$ are determined so that the exponent in (4.1) becomes stationary. Up to the one-loop approximation, these stationary fields are the arguments of the effective action Γ.[26] We change the integration variables from $(\underline{A}_\mu, \underline{\pi}, \underline{C}, \underline{\overline{C}})$ to $(\underline{B}_\mu, \underline{\chi}, \underline{\psi}, \underline{\overline{\psi}})$ and henceforth denote $(\underline{A}_\mu^{\rm st}, \underline{\pi}^{\rm st}, \underline{C}^{\rm st}, \underline{\overline{C}}^{\rm st})$ simply by $(\underline{A}_\mu, \underline{\pi}, \underline{C}, \underline{\overline{C}})$. Then we expand the exponent around the stationary fields and extract the term $L^{(1)}[\underline{B}, \underline{\chi}, \underline{\psi}, \underline{\overline{\psi}}]$ which is quadratic in $\underline{B}_\mu, \underline{\chi}, \underline{\psi}, \underline{\overline{\psi}}$. The one-loop contribution to the effective action Γ is obtained by evaluating the Gaussian functional integral

$$\int [d\underline{B}][d\underline{\chi}][d\underline{\psi}][d\underline{\overline{\psi}}] \exp\left(i \int dx\, L^{(1)}[\underline{B}, \underline{\chi}, \underline{\psi}, \underline{\overline{\psi}}]\right) , \tag{6.2}$$

where we have set the weight function $w[\underline{\pi}]$ (3.8) equal to one by assuming the use of dimensional regularization. Since $L^{(1)}[\underline{B}, \underline{\chi}, \underline{\psi}, \underline{\overline{\psi}}]$ contains, in particular, a term of the form

$$\tfrac{1}{2}(\partial_\mu \underline{\chi}^a) W^{ab}[\underline{\pi}] \partial^\mu \underline{\chi}^b , \tag{6.3}$$

with

$$W^{ab}[\underline{\pi}] = \delta^{ab} + \underline{\pi}^a \underline{\pi}^b / \sigma^2 , \tag{6.4}$$

we make a further change of variables $\underline{\chi}^a \to \underline{\phi}^a = V^{ab}[\underline{\pi}]\underline{\chi}^b$ so that (6.3) is brought to the form $\frac{1}{2}(\partial^\mu \underline{\phi})^2$. That is, we choose $V^{ab}[\underline{\pi}] = \delta^{ab} + \beta \underline{\pi}^a \underline{\pi}^b$, where $\beta[\underline{\pi}] = 1/[\sigma(\sigma + F)]$. The Jacobian associated with this change of variables is given by $\partial(\underline{\phi})/\partial(\underline{\chi}) = \prod_x \det V_{ab}[\underline{\pi}(x)] = \prod_x \{f/\sigma[\underline{\pi}(x)]\}$ which is again set equal to one by use of dimensional regularization. (As a matter of fact, this Jacobian exactly cancels the previous weight function $w[\underline{\pi}]$.) Expressed in terms of $\underline{\phi}$, $L^{(1)}$ takes the form

$$\begin{aligned} L^{(1)} = &- \tfrac{1}{4}(\partial_\mu \underline{B}_\nu - \partial_\nu \underline{B}_\mu)^2 + \tfrac{1}{2} m^2 \underline{B}_\mu{}^2 - \frac{1}{2\alpha}(\partial^\mu \underline{B}_\mu)^2 - m \underline{B}^\mu \cdot \partial_\mu \underline{\phi} \\ &+ \tfrac{1}{2}(\partial_\mu \underline{\phi})^2 + \underline{\phi}^a N_\mu^{ab} \partial^\mu \underline{\phi}^b + \tfrac{1}{2}\underline{\phi}^a M^{ab} \underline{\phi}^b + \partial^\mu \underline{B}^{\nu a} T_{\mu\nu\rho}^{ab} \underline{B}^{\rho b} \\ &+ \tfrac{1}{2}\underline{B}^{\mu a} U_{\mu\nu}^{ab} \underline{B}^{\nu b} + \underline{B}^{\mu a}(Q_{\mu\nu}^{ab} \partial^\nu \underline{\phi}^b + R_\mu^{ab} \underline{\phi}^b) - \underline{\overline{\psi}} \cdot \partial^\mu \nabla_\mu [\underline{A}] \underline{\psi} , \end{aligned} \tag{6.5}$$

where $N_\mu^{ab}, M^{ab}, T_{\mu\nu\rho}^{ab}, U_{\mu\nu}^{ab}, Q_{\mu\nu}^{ab}, R_\mu^{ab}$ are functionals of $\underline{A}_\mu$ and $\underline{\pi}$. Expressions for these quantities are rather complicated; accordingly we simply remark that $T_{\mu\nu\rho}^{ab}$ and $U_{\mu\nu}^{ab}$ are functionals of $\underline{A}_\mu$ alone and quote the following result:

$$\begin{aligned} \mathrm{Tr}[M + \tfrac{1}{4}(N_\mu - N_\mu^{\rm tr})^2] = &(1/F^2)[(\partial_\mu \sigma)^2 + (\partial_\mu \underline{\pi})^2] + 3[(\partial_\mu \sigma / \sigma)^2 - H^0/\sigma] \\ &- 2(m/F)^2 \underline{A}_\mu{}^2 + 2(m/F^3)\underline{A}^\mu \cdot (\underline{\pi}\partial_\mu \sigma - \sigma \partial_\mu \underline{\pi} + \underline{\pi} \times \partial_\mu \underline{\pi}) + 3(m/F\sigma^2)\underline{A}_\mu \cdot \partial^\mu(\sigma \underline{\pi}) , \end{aligned} \tag{6.6}$$

$$Q^{ab} = \tfrac{1}{2} g[(F - \sigma)\delta^{ab} + \epsilon^{acb}\underline{\pi}^c - \underline{\pi}^a \underline{\pi}^b/(F + \sigma)] . \tag{6.7}$$

There are five types of diagrams which give rise to the one-loop ultraviolet divergences; see Fig. 2, diagrams (i)–(v). Since $T_{\mu\nu\rho}^{ab}$ and $U_{\mu\nu}^{ab}$ involve $\underline{A}_\mu$ alone, diagrams (i) and (ii) are common to the massless pure Yang-Mills theory and give rise to divergent terms proportional to $\underline{A}_\mu{}^2$. In the pure Yang-Mills theory, however, there is no mass renormalization and therefore the ultraviolet divergences coming from (i) and (ii) must add up to vanish. Diagram (iii) yields the one-loop divergence of the form

$$\mathcal{L}^{\rm div}({\rm iii}) = I\, \mathrm{Tr}[M + \tfrac{1}{4}(N_\mu - N_\mu^{\rm tr})^2] , \tag{6.8}$$

where I stands for the ultraviolet-divergent part of $[i/2(2\pi)^n] \int d^n k (1/k^2)$, i.e., $I = 1/[4\pi(2-n)]$ as $n \to 2$.

Similarly, diagrams (iv) and (v) lead to the following one-loop divergences:

$$\mathcal{L}^{\rm div}({\rm iv}) = I\alpha\, \mathrm{Tr}(QQ^{\rm tr}) , \tag{6.9}$$

$$\mathcal{L}^{\rm div}({\rm v}) = -I\, 2\alpha m\, \mathrm{Tr}(Q) . \tag{6.10}$$

Substitution of the expression (6.7) for Q shows that these two expressions combine to vanish. Consequently, the one-loop renormalization counterterm is given by $\Delta\mathcal{L} = -\mathcal{L}^{\rm div}({\rm iii})$ with the substitution of (6.6). It is easy to see that this counterterm is compatible with the WT identities. In particular, the divergent renormalization constants Z and X are determined to be (set $N = 2$)

$$\begin{aligned} Z &= 1 - (2/F_r{}^2)[N - (2/N)]I , \\ X &= 1 + (4/F_r{}^2)[N - (1/N)]I , \end{aligned} \tag{6.11}$$

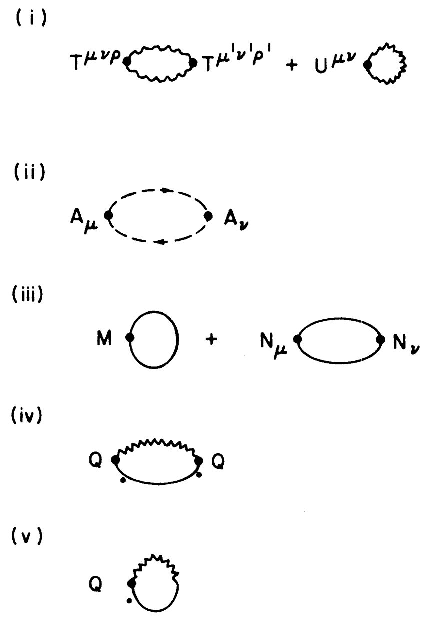

FIG. 2. Diagrams (i), (ii), and (iii) represent the propagation of B_μ, ψ, and ϕ, respectively. Diagram (v) represents a mixed B_μ-ϕ loop.

where we have recovered the subscript r. The above expressions with general N correspond to the SU(N) case; in this case we have determined them from the one-loop $\underline{A}_\mu \underline{A}_\nu$, $\underline{A}_\mu \underline{\pi}$, and $\underline{\pi}\underline{\pi}$ propagators using the power-series expression (B6) of Appendix B.

VII. ASYMPTOTIC FREEDOM

The O(N) nonlinear σ model is asymptotically free and has a nontrivial ultraviolet-stable fixed point in $n = 2+\epsilon$ dimensions[13-15] with $\epsilon > 0$ and infinitesimal. This short-distance feature is also true for the SU(N) nonlinear σ model and therefore common to massive Yang-Mills theory as well. To see this let us consider the renormalization-group equations,[27] which, e.g., for the renormalized inverse $\underline{\pi}^a \underline{\pi}^b$ propagator $\Gamma^{ab}(p^2) = \delta^{ab}\Gamma(p^2)$, takes the form

$$\left(\mu \frac{\partial}{\partial \mu} + \beta^{(m)} \frac{\partial}{\partial m_r} + \beta^{(e)} \frac{\partial}{\partial e} - 2\gamma^{(\pi)}\right) \Gamma(p^2; e, m_r, \alpha; \mu) = 0 , \tag{7.1}$$

where μ is an arbitrary reference mass (or, more definitely, one may regard μ^2 as denoting the renormalization point $p^2 = \mu^2$) and $e \equiv \mu^{n-2}/(4\pi F_r^{\,2})$. In $n = 2+\epsilon$ dimensions, the coupling constant $1/F_r^{\,2}$ and the $\underline{\pi}$ field have dimension $(2-n)$ and $(n-2)/2$, respectively, in units of mass; correspondingly, by means of the dimensionless coupling constant $e = \mu^{n-2}/(4\pi F_r^{\,2})$ we have defined the continuation of $1/F^2$ into n dimensions.

As is well known,[27] $\beta^{(e)}$ and $\gamma^{(\pi)}$ are given by

$$\beta^{(e)} = (\mu d/d\mu) e , \quad \gamma^{(\pi)} = \tfrac{1}{2}(\mu d/d\mu) \ln Z , \tag{7.2}$$

where $d/d\mu$ is a derivative to be taken with bare parameters m and F fixed. Noting the fact that $1/(4\pi F^2) = e\mu^{2-n}(XZ)^{-1}$ is a bare parameter independent of μ and using (6.11), one finds that

$$\beta^{(e)} = e(\epsilon - 2Ne) + O(e^2\epsilon) . \tag{7.3}$$

Similarly, if follows from (6.11) and (7.2) that the anomalous dimension $\gamma^{(\pi)}$ of the $\underline{\pi}_r$ field is given by

$$\gamma^{(\pi)} = [N - (2/N)]e + O(e\epsilon) . \tag{7.4}$$

As a consequence of $\mathrm{U}(N)_L \times \mathrm{U}(N)_R$-symmetric renormalization, the composite operators $\pi^0[\underline{\pi}]$ and $\sigma^a[\underline{\pi}]$ have the same anomalous dimension. It is clear from (7.3) that the theory is asymptotically free in two dimensions while it has a nontrivial ultraviolet-stable point $e_{\text{critical}} = \epsilon/(2N)$ of order ϵ in $2+\epsilon$ dimensions.

VIII. CONCLUSIONS

In this paper we have presented a systematic study of Yang-Mills field theories with explicit mass terms. We have shown that these theories may be considered within the conventional framework of mass generation via the Higgs-Kibble mechanism in gauge theories. The unique feature of this class of theories is that some or all of the Higgs fields are realized nonlinearly. For these nonlinear realizations the Higgs sector of the theory involves solely the Goldstone bosons which are absorbed by the Higgs-Kibble mechanism. In a particular (unitary) gauge, the entire nonlinear Higgs Lagrangian is reduced solely to the explicit mass term for the gauge field.

As an example of this mechanism, we have studied an SU(N) massive Yang-Mills theory. In the case where all masses are degenerate, the equivalent Higgs theory involves nonlinear chiral Higgs fields, having an $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ symmetry. The $\mathrm{SU}(N)_L$ symmetry is gauged while the $\mathrm{SU}(N)_R$ symmetry reflects the mass degeneracy and symmetry of the massive Yang-Mills theory.

With the fundamental structure of massive Yang-Mills theories determined, we have focused on the renormalizability and perturbative aspects of these theories. The renormalization problems were studied using the standard covariant gauges where the divergence structure is softer than that of the original unitary gauge. In this formulation, it is clear that the principal divergences are those associated with the nonlinear Higgs sector of the theory.

In four space-time dimensions the divergence structure remains untractable for the nonlinear

Higgs theory. However, in two dimensions the analogous O(N) nonlinear σ models were known to be renormalizable. We have extended this proof to the $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ nonlinear σ model.

Using these results, the nonlinear chiral gauge theories (massive Yang-Mills theories) were shown to be renormalizable in two dimensions. This proof requires the full use of the Ward-Takahashi identities for the global $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ symmetries as well as those associated with the local $\mathrm{SU}(N)_L$ symmetry.

We have further shown that these theories are asymptotically free field theories. This fact implies that these theories are governed by a nontrivial ultraviolet fixed point above two dimensions. Whether the theories may be extrapolated to four dimensions while maintaining control of the short-distance behavior remains an open question.

We may also speculate on the phase behavior of these theories in two and higher dimensions. In two dimensions, the SU(N) nonlinear σ model (chiral Higgs theory) is expected to exist in the nonperturbative symmetric phase. The perturbative Goldstone phase is expected to be unstable due to infrared instabilities as a result of Coleman's theorem.[28] This result could be established explicitly in the O(N) models at large N.[13-15] We have not been able to extend these explicit calculations to the SU(N) models. Above two dimensions these theories should exhibit a phase transition with the weak-coupling phase being the perturbative Goldstone phase.

Contrary to the σ model, the massive Yang-Mills theory is expected to remain in the perturbative broken-symmetry phase even in two dimensions. The Higgs mechanism is expected to prevent infrared instabilities at least for weak coupling. For strong coupling, a symmetric confinement phase may exist as in the case of O(N) gauge theory.[29] Above two dimensions we expect similar features to exist so long as the short-distance behavior remains under control of the ultraviolet-stable fixed point.

APPENDIX A

In terms of the "SU(N) notation," $(\tilde{A}B)^c = d^{cab}A^aB^b$ and $(A\times B)^c = f^{cab}A^aB^b$ read

$$\begin{aligned}(\tilde{A}B)^0 &= (2/N)^{1/2}(A\cdot B)\\ &= (2/N)^{1/2}(A^0B^0 + \underline{A}\cdot\underline{B}),\\ (\tilde{A}B)^k &= (2/N)^{1/2}(A^0\underline{B}^k + \underline{A}^kB^0) + (\tilde{\underline{A}}\underline{B})^k, \qquad \text{(A1)}\\ A\times B &= \underline{A}\times\underline{B}\quad [(A\times B)^0 = 0].\end{aligned}$$

In view of (2.7) the product $\hat{A}\hat{B}$ is rewritten as

$$\begin{aligned}\hat{A}\hat{B} &= [(\tilde{A} + iA\times)B]^a\lambda^a\\ &= (2/N)(A\cdot B)1 + \{(2/N)^{1/2}(A^0\underline{B}^a + \underline{A}^aB^0)\\ &\qquad + [(\tilde{\underline{A}} + i\underline{A}\times)\underline{B}]^a\}\lambda^a, \qquad \text{(A2)}\end{aligned}$$

where $\tilde{\underline{A}}\underline{B}$ expressed in terms of underlined vectors denotes an (N^2-1)-component vector, $(\tilde{\underline{A}}\underline{B})^k$ $(k\neq 0)$.

APPENDIX B

In this appendix we study the parametrization of the SU(N) matrix

$$M[\underline{\pi}]/F = (\hat{\sigma} + i\hat{\pi})/F, \qquad \text{(B1)}$$

normalized so that $M[\underline{\pi}] = F + i\hat{\pi} + O(\underline{\pi}^2/F)$.

In what follows we shall set $F = 1$ for simplicity.

Since $M[\underline{\pi}]$ is an SU(N) matrix connected to the unit matrix, it is uniquely written as

$$M[\underline{\pi}] = \hat{\sigma} + i\hat{\pi} = \exp(i\hat{\underline{u}}), \qquad \text{(B2)}$$

in terms of a traceless Hermitian $(N\times N)$ matrix $\hat{\underline{u}} = \underline{u}^a\lambda^a$, where $\underline{u}^a(x)$ $(a = 1,\ldots, N^2-1)$ are real functions. Repeated use of the basic formula

$$\hat{\underline{u}}\hat{\underline{u}} = (2/N)\underline{u}^2 1 + (\tilde{\underline{u}}\underline{u})^a\lambda^a \qquad \text{(B3)}$$

enables us to cast $\exp(i\hat{\underline{u}})$ into the desired form (B1); in particular, $\underline{\pi}(\underline{u})$ is given by

$$\underline{\pi} = [1 - (\underline{u}^2/3N)]\underline{u} - \tfrac{1}{6}\tilde{\underline{u}}\tilde{\underline{u}}\underline{u} + O(\underline{u}^5). \qquad \text{(B4)}$$

[In deriving this expression we have used the relation $\underline{u}\times(\tilde{\underline{u}}\underline{u}) = 0$ which follows from the identity[21]

$$f_{abk}d_{kce} + f_{ebk}d_{kca} + f_{cbk}d_{kae} = 0, \qquad \text{(B5)}$$

which is valid for SU(N).] We can solve (B4) for $\underline{u}$ in a power series in $\underline{\pi}$ and express $\pi^0(\underline{u})$ and $\underline{\sigma}^a(\underline{u})$ in powers of $\underline{\pi}$; this procedure defines the parametrization (B1) uniquely. The first few terms in $\pi^0[\underline{\pi}]$ and $\sigma^a[\underline{\pi}]$ are given by

$$\begin{aligned}(2/N)^{1/2}\pi^0 &= -(1/3N)\underline{\pi}\cdot\tilde{\underline{\pi}}\underline{\pi} + \cdots,\\ (2/N)^{1/2}\sigma^0 &= 1 - (1/N)\underline{\pi}^2 - (1/2N^2)(\underline{\pi}^2)^2\\ &\qquad - (1/4N)\underline{\pi}\cdot\tilde{\underline{\pi}}\tilde{\underline{\pi}}\underline{\pi} + \cdots, \qquad \text{(B6)}\\ \underline{\sigma} &= -\tfrac{1}{2}\tilde{\underline{\pi}}\underline{\pi} - (1/4N)(\underline{\pi}^2)\tilde{\underline{\pi}}\underline{\pi} - \tfrac{1}{8}\tilde{\underline{\pi}}^3\underline{\pi}\\ &\qquad + (1/12N)(\underline{\pi}\cdot\tilde{\underline{\pi}}\underline{\pi})\underline{\pi} + \cdots,\end{aligned}$$

where omitted terms are of a higher order in the $\underline{\pi}$ field. The overall normalization constant F is recovered if we replace (π^a, σ^a) by $(\pi^a/F, \sigma^a/F)$.

By making Hermitian conjugation in (B2) we learn that $\pi^a(-\underline{u}) = -\pi^a(\underline{u})$ while $\sigma^a(-\underline{u}) = \sigma^a(\underline{u})$. Correspondingly, $\pi^0[\underline{\pi}]$ is odd in $\underline{\pi}$ whereas $\sigma^a[\underline{\pi}]$ are even in $\underline{\pi}$.

APPENDIX C

In this appendix we calculate the change of the source Lagrangian (4.2) under an infinitesimal $SU(N)_R$ transformation. Since the first term on the right-hand side of (4.3) is trivial we shall confine ourselves to the second term.

Let us first note that, in $(N\times N)$-matrix form, (2.12) reads

$$\delta(\hat{\sigma}+i\hat{\pi})=\tfrac{1}{2}i[\delta\hat{\epsilon}_L(\hat{\sigma}+i\hat{\pi})-(\hat{\sigma}+i\hat{\pi})\delta\hat{\epsilon}_R]. \tag{C1}$$

The P and Σ terms in $\mathcal{L}_{\text{source}}$ (4.2) are written as

$$\mathcal{L}_s(P,\Sigma)=\tfrac{1}{8}g\,\mathrm{Tr}\{P[\hat{C}(\hat{\sigma}+i\hat{\pi})+(\hat{\sigma}-i\hat{\pi})\hat{C}] +i\hat{\Sigma}[\hat{C}(\hat{\sigma}+i\hat{\pi})-(\hat{\sigma}-i\hat{\pi})\hat{C}]\}. \tag{C2}$$

Applying the left transformation on (C2) gives

$$\begin{aligned}\delta_R\mathcal{L}_s(P,\Sigma)&=\tfrac{1}{16}ig\,\mathrm{Tr}[-(\hat{P}+i\hat{\Sigma})\hat{C}(\hat{\sigma}+i\hat{\pi})\delta\hat{\epsilon}_R\\&\qquad+(\hat{P}-i\hat{\Sigma})\delta\hat{\epsilon}_R(\hat{\sigma}-i\hat{\pi})\hat{C}]\\&=\tfrac{1}{16}g\,\mathrm{Tr}\{\delta\hat{\epsilon}_R[(\hat{\Sigma}-i\hat{P})\hat{C}(\hat{\sigma}+i\hat{\pi})\\&\qquad-(\hat{\sigma}-i\hat{\pi})\hat{C}(\hat{\Sigma}+i\hat{P})]\}.\end{aligned}$$

Finally repeated use of formula (A2) leads to the desired expression (4.3).

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$$\langle T^* A^a_\mu(p)\, A^b_\nu(p)\rangle = \frac{i}{p^2 - m^2}\left[- g_{\mu\nu} + \{(1-\alpha)(\kappa^2 - p^2) - \alpha m^2\} D(p^2)\, p_\mu p_\nu \right]\delta^{ab}$$

$$\langle T^* A^a_\mu(p)\, \pi^b(-p)\rangle = \alpha m\, p_\mu D(p^2)\,\delta^{ab}$$

$$\langle T^* \pi^a(p)\, \pi^b(-p)\rangle = i(\alpha m^2 - p^2)\, D(p^2)\,\delta^{ab}$$

$$\langle T^* C^a(p)\, \bar{C}^b(-p)\rangle = i/p^2$$

$$D(p^2) = 1/[\, p^2(\kappa^2 - p^2) - \alpha\kappa^2 m^2]$$

FIG. 1. Free propagators. All the fields and parameters are renormalized quantities; the subscript r is suppressed. The parameter κ^2 is an infrared cutoff.

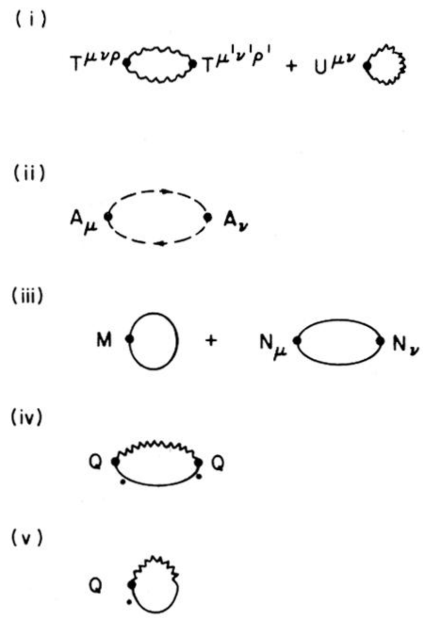

FIG. 2. Diagrams (i), (ii), and (iii) represent the propagation of B_μ, ψ, and ϕ, respectively. Diagram (v) represents a mixed B_μ-ϕ loop.