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Particles of half-integral or integral helicity by quantization of a nonrelativistic free particle, and related topics

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It is known that the equations of motion of a classical system do not in general determine its canonical description uniquely. Thus the corresponding quantum system obtained by canonical quantization is also ambiguous. We use this freedom to quantize a free nonrelativistic particle so that the corresponding quantum particle has half-integral or integral helicity. Methods are developed for finding inequivalent canonical descriptions of a given classical system. It is emphasized that classical symmetries can be broken at the quantum level by a suitable choice of the canonical formalism prior to quantization. It is suggested that this may provide a new mechanism for breaking internal symmetries at the quantum level.

I. INTRODUCTION

In this paper, we adopt the view that a system in classical mechanics is completely specified by (a) the equations of motion and (b) the precise empirical meaning (during a measurement) of the variables in the equations of motion. The state of such a system is thus determined by a point $\xi = (\xi_1, \xi_2, \ldots, \xi_{2n})$ on a $2n$-dimensional manifold M (the phase space). The physical meaning of the variables ξ_i are *a priori* given. The time evolution of ξ_i is fixed by the first-order equations

$$\frac{d\xi_i(t)}{dt} \equiv \dot{\xi}_i(t) = \alpha_i(\xi(t)), \tag{1.1}$$

where $\alpha(\xi)$ is a known vector field.[1]

In this point of view, the canonical formalism is a superstructure which is not necessary for a complete description of the *classical* system [as defined by (a) and (b)]. Further, as is known,[2,3] we can in general find several (generalized) Poisson brackets (PB's)

$$\{\xi_i, \xi_j\}^{(\nu)} = \omega_{ij}^{(\nu)}(\xi) \tag{1.2}$$

and associated Hamiltonians $H^{(\nu)}(\xi)$, all of which yield the same equations of motion (1.1)[4]:

$$\{\xi_i, H^{(\nu)}(\xi)\}^{(\nu)} = \alpha_i(\xi). \tag{1.3}$$

The superscripts ν on the (generalized) PB symbols are to emphasize that the latter are canonically inequivalent for different ν. The classical system, by definition, is the same regardless of which Hamiltonian and PB's we use in the canonical formalism.

However, when we canonically quantize the classical system, the resultant quantum system depends critically on the particular canonical description we use for the classical system.[2,3] Thus, there is a deep ambiguity in the passage from classical to quantum theory. This ambiguity is distinct from the ambiguities due to factor-ordering problems.

In Sec. II, we show how this ambiguity can be used to quantize a free nonrelativistic (NR) particle with no internal degrees of freedom, and obtain a quantum-mechanical particle with an intrinsic helicity. This helicity can be half-integral or integral. Thus the quantum particle can have half-integral or integral angular momenta. It can be a fermion or a boson. Such a possibility arises because in quantum mechanics helicity is a property characteristic of the generators of geometric rotations on the states of the system. The form of these generators depends on the commutation relations between coordinates and momenta. Note however that for a *classical* system as defined by (a) and (b), the "generator of rotations" has no intrinsic meaning and depends on the choice of the canonical description.

In the next three sections, we discuss three methods for finding inequivalent canonical descriptions of classical systems. They are illustrated by the example in Sec. II. The basic ideas behind the first method are contained in Ref. 2. We further develop these ideas due to Ref. 2 in Sec. III.

The equations of motion of a NR free particle are invariant under the Galilei group $\mathcal{G}$.[5] They are also invariant under the larger group wherein the spatial rotations in $\mathcal{G}$ are replaced by $GL(3,R)$. In the conventional canonical formalism given in textbooks,[5] only $\mathcal{G}$ is canonically implementable. When we pass to quantum mechanics, there are no unitary operators to implement those symmetries which are not in $\mathcal{G}$.[6] Such symmetries are thus broken in the conventional quantum mechanics of a free particle. This situation in the theory of a free NR particle is generic. For a classical system, the full symmetry group of the equations of motion is, as a rule, larger than that which is canonically implementable. The noncanonical symmetries are then broken at the quantum level. Further, the notion of canonicity depends on our choice of the PB's for the classical system. It follows that different subgroups of the full classical symmetry group may become canonical when the PB's are changed.[2] Thus, it becomes possible to break appropriate classical symmetries at the quantum level by a suitable choice of the canonical description for the classical system. This is a novel possibility for breaking symmetries at the quantum level. In the final section, we indicate methods to find canonical descriptions such that the corresponding quantum theories will break appropriate symmetries of the classical equations. These methods may be useful in breaking internal symmetries like SU(3). (See in this connection Santilli.[3])

The existence of ambiguities in the Lagrangian and Hamiltonian descriptions of classical systems has long been known.[7] At the quantum level, it has also been known for some time that the Heisenberg equations of motion do not fully determine the commutation relations (CR's) of ξ_i. Wigner and others investigated such quantum-mechanical ambiguities in the early 50's.[8] Further, the realization that the equations of motion permit more general algebraic structures than CR's led to the development of parastatistics.[9] However, a systematic study of the ambiguities in the Lagrangian and Hamiltonian formalisms of classical systems has been initiated only recently.[2,3]

II. HELICITY FOR A CLASSICAL NONRELATIVISTIC FREE PARTICLE

The conventional canonical description of a NR free particle assumes the PB's

$$\{q_i, q_j\}^{(0)} = \{p_i, p_j\}^{(0)} = 0\,, \qquad \{q_i, p_j\}^{(0)} = \delta_{ij}\,, \tag{2.1}$$

and defines the Hamiltonian to be[10]

$$H^{(0)} = \frac{p^2}{2m}\,. \tag{2.2}$$

Here, m is the mass of the particle, q_i are its Cartesian coordinates, and p_i/m are the corresponding velocities. Thus, $\xi_i = q_i$ and $\xi_{i+3} = p_i$ $(i \leq 3)$. The equations of motion which follow from (2.1) and (2.2) are

$$\dot{q}_i(t) = \frac{p_i(t)}{m}\,, \qquad \dot{p}_i(t) = 0\,. \tag{2.3}$$

An alternative canonical description which yields the equations of motion (2.3) is obtained by replacing (2.1) by

$$\{q_i, q_j\}^{(1)} = -\lambda \epsilon_{ijk} p_k / p^3\,, \qquad \{p_i, p_j\}^{(1)} = 0\,, \qquad \{q_i, p_j\}^{(1)} = \delta_{ij}\,, \tag{2.4}$$

where λ is a constant. The new Hamiltonian $H^{(1)}$ is the same as (2.2):

$$H^{(1)} = \frac{p^2}{2m}\,. \tag{2.5}$$

It is trivial to verify the validity of (2.3). Note that we get the same classical system for any value of λ. Further, this value is unrestricted at the classical level. However, the situation is different for the quantum system as we shall see below. (See also the remarks in Sec. I.)

The spatial rotation group SO(3) acts on q and p as follows:

$$q_i \to R_{ij} q_j\,, \qquad p_i \to R_{ij} p_j\,, \quad R \in \mathrm{SO}(3)\,. \tag{2.6}$$

An inspection of (2.4) shows that this group is canonically implementable. However, its generators are no longer $q \times p$. They are, instead,[10,11]

$$J = q \times p + \lambda \hat{p} \tag{2.7}$$

as a straightforward calculation shows. Thus, the particle has a helicity

$$\hat{p} \cdot J = \lambda \tag{2.8}$$

in the direction of the momentum. It follows that (a) quantization is possible only if λ is half-integral or integral, (b) the quantum system is fermionic or bosonic accordingly as λ is half-integral or integral.[11]

In the new canonical description, for the classical system as well, the generators of rotations are given by (2.7). Thus, one may define the helicity of the classical NR particle in the new canonical description to be λ. However, since the classical NR particle is the same regardless of its canonical description, such a definition is of doubtful utility.

In the following sections, we will discuss the close resemblance between this formalism and Dirac's theory of magnetic monopoles.

For the PB's (2.4), the following symmetries of the classical equations of motion are not canonical transformations; (i) spatial inversion, (ii) the

Galilei transformations[12,5]

$$q_i \to q_i + w_i t\ ,$$
$$p_i \to p_i + m w_i\ . \tag{2.9}$$

The situation with regard to spatial inversions is similar to that in Dirac's theory. As regards (ii), we note that Galilei invariance can be restored if q_i and p_i are interpreted as relative coordinates and momenta of two noninteracting particles, since these coordinates are invariant under such Galilei transformations.

We show in Sec. V that such a method of quantization is possible in the presence of a certain class of interactions as well. The Galilei transformations (2.9) are not symmetries of the equations of motion when the particle moves, for instance, in an external central potential. (Here, as usual, q_i and p_i/m are interpreted as Cartesian coordinates and velocities.) Their failure to be canonical is thus less striking for these systems.

It is of course not evident that there is any physical system in nature for which the method of quantization of this section is the correct one. However, such ambiguities in canonical description and quantization seem to merit further investigation, particularly since they offer the possibility of discovering novel and physically relevant quantization methods. In the sections which follow, we study some schemes for producing inequivalent canonical descriptions, and also touch upon their physical implications.

III. METHOD 1 FOR FINDING INEQUIVALENT CANONICAL DESCRIPTIONS

We assume throughout the paper that we are initially given a set of PB's

$$\{\xi_i, \xi_j\}^{(0)} = \omega_{ij}^{(0)}(\xi) \tag{3.1}$$

and a Hamiltonian $H^{(0)}(\xi)$ which yield the equations of motion (1.1).[1] It is of course also assumed that (3.1) fulfills the Jacobi identity.

Let $\{s\}$ denote the set of symmetry transformations of the given classical system. Thus, $\xi \to s(\xi)$ is an invertible transformation of M onto M such that if $\xi(t)$ fulfills (1.1), then

$$\frac{d}{dt} s_i(\xi(t)) = \alpha_i(s(\xi(t)))\,. \tag{3.2}$$

Equations (3.2) are entirely equivalent to (1.1).

If we now set[13]

$$\{s_i(\xi), s_j(\xi)\}^{(1)} = \omega_{ij}^{(0)}(s(\xi))\,, \tag{3.3}$$

and define the new Hamiltonian

$$H^{(1)}(\xi) = H^{(0)}(s(\xi))\,, \tag{3.4}$$

the equations of motion (3.2), and hence (1.1), are evidently reproduced. [Note that $\alpha_i(\xi) = \{\xi_i, H^{(0)}(\xi)\}^{(0)} = \omega_{ij}^{(0)}(\xi)\partial H^{(0)}(\xi)/\partial \xi_j$.] The PB's (3.3) can be rewritten as

$$\{\xi_i, \xi_j\}^{(1)} = \omega_{ij}^{(1)}(\xi) \tag{3.5}$$

for a suitable $\omega^{(1)}$.

If s is a canonical transformation relative to the PB's (3.1), $\omega^{(1)}$ and $\omega^{(0)}$ are equal, while $H^{(1)}$ and $H^{(0)}$ differ at most by a constant by a well-known theorem. Hence, (3.3), (3.4), and (3.5) lead to nothing new. Thus we are led to the first method for finding new canonical descriptions: *Let s be a symmetry transformation which is not canonical relative to the PB's (3.1). Then, the constructions (3.3), (3.4), and (3.5) give a new canonical description of the same classical system.*

The authors of Ref. 2 were well aware of this method. (See also Ref. 3.) The new contribution in this paper will be to develop ways to find such transformations s. However, we will first illustrate the idea by a simple example due to these authors. For a free particle, let s be any constant nonsingular real 3×3 matrix which is not orthogonal. If q_i and p_i/m are Cartesian coordinates and velocities, the transformation

$$q_i \to s_{ij} q_j\ ,$$
$$p_i \to s_{ij} p_j \tag{3.6}$$

is a symmetry. It is also not canonical relative to the PB's (2.1). Let

$$\{s_{ik}q_k, s_{jl}q_l\}^{(1)} = \{s_{ik}p_k, s_{jl}p_l\}^{(1)} = 0\ ,$$
$$\{s_{ik}q_k, s_{jl}p_l\}^{(1)} = \delta_{ij}\ , \tag{3.3$'$}$$

$$H^{(1)}(\xi) = \frac{1}{2m}(s_{ik}p_k)(s_{il}p_l)\,. \tag{3.4$'$}$$

We can rewrite (3.3′) as

$$\begin{aligned}&\{q_i, q_j\}^{(1)} = \{p_i, p_j\}^{(1)} = 0\ ,\\ &\{q_i, p_j\}^{(1)} = s_{ik}^{-1}s_{jk}^{-1}\,.\end{aligned} \tag{3.5$'$}$$

Then, (3.4′) and (3.5′) give the new canonical description.

We will now describe a method for finding noncanonical symmetries. We will then apply it to derive (2.4) and (2.5).

Let $\tau^{(A)}$ be a *known* symmetry of the given classical system which depends on a set of continuous parameters $A = (A_1, A_2, \ldots, A_k)$. It is immaterial whether $\tau^{(A)}$ is canonical or not. For a free particle, $\tau^{(A)}$ can be a spatial translation:

$$\begin{aligned}&\tau^{(A)}(q,p)_i = q_i + A_i\,, \quad \tau^{(A)}(q,p)_{i+3} = p_i\,,\\ &A = (A_1, A_2, A_3),\ i = 1, 2, 3\,.\end{aligned} \tag{3.7}$$

For a central potential problem, $\tau^{(A)}$ can be a spatial rotation:

$$\tau^{(A)}(q,p)_i = R_{ij}^{(A)} q_j\,,$$
$$\tau^{(A)}(q,p)_{i+3} = R_{ij}^{(A)} p_j\,, \tag{3.8}$$
$$R^{(A)} \in \mathrm{SO}(3),\quad i=1,2,3\,.$$

The empirical meaning of q_i and p_i in (3.7) and (3.8) is as in Sec. II. In (3.8), $A=(A_1,A_2,A_3)$ where A_i are the Euler angles.

Let $c=(c_1,c_2,\ldots,c_m)$ be a set of *known* constants motion. They are functions of ξ. For a central potential problem, we can for instance take c_i $(i=1,2,3)$ to be the components of angular momenta and c_4 to be energy.

Now let A be a function of $c(\xi)$. Then the claim is that the transformation $\xi \to s(\xi)$, where

$$s(\xi) = \tau^{(A(c(\xi)))}(\xi)\,, \tag{3.9}$$

is a symmetry transformation which is in general not canonical.[14,15]

The example below will show that (3.9) in general is not canonical. We will now prove that it is a symmetry of the equations of motion. When A is a constant, (3.2) (with $s=\tau^{(A)}$) and (1.1) are equivalent. For constant A, (3.2) can be rewritten as

$$J_{ij}^{(A)}(\xi(t))\frac{d\xi_j(t)}{dt} = \alpha_i(\tau^{(A)}(\xi(t)))\,, \tag{3.10}$$

where

$$J_{ij}^{(A)}(\xi) = \frac{\partial \tau_i^{(A)}(\xi)}{\partial \xi_j}\,. \tag{3.11}$$

Comparison of (3.10) and (1.1) shows that

$$J_{ij}^{(A)}(\xi)\alpha_j(\xi) = \alpha_i(\tau^{(A)}(\xi))\,. \tag{3.12}$$

Now if A is made a function of $c(\xi)$, since $c(\xi)$ is constant on classical trajectories, we find

$$\frac{d\tau_i^{(A)}(\xi)}{dt} = J_{ij}^{(A)}(\xi(t))\alpha_j(\xi(t))$$
$$= \alpha_i(\tau^{(A)}(\xi(t))) \tag{3.13}$$

by (3.12), which is the required result. In (3.13), it is understood that $A=A(c(\xi(t)))$.

We now apply this method to the example of Sec. II. For a free particle, spatial translations are symmetries, and the Cartesian velocities p_i/m are constants of motion. So choose $\tau^{(A(p))}$ to be a velocity-dependent translation:

$$\tau_i^{(A(p))}(q,p) = q_i + A_i(p)\,,$$
$$\tau_{i+3}^{(A(p))}(q,p) = p_i\,,\quad i=1,2,3\,. \tag{3.14}$$

Then the new PB's [derived from (2.1) via (3.3)] are

$$\{q_i,q_j\}^{(1)} = -[\partial_i A_j(p) - \partial_j A_i(p)]\,,$$
$$\{p_i,p_j\}^{(1)} = 0\,, \tag{3.15}$$
$$\{q_i,p_j\}^{(1)} = \delta_{ij}\,,$$

while the new Hamiltonian is

$$H^{(1)}(q,p) = p^2/2m\,. \tag{3.16}$$

When A is taken to be the solution of the equation[16]

$$\partial_i A_j(p) - \partial_j A_i(p) = \lambda \epsilon_{ijk}\frac{p_k}{p^3}\,, \tag{3.17}$$

we recover (2.4) and (2.5).

If A is so chosen that its curl is not a second-rank antisymmetric tensor under spatial rotations, the latter will become noncanonical relative to the new PB's. This is a way to break rotational invariance at the quantum level.

IV. METHOD 2 FOR FINDING INEQUIVALENT CANONICAL DESCRIPTIONS

Initially, the classical system is described by the dynamical variables ξ_i, the PB's indexed by zero, and the Hamiltonian $H^{(0)}(\xi)$. [Recall that $H^{(0)}(\xi)$ is not the free Hamiltonian. It is the Hamiltonian which generates the equations of motion for the PB's (3.1).]

Our second method consists in first enlarging this system by introducing suitable additional variables η_α. The Hamiltonian for this enlarged system is taken to be $H^{(0)}(\xi)$ while the PB's between q's are also unchanged. So, the ξ's continue to fulfill the original equations of motion (1.1). The PB's involving η's are taken to be

$$\{\eta_\alpha,\eta_\beta\}^{(0)} = \Omega_{\alpha\beta}(\eta)\,, \tag{4.1}$$
$$\{\xi_i,\eta_\alpha\}^{(0)} = 0 \tag{4.2}$$

for a suitable choice of $\Omega_{\alpha\beta}$ (consistent with Jacobi identities). Thus,

$$\dot\eta_\alpha = 0\,. \tag{4.3}$$

Next, we impose constraints on the system:

$$\phi_\sigma(\xi,\eta)=0,\quad \sigma=1,2,\ldots,\nu\,. \tag{4.4}$$

Here, we require the following: (i) The constrained hypersurface must be invariant under time evolution:

$$\{\phi_\sigma(\xi,\eta),H^{(0)}(\xi)\}^{(0)}=0 \text{ modulo constraints;} \tag{4.5}$$

(ii) Using the constraints, we should be able to eliminate all the η's (and no more degrees of freedom) by applying Dirac's method for constrained Hamiltonian systems.[17]

When the η's are so eliminated, the resultant PB's [or rather Dirac brackets (DB's)[17]] $\omega^{(1)}$ between ξ's will in general differ from $\omega^{(0)}$. Also, due to (4.5), $H^{(0)}$ is a first-class variable.[17] Hence, on the constrained hypersurface, its DB with any variable is equal to its PB with superscript zero:

$$\{\xi_i, H^{(0)}(\xi)\}^{(1)} \equiv \omega_{ij}^{(1)}(\xi) \frac{\partial H^{(0)}(\xi)}{\partial \xi_j}$$

$$= \{\xi_i, H^{(0)}(\xi)\}^{(0)} . \tag{4.6}$$

The equations of motion of ξ's are thus unchanged by the process of elimination of η's.

The result of these manipulations is that (i) we have eliminated the extra variables η_α and are left with the original variables ξ_i, (ii) the PB $\omega_{ij}^{(1)}(\xi)$ between ξ_i and ξ_j is in general different from the PB $\omega_{ij}^{(0)}(\xi)$ we started out with, and (iii) the original equations of motion (1.1) are still generated by $H^{(0)}(\xi)$ and the new PB's.

We shall now apply this method to derive (2.4) and (2.5).[18] Enlarge the free particle system [given by (2.1) and (2.2)] by introducing the additional "isospin" vector $I=(I_1, I_2, I_3)$. The PB's involving I_α's are assumed to be

$$\{I_\alpha, I_\beta\}^{(0)} = \epsilon_{\alpha\beta\gamma} I_\gamma , \tag{4.7}$$

$$\{\xi_i, I_\alpha\}^{(0)} = 0 . \tag{4.8}$$

Next we impose the constraints

$$\phi_1 \equiv I_\alpha I_\alpha - \mu = 0 , \tag{4.9}$$

$$\phi_2 \equiv \hat{p}_\alpha I_\alpha - \lambda = 0 , \tag{4.10}$$

where μ and λ constants.

Both these constraints are preserved by time evolution. Further, they can be used to eliminate all the I_α's (and no more extra variables). For: (a) ϕ_1 has zero PB's with all the variables ξ_i, I_α. It is in the center of the PB algebra. Thus we can eliminate one degree of freedom by setting $\phi_1 = 0$. (b) ϕ_2 is a first-class constraint.[17] Further, it generates nontrivial canonical transformations on the enlarged phase space. So (4.10) eliminates two degrees of freedom. Combining (4.9) and (4.10), we thus eliminate three degrees of freedom which can be taken to be the I_α's.

It is easily verified that p_i and[19]

$$q_i^* = q_i + \frac{1}{p^2} \epsilon_{ij\alpha} p_j I_\alpha \tag{4.11}$$

have vanishing PB's (with superscript zero) with ϕ_2 (and, of course, with ϕ_1). Thus they are first-class variables[17] and describe the reduced phase space, which fulfills (4.9) and (4.10). Their PB's on the reduced phase space are

$$\{q_i^*, q_j^*\}^{(0)} = -\lambda \epsilon_{ijk} \frac{p_k}{p^3} , \tag{4.12}$$

$$\{p_i, p_j\}^{(0)} = 0 , \tag{4.13}$$

$$\{q_i^*, p_j\}^{(0)} = \delta_{ij} . \tag{4.14}$$

Now instead of working with q^* and p and their PB's indexed by zero, we can equally well work with q and p and their Dirac brackets indexed by one:

$$\{q_i, q_j\}^{(1)} = -\lambda \epsilon_{ijk} \frac{p_k}{p^3} , \tag{4.15}$$

$$\{p_i, p_j\}^{(1)} = 0 , \tag{4.16}$$

$$\{q_i, p_j\}^{(1)} = \delta_{ij} . \tag{4.17}$$

These are the same as (2.4). Further, for the Hamiltonian $p^2/2m$, the equations of motion (2.3) are recovered.

V. METHOD 3 FOR FINDING INEQUIVALENT CANONICAL DESCRIPTIONS

In the last three sections, we have described a few methods for constructing inequivalent canonical descriptions. As we have described them, however, some of these methods [for example that of Sec. II or Eqs. (3.3′)–(3.5′)] seem applicable only to free particles. In this section, we will show how all the methods available for the free system can be applied to many interacting systems as well.

It is well known that for a large class of interacting systems, there exist coordinates

$$\begin{aligned} &\xi_i^{\mathrm{IN}} \equiv q_i^{\mathrm{IN}} = q_i^{\mathrm{IN}}(\xi) , \\ &\xi_{i+n}^{\mathrm{IN}} \equiv p_i^{\mathrm{IN}} = p_i^{\mathrm{IN}}(\xi) , \quad i = 1, 2, \ldots, n \end{aligned} \tag{5.1}$$

which obey the free equations

$$\begin{aligned} &\frac{dq_i^{\mathrm{IN}}(t)}{dt} = \frac{p_i^{\mathrm{IN}}(t)}{m} , \\ &\frac{dp_i^{\mathrm{IN}}(t)}{dt} = 0 . \end{aligned} \tag{5.2}$$

Here, n denotes the number of degrees of freedom. Thus we can introduce new canonical structures for such systems by first introducing the coordinates ξ_i^{IN} and then applying the methods available for free systems. The new descriptions can of course be finally rewritten in the variables ξ_i if desired. Note also that for every system which admits such coordinates ξ^{IN}, the symmetry group of the equations of motion is isomorphic to the corresponding symmetry group of the free equations. In the usual formalism, only a subgroup thereof is canonically realizable.

The construction of ξ^{IN} is analogous to the construction of the "in" variables in the quantum theory of scattering.[20] We assume that the classical system admits no closed orbits and that the interaction is sufficiently short range.[21] Then the classical Møller operator

$$\Omega^+ = \lim_{t \to -\infty} U_{H^{(0)}}(-t) U_{H_F^{(0)}}(t) \tag{5.3}$$

exists and is invertible. (This definition of Ω^+ is similar to that in quantum scattering theory.[20]) We have used the notation

$$H^{(0)}(\xi)=\frac{p^2}{2m}+H_I^{(0)}(q,p)$$
$$\equiv H_F^{(0)}(p)+H_I^{(0)}(q,p), \quad \xi=(q,p), \tag{5.4}$$

where $H_F^{(0)}$ and $H_I^{(0)}$ are the "free" and "interaction" Hamiltonians, respectively. (Note that $H^{(0)}$ denotes the full and not the free Hamiltonian.) The variables q and p obey the usual PB's [cf. (2.1)]. The classical time evolution operators associated with $H^{(0)}$ and $H_F^{(0)}$ have been denoted by $U_{H^{(0)}}(t)$ and $U_{H_F^{(0)}}(t)$. Their meaning is illustrated by the action of the operator $U_{H^{(0)}}(t)$ on a point $\bar{\xi}$ in the phase space M:

$$[U_{H^{(0)}}(t)\bar{\xi}]_i=\sum_{n=0}^{\infty}\frac{t^n}{n!}\{\{\ldots\{\{\xi_i,H^{(0)}(\xi)\}^{(0)},H^{(0)}(\xi)\}^{(0)}, \ldots\}^{(0)},H^{(0)}(\xi)\}^{(0)}\Big|_{\xi=\bar{\xi}}. \tag{5.5}$$

As in the quantum-mechanical scattering theory, we have the properties

$$U_{H^{(0)}}(t_1+t_2)=U_{H^{(0)}}(t_1)U_{H^{(0)}}(t_2), \tag{5.6}$$
$$U_{H_F^{(0)}}(t_1+t_2)=U_{H_F^{(0)}}(t_1)U_{H_F^{(0)}}(t_2), \tag{5.7}$$
$$U_{H^{(0)}}(0)=U_{H_F^{(0)}}(0)=1. \tag{5.8}$$

Thus,

$$\Omega^+=\lim_{t'\to-\infty}U_{H^{(0)}}(-(t'-t))U_{H_F^{(0)}}((t'-t))$$
$$=U_{H^{(0)}}(t)\Omega^+U_{H_F^{(0)}}(-t), \tag{5.9}$$

or

$$[\Omega^+]^{-1}U_{H^{(0)}}(t)=U_{H_F^{(0)}}(t)[\Omega^+]^{-1}. \tag{5.10}$$

The new coordinates $\xi^{\rm IN}$ are defined by

$$\xi^{\rm IN}=[\Omega^+]^{-1}\xi$$
$$\equiv\lim_{t\to-\infty}U_{H_F^{(0)}}(-t)U_{H^{(0)}}(t)\xi. \tag{5.11}$$

Under time evolution, ξ evolves according to

$$\xi\xrightarrow{U_{H^{(0)}}(t)}U_{H^{(0)}}(t)\xi. \tag{5.12}$$

Thus, by (5.10), the time evolution of $\xi^{\rm IN}$ is given by

$$\xi^{\rm IN}\to U_{H_F^{(0)}}(t)\xi^{\rm IN}. \tag{5.13}$$

The interpretation of the right-hand side is similar to (5.5). It follows that $\xi_i^{\rm IN}$ obey the free equations (5.2). The equations of motion of ξ_i can be recovered by expressing (5.2) in terms of ξ_i using (5.11). The two sets of equations are entirely equivalent.

Note also the following facts: (i) Since $U_{H^{(0)}}(t)$ and $U_{H_F^{(0)}}(-t)$ are canonical transformations relative to the PB's (2.1), $[\Omega^+]^{-1}$ generates a canonical transformation relative to these PB's. Therefore,

$$\{q_i^{\rm IN},q_j^{\rm IN}\}^{(0)}=\{p_i^{\rm IN},p_j^{\rm IN}\}^{(0)}=0,$$
$$\{q_i^{\rm IN},p_j^{\rm IN}\}^{(0)}=\delta_{ij}. \tag{5.14}$$

(ii) In view of (5.14), the Hamiltonian $H_F^{(0)}(p^{\rm IN})=(p^{\rm IN})^2/2m$ generates the equations of motion (5.2) for the PB's (2.1). But so does $H^{(0)}(\xi)$. Thus, $H^{(0)}(\xi)$ and $H_F^{(0)}(p^{\rm IN})$ can differ at most by a constant[20]:

$$H^{(0)}(\xi)=H_F^{(0)}(p^{\rm IN})+\text{constant}. \tag{5.15}$$

New PB's and Hamiltonians can be introduced to describe (5.2). They can then be reexpressed in terms of ξ.[22] Of course, relative to these new PB's, the change of variable (5.11) will not in general be canonical.

We now discuss how the variables $\tau_i^{A(p)}(q,p)$ introduced in (3.14) are intimately related to the "in" variables for the magnetic monopole system. The discussion should further clarify the origin of the PB's $\{\cdot,\cdot\}^{(1)}$ and the Hamiltonian $H^{(1)}$ of Sec. II.

A NR charged particle in the field of a magnetic monopole is described by the Hamiltonian

$$H^{(0)}=[p-B(q)]^2/2m, \tag{5.16}$$

where B is a solution of the equation

$$\partial_iB_j(q)-\partial_jB_i(q)=\lambda\epsilon_{ijk}\frac{q_k}{q^3}. \tag{5.17}$$

Here λ is a suitable constant. A solution for B is[10]

$$B(q)=-\lambda\frac{(\hat{n}\cdot q)(\hat{n}\times q)}{q[q^2-(\hat{n}\cdot q)^2]}, \tag{5.18}$$

where the constant unit vector $\hat{n}$ gives the direction of the Dirac string (cf. Ref. 11). The PB's of q and p are given by (2.1). The relation of the Cartesian velocity $\dot{q}$ to p is

$$\dot{q}=[p-B(q)]/m. \tag{5.19}$$

Therefore,

$$\{q_i,\dot{q}_j\}^{(0)}=\frac{1}{m}\delta_{ij},$$
$$\{\dot{q}_i,\dot{q}_j\}^{(0)}=\frac{\lambda}{m^2}\epsilon_{ijk}\frac{q_k}{q^3}. \tag{5.20}$$

Suppose that the position and velocity of the particle at time zero are $q(0)$ and $\dot{q}(0)$. We now study the action of $[\Omega^+]^{-1}$ on $(q(0),\dot{q}(0))$.[23]

Let $(q(0),\dot{q}(0))$ become $(q(t),\dot{q}(t))$ after time t when evolved by the full Hamiltonian $H^{(0)}$:

$$U_{H^{(0)}}(t)(q(0),\dot{q}(0))=(q(t),\dot{q}(t)). \tag{5.21}$$

We are interested in this expression as $t \to -\infty$. Then the particle moves almost in a straight line[24]:

$$q(t) = t\dot{q}(\infty) + s(\infty) + O(t^{-1}) \text{ as } t \to -\infty . \tag{5.22}$$

Thus,

$$\dot{q}(t) = \dot{q}(\infty) + O(t^{-2}) \text{ as } t \to -\infty . \tag{5.23}$$

The momentum $p(t)$ conjugate to $q(t)$ is given by (5.19):

$$p(t) = m\dot{q}(t) + B[q(t)] . \tag{5.24}$$

Therefore, by (5.22), (5.23), and (5.18),

$$p(t) = m\dot{q}(\infty) + \frac{1}{|t|} B[\dot{q}(\infty)] + O(t^{-2}) \text{ as } t \to -\infty . \tag{5.25}$$

Next we evolve $q(t)$ and $p(t)$ with the free Hamiltonian $H_F^{(0)}$ for a time $-t$. Then, we let $t \to -\infty$ to find q^{IN}, p^{IN}. Thus,

$$\begin{aligned} q^{\mathrm{IN}} &= \lim_{t \to -\infty} \left[q(t) - \frac{t}{m} p(t)\right] \\ &= s(\infty) + \frac{1}{m} B[\dot{q}(\infty)] , \end{aligned} \tag{5.26}$$

$$\begin{aligned} p^{\mathrm{IN}} &= p(\infty) \\ &= m\dot{q}(\infty) . \end{aligned} \tag{5.27}$$

The full Hamiltonian $H^{(0)}$ can be written as $\frac{1}{2} m\dot{q}(0)^2$. Since the magnitude of velocity is conserved for this system,[24] it follows that

$$\begin{aligned} H^{(0)} &= \tfrac{1}{2} m\dot{q}(\infty)^2 \\ &= (p^{\mathrm{IN}})^2/2m . \end{aligned} \tag{5.28}$$

This is consistent with the general result (5.15).

The "in" variables obey (5.14). Therefore,

$$\begin{aligned} \{s_i(\infty), s_j(\infty)\}^{(0)} &= -\lambda \epsilon_{ijk} \frac{p_k(\infty)}{p(\infty)^3} , \\ \{p_i(\infty), p_j(\infty)\}^{(0)} &= 0 , \\ \{s_i(\infty), p_j(\infty)\}^{(0)} &= \delta_{ij} . \end{aligned} \tag{5.29}$$

Comparison of Eqs. (3.14)–(3.17) with (5.26)–(5.29) shows that (i) the "impact parameter" $s(\infty)$ and "asymptotic momentum" $p(\infty)$ correspond to q and p of (3.14), (ii) q^{IN} and p^{IN} correspond to $\tau^{[A(p)]}(q,p)$ of (3.14). Thus the impact parameter $s(\infty)$ and asymptotic momentum $p(\infty)$ of the magnetic monopole system obey the free equations of motion, but their canonical description is not the usual canonical description. It is the description with the superscript 1.

It is possible to introduce new canonical descriptions for the monopole system as described earlier in this section. We will not pursue this point here.

VI. METHOD FOR BREAKING CLASSICAL SYMMETRIES AT THE QUANTUM LEVEL

We remarked in the introduction on the following points: (i) The full symmetry group of the classical equations of motion is as a rule larger than that which is canonically implementable. (ii) The set of canonically implementable symmetries depends on our choice of the canonical description. (iii) The noncanonical symmetries can not be unitarily implemented at the quantum level. They are thus expected to be broken at the quantum level.

Thus, it becomes possible to break appropriate classical symmetries at the quantum level. Such a method may be relevant for breaking internal symmetries.

In this section, we will illustrate the preceding points in terms of the free particle system and the canonical description (3.4′)–(3.5′). The papers cited in Ref. 2 should be consulted for more thorough discussions of points (i) and (ii). We will also discuss which symmetries are expected to survive as canonical symmetries when the methods of Secs. III and IV are used to introduce new canonical descriptions. This will also show how to control symmetry breaking in these methods.

The NR free particle equations admit GL$(3,R)$ as a symmetry group. An element $g \in \mathrm{GL}(3,R)$ acts on Cartesian coordinates and momenta q and p as follows:

$$\begin{aligned} q_i &\xrightarrow{g} g_{ij} q_j , \\ p_i &\xrightarrow{g} g_{ij} p_j . \end{aligned} \tag{6.1}$$

For the PB's (3.5′), g is canonical only if

$$g[s^T s]^{-1} g^T = [s^T s]^{-1} . \tag{6.2}$$

We have used a matrix notation and M^T denotes the transpose of the matrix M. Thus the canonical subgroup of GL$(3,R)$ depends on the choice of s.

Now suppose that $[s^T s]^{-1}$ is not a multiple of the identity. Then spatial rotations are not canonical. At the quantum level, let $|q'\rangle$ and $|q''\rangle$ denote states with eigenvalues q_i' and q_i'' for q_i. Then we find[25]

$$\langle q''| \exp\left[-\frac{i}{\hbar} H^{(1)} t\right] |q'\rangle \neq \langle Rq''| \exp\left[-\frac{i}{\hbar} H^{(1)} t\right] |Rq'\rangle . \tag{6.3}$$

Here, $t \neq 0$ and R is a spatial rotation matrix $\neq 1$. In this particular example, the two sides differ by a phase which depends on q', q'', and R.[25,26] Thus if we replace the initial and final states on the left-hand side by general wave packets

$$|\phi\rangle = \int |q'\rangle \rho(q') d^3q' ,$$
$$|\psi\rangle = \int |q''\rangle \tilde{\rho}(q'') d^3q'' , \tag{6.4}$$

and denote the rotated wave packets by $|\phi^R\rangle$ and $|\psi^R\rangle$,

$$|\phi^R\rangle = \int |Rq'\rangle \rho(q') d^3q' ,$$
$$|\psi^R\rangle = \int |Rq''\rangle \tilde{\rho}(q'') d^3q'' , \tag{6.5}$$

we find that the transition probabilities are not rotationally invariant:

$$\left|\left\langle \psi \left| \exp\left[-\frac{i}{\hbar} H^{(1)} t\right] \right| \phi \right\rangle\right|^2 \neq \left|\left\langle \psi^R \left| \exp\left[-\frac{i}{\hbar} H^{(1)} t\right] \right| \phi^R \right\rangle\right|^2 . \tag{6.6}$$

Therefore, this quantum mechanics of the free particle breaks spatial rotational invariance.

The action in the phase space which is appropriate for the canonical description (3.3′)–(3.5′) is

$$\int (P\dot{Q} - P^2/2m) dt$$

(cf. Refs. 4 and 25). As is well known, the phase of $\langle q'' | \exp\{-iH^{(1)}t/\hbar\} | q' \rangle$ [modulo the contribution from $i^{-3/2}$ (Ref. 25)] is given by

$$\int_0^t (P\dot{Q} - P^2/2m) dt$$

where the integral is along the actual classical trajectory with Cartesian coordinates q' and q'' at times zero and t respectively. This is just the appropriate Hamilton's principal function for this canonical description. The failure of rotational invariance in this example is due to the fact that this phase does not transform properly under rotations. These observations can be easily generalized. Thus, suppose that a symmetry transformation of any classical system (not necessarily the free particle) can not be canonically implemented for a particular choice of the canonical description. Then the corresponding phase-space action[4] fails to change only by a total time derivative of a function under this transformation. Now consider the quantum theory obtained by canonical quantization starting from this particular canonical description. In the semiclassical approximation, the phase of a quantum amplitude in this quantum theory is governed by the value of this action along an appropriate classical trajectory. Thus it fails to transform properly under this transformation. This shows (at the level of the semiclassical approximation) that this symmetry is broken in this quantum system.

We now examine Sec. III in the context of symmetry breaking. Suppose that $\xi \to f(\xi)$ is a canonical symmetry transformation relative to a canonical description with superscript zero. Then it is clear from (3.3) and (3.4) that if f commutes with s,

$$f[s(\xi)] = s[f(\xi)] , \tag{6.7}$$

then f will be a canonical symmetry for the new description with superscript 1. Otherwise, as a rule, it will fail to be canonical.[27] Thus for a particle in a central potential, if s is a spatial rotation around the third axis which depends on energy, rotations around the third axis will be canonical symmetries in the new description. Rotations around 1 and 2 axes will in general fail to be canonical.[28]

Next consider Sec. IV. Suppose that the original system described by the variables ξ has the canonical symmetry group G_ξ relative to the PB's with index zero. Let G_η be the canonical symmetry group which acts nontrivially only on η relative to the PB's (4.1) and (4.2). Then the enlarged system has the canonical symmetry group $G_\xi \times G_\eta$ relative to the description indexed by zero. Now suppose we impose the constraints (4.4) and eliminate η. Then only the subgroup of $G_\xi \times G_\eta$ which leaves the constrained surface invariant will survive as a canonical symmetry.[29]

Let us illustrate the above for the free particle system enlarged by isospin. For our purposes, the group G_ξ can be taken to be the Euclidean group in three dimensions.[30] It is generated by

$$L_\alpha = \epsilon_{\alpha\beta\gamma} q_\beta p_\gamma \quad \text{and } p_i , \tag{6.8}$$

under the PB's indexed by zero. The group G_η can be taken to be another Euclidean group with generators

$$I_\alpha \quad \text{and} \quad 0 . \tag{6.9}$$

The translations are trivially represented in the latter. The "little group" which leaves the constraints invariant is the diagonal subgroup G of $G_\xi \times G_\eta$ with generators

$$J_\alpha = L_\alpha + I_\alpha \quad \text{and } p_i . \tag{6.10}$$

It is also a three-dimensional Euclidean group. It is the only subgroup of $G_\xi \times G_\eta$ which survives as a canonical symmetry when the constraints are eliminated and the new PB's are introduced.

Consider the quantization of this system when only the constraint (4.9) is imposed. The system then describes a free particle with an internal isospin degree of freedom where the total isospin is fixed. The generators become Hermitian operators. Quantization is possible only if

$$\mu = I(I+1), \quad I = 0, \tfrac{1}{2}, 1, \ldots . \tag{6.11}$$

The Casimir invariants of G are[30]

$$\begin{aligned} M^2 &= p_i p_i , \\ X &= \hat{p}_\alpha J_\alpha \\ &= \hat{p}_\alpha I_\alpha . \end{aligned} \tag{6.12}$$

It is evident that the quantum-mechanical Hilbert space carries irreducible representations of G with the helicity X taking the values

$$-I, -I+1, \ldots, +I. \tag{6.13}$$

We can now impose (4.10) as a condition on the states.[18] Consistency requires that λ has one of the values allowed by (6.13):

$$\lambda = \text{ an eigenvalue of } X. \tag{6.14}$$

The condition (4.10) then picks out the representation of G with helicity λ. A simple calculation also shows that the form of J_α on the states picked out by (4.10) is

$$J_\alpha = \epsilon_{\alpha\beta\gamma}\, q^*_\beta \hat{p}_\gamma + \lambda \hat{p}_\alpha , \tag{6.15}$$

where q^* is defined by (4.11) and fulfills (4.12) and (4.14). These observations are consistent with the discussion in Sec. II.

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[1]We assume throughout the paper that the forces (and hence the vector field α and the Hamiltonians) have no explicit time dependence.

[2]D. G. Currie and E. J. Saletan, J. Math. Phys. 7, 967 (1966); E. J. Saletan and A. H. Cromer, *Theoretical Mechanics* (Wiley, New York, 1971), Y. Gelman and E. J. Saletan, Nuovo Cimento 18B, 53 (1973); G. Marmo and A. Simoni, Lett. Nuovo Cimento 15, 179 (1976); G. Marmo and E. J. Saletan, at the Proceedings of the V International Colloquium on Group Theoretical Methods in Physics, Montreal, 1976 (unpublished); G. Marmo and E. J. Saletan, Nuovo Cimento 40B, 67 (1977), and references cited in these papers.

[3]P. Havas, Nuovo Cimento 5, 363 (1957); J. F. Kennedy and E. H. Kerner, Am. J. Phys. 33, 463 (1965); G. Rosen, *Formulation of Classical and Quantum Dynamical Theory* (Academic, New York, 1969); A. M. Wolsky, Am. J. Phys. 39, 529 (1971); R. M. Santilli, Ann. Phys. (N.Y.) 103, 354 (1977); 103, 409 (1977); 105, 227 (1977); MIT-CTP Report No. 609 and 610, 1977 (unpublished); *The Inverse Problem in Newtonian Mechanics*, Vol. I and II (Springer, New York, to be published); Harvard Report No. HUTP-77/A066, 1977 (unpublished); Hadronic J. 1, 223 (1978).

[4]A variational principle and a Lagrangian also exist for these different canonical descriptions. The analog of the usual phase-space action $\int (pdq - Hdt)$ is $\int (\theta^{(\nu)} - H^{(\nu)}dt)$, where $\theta^{(\nu)}$ is the canonical one-form for the PB (1.2). Further, there always exist coordinates $Q^{(\nu)}, P^{(\nu)}$ (at least locally) such that

$$\{Q_i^{(\nu)}, Q_j^{(\nu)}\}^{(\nu)} = \{P_i^{(\nu)}, P_j^{(\nu)}\}^{(\nu)} = 0,$$
$$\{Q_i^{(\nu)}, P_j^{(\nu)}\} = \delta_{ij}.$$

(This is Darboux's theorem.) Thus, a Lagrangian can be found by writing $H^{(\nu)}$ in these coordinates and performing a Legendre transformation. For requisite information on symplectic geometry see, for example, Shlomo Sternberg, *Lectures on Differential Geometry* (Prentice-Hall, Englewood Cliffs, New Jersey, 1964).

[5]The Galilei group is fully discussed, for example, in E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (Wiley, New York, 1974).

[6]By a symmetry of the classical equations of motion, we mean as usual an invertible onto mapping $\xi \to s(\xi)$ such that, if $\xi(t)$ is a solution of these equations, then so is $s[\xi(t)]$ (cf. Sec. III).

[7]S. Lie, Arch. Math. Naturvidenskab, 129 (1877); E. T. Whittaker, *Analytical Dynamics* (Dover, New York, 1944), p. 305.

[8]E. P. Wigner, Phys. Rev. 77, 711 (1950); L. M. Yang, Phys. Rev. 84, 788 (1951). There are undoubtedly more papers on the subject that we are not aware of.

[9]See, for example, O. W. Greenberg and A. Messiah, Phys. Rev. 138, B1155 (1965); S. N. Biswas, *Statistical Physics, Lectures given at the Symposium Celebrating Fifty Years of Bose Statistics* (Indian Institute of Science, Bangalore, India, 1974), and references cited in these papers.

[10]In the text, we do not put arrows on vectors for simplicity of notation. Sometimes the magnitude of a vector is also denoted by the same symbol as the vector. This should cause no confusion. A unit vector is distinguished by a hat. Thus, $\hat{A} = \vec{A}/|\vec{A}|$.

[11]The PB's (2.4) and the rotation generator (2.7) are

similar to the expressions which occur in the theory of magnetic monopoles. See, for example, M. Fierz, Helv. Phys. Acta 17, 27 (1944); C. A. Hurst, Ann. Phys. (N.Y.) 50, 51 (1968); H. J. Lipkin, W. I. Weisberger, and M. Peshkin, *ibid.* 53, 203 (1969); M. Peshkin, *ibid.* 66, 542 (1971); V. I. Strazhev and L. M. Tomilichik, Fiz. Elem. Chastits At. Yad. 4, 187 (1973) [Sov. J. Part. Nucl. 4, 78 (1973)]; Richard A. Carrigan, Jr., Fermilab. Report No. 77/42 2000, 1977 (unpublished), and references therein. The results of these papers can be readily adapted to justify the remarks (a) and (b) which follow (2.8).

[12]This was pointed out by N. Mukunda.

[13]Note the following: (a) $\{s_i(\xi), s_j(\xi)\}^{(\nu)} = [\partial s_i(\xi)/\partial \xi_k] \times [\partial s_j(\xi)/\partial \xi_l]\omega_{kl}^{(\nu)}(\xi)$; (b) if s is a canonical transformation relative to the PB's with superscript ν, then by definition, $[\partial s_i(\xi)/\partial \xi_k]\, [\partial s_j(\xi)/\partial \xi_l]\, \omega_{kl}^{(\nu)}(\xi) = \omega_{ij}^{(\nu)}[s(\xi)]$; (c) the PB's (3.3) with superscript one fulfill Jacobi identities if the PB's with superscript zero do so, even if s is not a canonical transformation.

[14]Such transformations (3.9) bear a resemblance to gauge transformations. The symmetry transformations in (3.7) are made to depend on constants of motion and hence on ξ to get (3.9). The symmetry transformations are made to depend on space-time coordinates to get gauge transformations.

[15]We have no general and simple criteria as to when the transformation (3.9) is invertible. Invertibility can of course be checked when dealing with specific cases.

[16]If p is replaced by q, this is the equation which defines the vector potential in Dirac's theory of magnetic monopoles (cf. the references cited in Ref. 11).

[17]For excellent discussions of constrained Hamiltonian systems, see for example, P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monographs Series No. 2 (Yeshiva University, New York, 1964); E. C. G. Sudarshan and N. Mukunda, Ref. 5; A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Rome, 1976).

[18]This discussion resembles the work of A. S. Goldhaber, Phys. Rev. 140, B1407 (1965) on magnetic monopoles. See also R. Jackiw and C. Rebbi, Phys. Rev. Lett. 36, 1116 (1976); P. Hasenfratz and G. 't Hooft, Phys. Rev. Lett. 36, 1119 (1976).

[19]We will derive the DB's by first starring the variables in the sense of P. G. Bergmann and I. Goldberg, Phys. Rev. 98, 531 (1955).

[20]See, for example, R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966), Chap. 6. In what follows, we can equally well work with the "out" variables.

[21]These are similar to the assumptions in quantum scattering theory (Ref. 20). In the classical context, there is a simple explanation as to why $[\Omega^+]^{-1}$ does not exist if there are closed orbits. For, expressed in the "in" variables, the trajectories are straight lines, cf. (5.2). But no well-defined map can change closed orbits to straight lines.

[22]The construction of the "in" variables requires the solution of the equations of motion. Thus, suitable approximation schemes have to be developed to use the method of this section for general systems.

[23]For a related discussion, see D. Zwanziger, Phys. Rev. 174, 1484 (1968); Phys. Rev. D 6, 458 (1972).

[24]See, for example, I. R. Lapidus and J. L. Pietenpol, Am. J. Phys. 28, 17 (1960); G. Nadeau, *ibid.* 28, 566 (1966).

[25]Since $Q_i = s_{ik}q_k$, $P_i = s_{il}p_l$ fulfill $\{Q_i, Q_j\}^{(1)} = \{P_i, P_j\}^{(1)} = 0$, $\{Q_i, P_j\}^{(1)} = \delta_{ij}$, quantization in these variables is standard. Let $|Q')$ and $|Q'')$ denote states with eigenvalues Q_i' and Q_i'' for Q_i. The Hamiltonian is $H^{(1)} = P^2/2m$. Thus,

$$\left(Q''\left|\exp\left(-\frac{i}{\hbar}H^{(1)}t\right)\right|Q'\right) = (m/2\pi i\hbar t)^{3/2}\exp\left(\frac{i}{\hbar}\,\frac{m(Q''-Q')^2}{2t}\right),$$

cf. K. Gottfried, *Quantum Mechanics* (Benjamin, New York, 1966), Vol. I, p. 22. But $q_i|Q') = s^{-1}{}_{ik}Q_k'|Q')$ or $|q'\rangle = |sq')$, $|q''\rangle = |sq'')$. Hence,

$$\left\langle q''\left|\exp\left(-\frac{i}{\hbar}H^{(1)}t\right)\right|q'\right\rangle = (m/2\pi i\hbar t)^{3/2}\exp\left(\frac{i}{\hbar}\,\frac{m(sq''-sq')^2}{2t}\right)$$

and (6.3) follows.

[26]The phase difference is not of the form $\exp\{i[\phi(q'',R) - \phi(q',R)]\}$, cf. Ref. 25. If it had been of that form, we could have defined a unitary operator $U(R)$ for the rotation $R \in SO(3)$ by $U(R)|q'\rangle = |Rq'\rangle e^{i\phi(q')}$. Then $U(R)$ will commute with $H^{(1)}$:

$$\left\langle q''\left|\exp\left(-\frac{i}{\hbar}H^{(1)}(t)\right)\right|q'\right\rangle = \left\langle q''\left|U(R)^{-1}\exp\left(-\frac{i}{\hbar}H^{(1)}t\right)U(R)\right|q'\right\rangle.$$

[27]This is not always true. For instance, the transformation (3.14) where A fulfills (3.17) does not commute with rotations. The reason is that A has an explicit dependence on the direction of the Dirac string [cf. (5.17) and (5.18)]. Still, rotations can be canonically implemented. See Sec. II and also the references cited in Ref. 11.

[28]Note however that for the method of Sec. III, if $\xi \to f(\xi)$ is a canonical symmetry for the original description, $s(\xi) \to f[s(\xi)]$ is a canonical symmetry for the new description. Thus, in this method, there are isomorphic groups which are canonical symmetries in both descriptions. The physical meanings of these isomorphic groups are in general quite different. For instance, for (3.14) and (3.17), the isomorphic parity transformations $\xi \to -\xi$ and $\tau^A(\xi) \to -\tau^A(\xi)$ are physically different since $\tau^A(-\xi) \neq -\tau^A(\xi)$, cf. (5.17) and (5.18). Here $\xi = (q, p)$. See also Sec. II.

[29]This was pointed out by N. Mukunda. The discussion which follows is based on Mukunda's remarks. See in this context, P. G. Bergmann, Phys. Rev. 75, 680 (1949); P. G. Bergmann and J. H. M. Brunings, Rev. Mod. Phys. 21, 480 (1949); the reference cited in Ref. 19 and N. Mukunda, Ann. Phys. (N.Y.) 99, 408 (1976).

[30]See, for example, the reference cited in Ref. 5.