Some remarks on the Green's function formalism of Klauder's augmented quantum field theory: Yukawa model and "σ model"

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Equations for Green's functions of the augmented quantum field theory with Yukawa-type interaction are rewritten in terms of irreducible many-point functions. "Minimal" equations for many-point functions are derived which do not contain the coupling constant explicitly, while the coupling constant appears in "constraints" explicitly. The problem of finding lower many-point functions for given higher many-point functions is also discussed. It is also shown that the "σ model" is not equivalent to the ϕ^4 model.

I. INTRODUCTION

In this note we investigate the structure of Green's function equations of augmented quantum field theory of the Yukawa-type interaction.[1] Though the equations for truncated Green's functions are apparently linear, they become nonlinear after separation of the reducible part of four-point and higher many-point functions, as is the case for the ϕ^4 model considered in our previous paper.[2] In the present case the situation is further complicated by the fact that there are two equations for each G_{mn}, where m is the number of fermion variables and n the number of boson variables. Some of those equations degenerate to become constraints. On the other hand, the "minimal" equation for every G_{mn} does not explicitly contain the coupling constant λ.

II. FUNDAMENTAL EQUATIONS

The action in this model reads

$$\mathcal{S} = \int [\bar\psi(i\gamma^\mu\partial_\mu - M)\psi + \tfrac{1}{2}(\phi_{,\mu}{}^2 - m^2\phi^2) - \lambda\phi\bar\psi\psi]\, d^4x\,. \tag{2.1}$$

Then the augmented field equations to be satisfied by ϕ and ψ are

$$\phi(\Box + m^2)\phi + \lambda\phi\bar\psi_a\psi_a = 0 \tag{2.2}$$

and

$$\bar\psi_a(i\gamma^\mu_{bc}\partial_\mu\psi_c - M\psi_b) - \lambda\phi\bar\psi_a\psi_b = 0\,. \tag{2.3}$$

Defining truncated Green's functions by

$$\tau^T_{2m,n}(x_1, \ldots, x_m;\, y_1, \ldots, y_m;\, z_1, \ldots, z_n) \equiv \langle 0 | T\psi(x_1)\cdots\psi(x_m)\bar\psi(y_1)\cdots\bar\psi(y_m) \times \phi(z_1)\cdots\phi(z_n)|0\rangle^T, \tag{2.4}$$

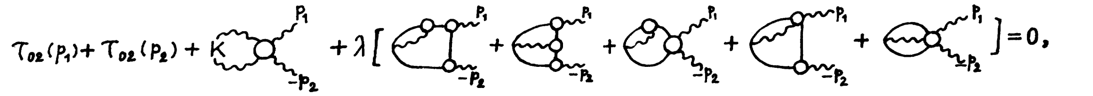
one gets the following equations for Green's functions:

$$i\Big[\sum_{r=1}^{n}\delta(x - z_r)\Big]\tau^T_{2m,n}(x_1, \ldots, x_m; y_1, \ldots, y_m; z_1, \ldots, z_n) + \lim_{x'\to x}(\Box_x + m^2)\tau^T_{2m,n+2}(x_1, \ldots, x_m; y_1, \ldots, y_m; z_1, \ldots, z_n, x, x') + \lambda\tau^T_{2m+2,n+1}(x_1, \ldots, x_m, x; y_1, \ldots, y_m, x; z_1, \ldots, z_n, x) = 0\,, \tag{2.5}$$

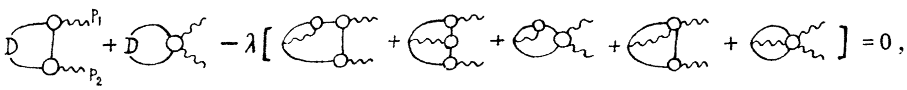
$$i\Big[\sum_{r=1}^{m}\delta(x - x_r)\Big]\tau^T_{2m,n}(x_1, \ldots, x_m; y_1, \ldots, y_m; z_1, \ldots, z_n) + \lim_{x'\to x}(i\gamma\cdot\partial_x - M)\tau^T_{2m+2,n}(x_1, \ldots, x_m, x; y_1, \ldots, y_m, x'; z_1, \ldots, z_n) - \lambda\tau^T_{2m+2,n+1}(x_1, \ldots, x_m, x; y_1, \ldots, y_m, x; z_1, \ldots, z_n, x) = 0\,. \tag{2.6}$$

III. STRUCTURE OF EQUATIONS IN TERMS OF IRREDUCIBLE MANY-POINT FUNCTIONS

After Fourier transformation, one gets the following equations for the two-point functions (from now on, superscript T will be omitted):



(3.1)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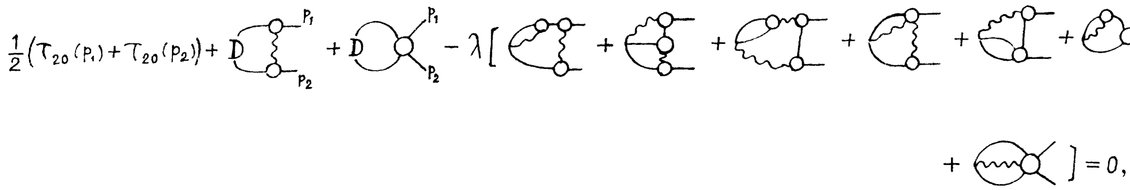
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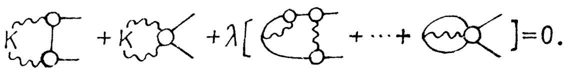
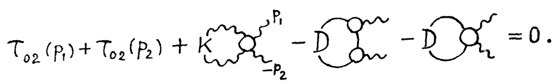
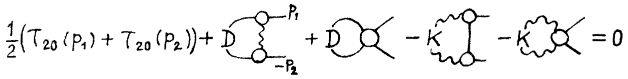

(3.4)

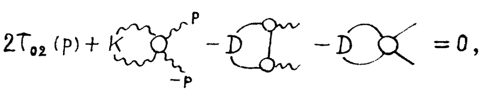
Here, combinatorial factors and transposed diagrams are omitted. Adding (3.2) to the appropriate combination of (3.1) with respect to spinor indices, one gets

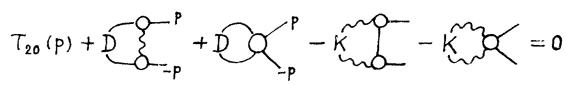

(3.5)

Similarly, from (3.3) and (3.4), one gets


(3.6)

Putting $p_1 = p_2 = p$ in (3.5) and (3.6), one gets the "minimal" equations for the two-point functions

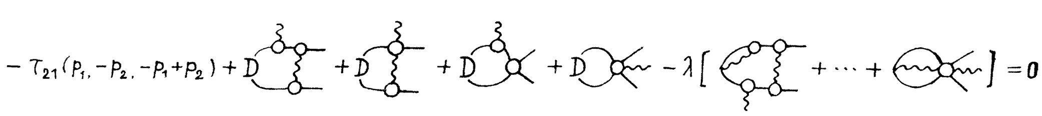

(3.7)

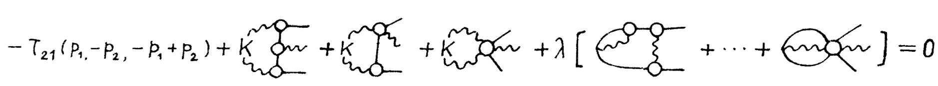

(3.8)

A strange feature of these equations is that they do not contain the coupling constant λ explicitly. Now, Eqs. (3.2) and (3.4) can be regarded as constraints on τ_{23} and τ_{41}, respectively. So far, we have eight functions τ_{02}, τ_{04}, τ_{20}, τ_{21}, τ_{22}, τ_{23}, τ_{40}, and τ_{41}, but not enough equations. This is a rather awkward situation. Operating with $\partial/\partial x_\mu$ on $\mathrm{Tr}[\gamma_\mu G_{21}(x,x;y)]$ one gets the following constraint on G_{21}:

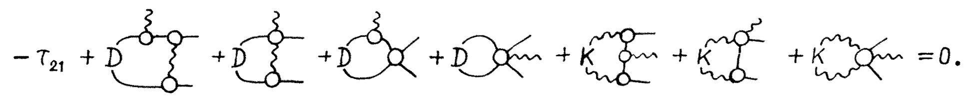
$$k_\mu \int d^4p\, \mathrm{Tr}[\gamma_\mu G_{21}(p, -p-k; k)] = 0\,, \tag{3.9}$$

which is not trivial in the scalar-spinor model but is automatically satisfied in the pseudoscalar-spinor model. Postponing further discussion of this problem, let us proceed to the equations for τ_{21} and τ_{40}. The two equations for τ_{21} read

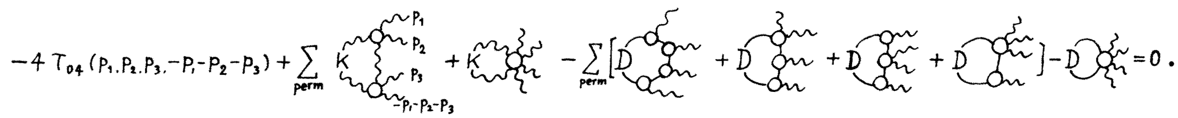

(3.10)


(3.11)

Here, use has been made of Eqs. (3.7) and (3.8). Adding (3.10) to (3.11), one gets


(3.12)

The minimal equation for τ_{04} reads


(3.13)

Again we have too many unknown functions but not enough equations and constraints.

IV. DESCENDING PROBLEM

Now, let us return to the problem with equations for τ_{02} and τ_{20}. Can one determine τ_{02} and τ_{20} if τ_{21}, τ_{22}, τ_{04}, and τ_{40} are given? In Eqs. (3.7) and (3.8), τ_{04}, τ_{22}, and τ_{40} appear with arguments $(p,-p,q,-q)$ (forward scattering). If τ_{04}, τ_{22}, and τ_{40} with arguments $(p,-p,q,-q)$ as well as τ_{21} subject to constraint (3.9) are given, Eqs. (3.7) and (3.8) can be regarded as coupled nonlinear integral equations. Defining

$$\sigma_{02}(p)=\int d^4q[(q^2-m^2)\tau_{04}(q,-q,p,-p)+(\gamma\cdot q-M)\tau_{22}(q,-q;p,-p)]\,, \tag{4.1}$$

$$\sigma_{20}(p)=\int d^4q[(q^2-m^2)\tau_{22}(q,-q,p,-p)+(\gamma\cdot q-M)\tau_{40}(q,-q,p,-p)]\,, \tag{4.2}$$

$$T_{02}(p)=[(p^2-m^2)\tau_{02}(p)]^{-1}\,, \tag{4.3}$$

$$T_{20}(p)=[(p^2-M^2)\tau_{20}(p)]^{-1}\,, \tag{4.4}$$

one gets the following equations for T_{02} and T_{20}:

$$T_{02}(p)+[\sigma_{02}(p)+\rho_{02}(\tau_{21},T_{20};p)]^{-1}(p^2-m^2-i\epsilon)^{-1}=0\,, \tag{4.5}$$

$$T_{20}(p)+[\sigma_{20}(p)+\rho_{20}(\tau_{21},T_{02},T_{20};p)]^{-1}(p^2-M^2-i\epsilon)^{-1}=0\,, \tag{4.6}$$

where

$$\rho_{02}(\tau_{21},T_{20};p)=\int d^4q(\gamma\cdot q-M)\{\tau_{21}(q,-q-p;p)[(p+q)^2-M^2]T_{20}(p+q)\tau_{21}(p+q)\tau_{21}(p+q,-q;-p) + \text{reversed loop}\}\,, \tag{4.7}$$

$$\rho_{20}(\tau_{21},T_{02},T_{20};p)=\int d^4q(\gamma\cdot q-M)\tau_{21}(p,q,-p-q)[(p+q)^2-m^2]T_{02}(p+q)\tau_{21}(-q,-p;p+q) + \int d^4q\,(q^2-m^2)\tau_{21}(p,q;-p-q)\,[(p+q)^2-M^2]T_{20}(p+q)\tau_{21}(p+q,-p;-q)\,. \tag{4.8}$$

Some comments are in order. For the integrals (4.1), (4.2), (4.7), and (4.8) to have the right behavior, those integrals ought to be interpreted as subtracted according to the following recipe:

$$\sigma_{02}^{r}(p)=\int_{m^2}^{p^2} d(p'^2)\,\frac{d}{d(p'^2)}\left\{\int d^4q[(q^2-m^2)\tau_{04}(q,-q,p',-p')+(\gamma\cdot q-M)\tau_{22}(q,-q,p',-p')](p'^2-m^2)^2\right\} \times(p^2-m^2-i\epsilon)^{-1}\,, \tag{4.9}$$

etc. Let us write the system of equations (4.5) and (4.6) abstractly

$$W[T_{02},T_{20};\sigma_{02},\sigma_{20},\tau_{21}]=0\,. \tag{4.10}$$

Our next task is to find sets of sufficient conditions for the existence of the solution to Eq. (4.10). So, let us try to apply Altman's theory of contractor directions. Denote by B the case of increasing continuous functions $B(s)$ such that $B(s)>0$ for $s>0$ and the integral

$$\int_0^a s^{-1}B(s)ds \tag{4.11}$$

exists for some $a>0$. Denote by $\mathfrak{X}$ the direct sum of complete metric spaces $\mathfrak{X}_{02}$ of candidates for T_{02} and $\mathfrak{X}_{20}$ of candidates for T_{20}. Then $\mathfrak{X}$ is a complete metric space, and the domain of W is a subset of $\mathfrak{X}$. If there exists a positive constant $q<1$ and a function $B\in \mathrm{B}$ such that for each $y \in \Gamma_T(W)$ there exists a positive number $\epsilon(T,y) \leq 1$ and an element $\overline{T}\in \mathfrak{D}(W)$ such that

$$\|W\overline{T} - WT - \epsilon y\| \leq q\epsilon \|y\| , \tag{4.12}$$

$$d(\overline{T},T) \leq \epsilon B(\|y\|) , \tag{4.13}$$

then the set $\Gamma_T(W)\subset Y$ is called a set of special contractor directions for W at T.

Let $\mathfrak{X}_0$ be a subset of $\mathfrak{X}$ and put $S\equiv S(T_0,r) = [T\colon\ d(T,T_0)<r, T\in\mathfrak{X}]$ for a given $T_0\in\mathfrak{X}_0$ and $U\equiv\mathfrak{X}_0\cap\overline{S}$, where $\overline{S}$ is the closure of S. Then we have the following theorem.

Theorem. Suppose the following hypotheses are satisfied: (1) $W\colon\ U\to Y$ is closed on U; (2) For any $T\in U_0\equiv X_0\cap S$, the set $\Gamma_t(W)$ is dense in some ball with center in Y;

(3) $$r\geq (1-q)^{-1}\int_0^a s^{-1}B(s)ds ,$$

$$a=(1-q)(1-\overline{q})^{-1}\|W[T_0]\| ,$$

and $\overline{q}$ is arbitrary with $q<\overline{q}<1$. Then the equation $W[T]=0$ has a solution $T\in U$. For proof, see Altman.[3]

Condition (3) of the above theorem can be replaced by

$$r\geq (1-q)^{-1}\int_0^a s^{-1}B(s)ds , \quad a=e^{1-q}\|W[T_0]\| . \tag{4.14}$$

If input τ_{04}, τ_{22}, and τ_{40} are chosen so that $KKKK\tau_{04}$, $KKDD\tau_{22}$, and $DDDD\tau_{40}$ have sufficiently large norm, there exist r, q, and B such that the conditions of the above theorem are satisfied. The snag of this theorem is that it is purely existential and does not give an algorithm for construction of a solution. For a constructive approach, one has to impose further conditions upon input.

Now let us assume that W is Fréchet differentiable. Then the Fréchet derivative can be written as a supermatrix

$$W'(T'_{02},T'_{20}) = \begin{bmatrix} W'_{11} & W'_{12} \\ W'_{21} & W'_{22}\end{bmatrix} , \tag{4.15}$$

$$W'_{11}=I , \tag{4.16}$$

$$W'_{12}=-(p^2-m^2-i\epsilon)^{-1}\rho_{02}(\tau_{21},\cdot\,;p) \times[\sigma_{02}(p)+\rho_{02}(\tau_{21},T'_{20};p)]^{-2} , \tag{4.17}$$

$$W'_{21}=-(p^2-M^2-i\epsilon)^{-1}\rho_{20}^{(1)}(\tau_{21},\cdot\,;p) \times[\sigma_{20}(p)+\rho_{20}(\tau_{21},T'_{02},T'_{20};p)]^{-2} , \tag{4.18}$$

$$W'_{22}=I-(p^2-M^2-i\epsilon)^{-1}\rho_{20}^{(2)}(\tau_{21},T'_{02},\cdot\,;p) \times[\sigma_{20}(p)+\rho_{20}(\tau_{21},T'_{02},T'_{20};p)]^{-2} , \tag{4.19}$$

where

$$\rho_{20}^{(1)}(\tau_{21},T''_{02};p)=\left\{\int d^4q(\gamma\cdot q-M)\tau_{21}(p,q;-p-q)[(p+q)^2-m^2]T''_{02}(p+q)\tau_{21}(-p,-q;p+q)\right\}^{\text{ren}} , \tag{4.20}$$

$$\rho_{20}^{(2)}(\tau_{21},T''_{20};p)=\left\{\int d^4q(q^2-m^2)\tau_{21}(p,-p-q;q)[(p+q)^2-M^2]T''_{20}(p+q)\tau_{21}(p+q,-p;-q)\right\}^{\text{ren}} \tag{4.21}$$

If the input $\{\sigma_{02},\sigma_{20},\tau_{21}\}$ is such that the supermatrix W' has at least a one-sided inverse for $\{T_{02},T_{20}\}$ belonging to some convex subset of $\mathrm{B}_{02}\oplus\mathrm{B}_{20}$ (B_{mn} being Banach space of T_{mn}), and some other conditions are satisfied, one can apply Newton-Kantorovich–type techniques. An interesting feature is that if σ's are large enough, W' is invertible, while it is not invertible in the weak-coupling limit $\sigma_{02}\to 0$, $\sigma_{20}\to 0$, $\tau_{21}\to 0$.

If a solution $\{T^*_{02},T^*_{20}\}$ of Eqs. (4.5) and (4.6) is found for a given input $\{\sigma_{02},\sigma_{20},\tau_{21}\}$, then the original equations (3.2) and (3.4) become constraints on τ_{04}, τ_{22}, τ_{23}, τ_{40}, and τ_{41} with general arguments. If τ_{04}, τ_{22}, and τ_{40} with general arguments can be so chosen that the terms with explicit factor λ need not become large, i.e., if constraints with $\lambda=0$ are approximately satisfied, one need not worry about the overgrowth of higher many-point functions.

For uniqueness of solution with a given input, further conditions must be satisfied.

V. σ MODEL

In this section, we consider the question whether the actions

$$\mathcal{I}'=\int\ [\tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi-m^2\phi^2)-\lambda\phi^4]d^4x \tag{5.1}$$

and

$$\mathcal{G}'' = \int \left[\tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - 2\sqrt{\lambda}\,\sigma\phi^2 + \sigma^2\right] d^4x \tag{5.2}$$

are equivalent in the augmented quantum field theory.[4] The field equations in the augmented quantum field theory read

$$\phi(\Box + m^2)\phi + 4\sqrt{\lambda}\,\sigma\phi^2 = 0\,, \tag{5.3}$$

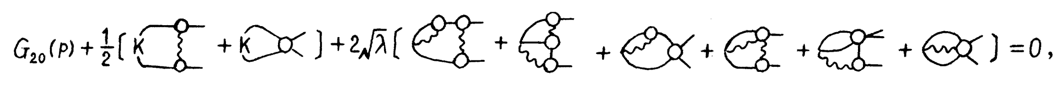
$$\sqrt{\lambda}\,\sigma\phi^2 - \sigma^2 = 0\,. \tag{5.4}$$

It is now obvious that Eq. (5.4) is not equivalent to the equation $\sigma = \sqrt{\lambda}\phi^2$. In the canonical quantum field theory, actions $\mathcal{G}'$ and $\mathcal{G}''$ are equivalent, and one can eliminate many-point functions involving σ from equations for many-point functions of the ϕ field. Green's functions with an external σ line can be expressed in terms of higher Green's functions of the ϕ field. But the situation is not so simple in the augmented quantum field theory.

Let us define

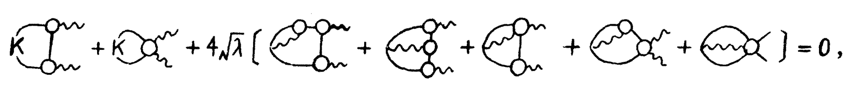
$$G_{mn}(x_1, \ldots, x_m; y_1, \ldots, y_n) = \langle 0|T\phi(x_1)\cdots\phi(x_m)\sigma(y_1)\cdots\sigma(y_n)|0\rangle\,. \tag{5.5}$$

Then the equations for G_{20} and G_{02} read

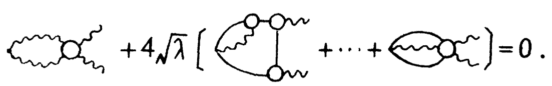


(5.6)

(5.7)

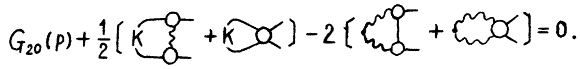


(5.8)



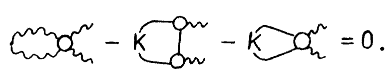
(5.9)

It should be noticed that we do not have an equation of the form $G_{02}(p) + \cdots = 0$. Subtracting (5.7) from (5.6), one gets



(5.10)

Similarly, from (5.8) and (5.9) one gets



(5.11)

Equations (5.10) and (5.11) are the minimal equations for G_{20} and G_{02}, respectively. Some comments are in order. If $G_{02}(p)$ has a pole at $p^2 = \mu^2$, the first term on the left-hand side of Eq. (5.9) has the form

$$\int d^4k \,\frac{\Gamma_{04}(k, -k, p, -k)}{(k^2 - \mu^2 - i\epsilon)^2 (p^2 - \mu^2 - i\epsilon)^2} = F_{04}(p)\,. \tag{5.12}$$

So, we define

$$F_{04}(p) = \frac{d}{d(\mu'^2)}\left[\int d^4k \,\frac{\Gamma_{04}(k, -k, p, -p)}{k^2 - \mu'^2 - i\epsilon}\right]\Bigg|_{\mu'^2 = \mu^2} \frac{1}{(p^2 - \mu^2 - i\epsilon)^2}\,. \tag{5.13}$$

In other words, we assumed that the function

$$F_{04}(p, \nu) = \int d^4k \,\frac{(k^2 - \mu^2)^2 G_{04}(k, -k, p, -p)(p^2 - \mu^2)^2}{k^2 - \nu - i\epsilon} \tag{5.14}$$

is differentiable with respect to ν at $\nu = \mu^2$. As we do not have the term G_{02} in Eq. (5.9), the integrals need not be interpreted as subtracted. In other words we have two alternatives. Either Eq. (5.9) is satisfied by the integrals as they stand or satisfied after subtraction of respective values at $p^2 = \mu^2$. On the other hand, integrals in Eq. (5.9) should be interpreted as subtracted. Moreover, for the last two terms on the left-hand side of Eq. (5.10) to make sense, we assumed that the functions

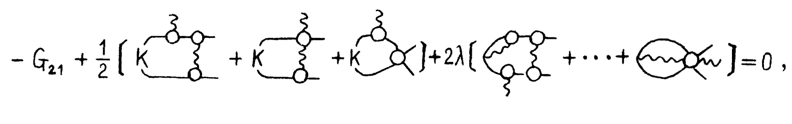
$$H_1(p,\nu)=\int d^4k\; \frac{(k^2-\mu^2)^2\, G_{21}(p,-p-k,k)\, G_{20}(-p-k)^{-1} G_{21}(p+k,-p,-k)}{k^2-\nu-i\epsilon}\,, \tag{5.15}$$

$$H_2(p,\nu)=\int d^4k\; \frac{(k^2-\mu^2)^2 G_{22}(p,-p,k,-k)}{k^2-\nu-i\epsilon} \tag{5.16}$$

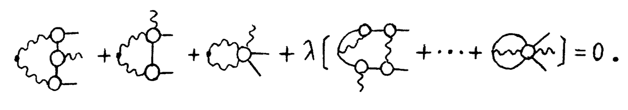
are differentiable at $\nu=\mu^2$.

If G_{02}, G_{21}, G_{22}, and G_{40} are given as input, Eq. (5.10) can be regarded as a nonlinear integral equation for G_{20}. In this case one cannot regard Eqs. (5.10) and (5.11) as simultaneous equations for G_{20} and G_{02}, because of the degeneracy of Eq. (5.11). If G_{20} is found, Eq. (5.11) can be regarded as a linear constraint on G_{04}, and Eqs. (5.7) and (5.9) can be regarded as linear constraints on G_{41} and G_{23}, respectively.

The equation for G_{21} reads



(5.17)



(5.18)

For the sum of the first three terms on the left-hand side of Eq. (5.18) to make sense, one has to assume that

$$\begin{aligned} J(p,q,\nu)=\int d^4k\; \frac{(k^2-\mu^2)^2}{k^2-\nu-i\epsilon}\big[&G_{21}(p,-p+k,-k)G_{20}(-p+k)^{-1}G_{21}(-p+k,q-k,p-q)G_{20}(q-k)^{-1}G_{21}(q-k,-q,k)\\ &+G_{22}(p,-q+k,q-p,-k)G_{20}(-q+k)^{-1}G_{21}(q-k,-q,k)\\ &+G_{21}(p,-p+k,-k)G_{20}(-p+k)^{-1}G_{22}(p-k,-q,q-p,k)+G_{04}(p,-q,-p+q,k,-k)\big] \end{aligned} \tag{5.19}$$

is differentiable with respect to ν at $\nu=\mu^2$. This is a restriction on G_{23}.

Anyway one cannot eliminate G_{mn}, $n\neq 0$ from the system of equations for G_{m0}.

VI. CONCLUDING REMARKS

The most important difference between a model with only one field, e.g., the ϕ^4 model and a model with two or more fields, e.g., the Yukawa model, is that the "minimal" equations in the latter do not contain the coupling constant explicitly, but are different from those in the pseudofree theory, while the coupling constant appears in the constraints. On the other hand, the absence of the bare vertex term in the equations is a good feature, and in particular implies that τ_{21} (and τ_{04}) can have nice asymptotic behavior without tricky cancellations. The latter feature is, of course, shared by the ϕ^4 model.

As in the σ model, if a field lacks a kinetic part in its field equation, the equation for its propagator is degenerate. Nevertheless, one cannot eliminate such a field from the augmented quantum field theory.

So far, we have considered the equations for Green's functions in the context of a descending problem. On the other hand, if one begins with given τ_{02}, τ_{20}, and τ_{21} (or G_{02}, G_{20}, G_{21}) in the Yukawa model (in the σ model), the equations under consideration become constraints on higher Green's functions. Those constraints seem to be satisfied by continuums of functions, but we do not have an algorithm to construct one or another sequence of higher Green's functions with proper symmetry.

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