$1/N$ expansion for general scalar interactions: Nonleading order and application to bounded interactions

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Arbitrary polynomial ineractions of the N-component scalar field are considered in one, two, and three dimensions. The effective potential is calculated up to the next-to-leading order in $1/N$. The result is applied to the case of bounded interactions. Under certain assumptions on the shape of the interaction Langrangian, the radiative corrections do not affect the large-φ^2 behavior of the effective potential.

I. INTRODUCTION

During the last couple of years there has been considerable interest in nonstandard expansion techniques in quantum field theory. The $1/N$ expansion[1] plays one of the central roles among these techniques. Progress toward application of this expansion to various models has been reported in Refs. 2–7, the ideas of which are central to the present note.

In the following we shall consider the N-component scalar field theory whose nonderivative part of the Lagrangian is the O(N)-invariant infinite polynomial of the form

$$V(\Phi^2/N) = N\sum_{k=1}^{\infty} \frac{1}{k!} V^{(k)}(0)(\Phi^2/N)^k . \qquad (1.1)$$

Careful analysis of the leading-order approximation to models of the form

$$L = \tfrac{1}{2}(\partial_\mu \Phi^a \partial^\mu \Phi^a) - V(\Phi^2/N) \qquad (1.2)$$

has been recently given by Schnitzer.[5] This note is intended to expand the scope of his findings to the next-to-leading order in $1/N$. Hence, whenever possible, we use notation which closely parallels that of Ref. 5.

The paper is ordered as follows: In Sec. II we derive the expression for the unrenormalized effective potential of the model (1.2) up to the next-to-leading order. We add a short remark on the one-dimensional case when our results are correct without need for renormalization. Sections III and IV are devoted to the renormalization carried out in two and three dimensions, respectively. Explicit expressions for counterterms are derived. Finally, Sec. V contains the discussion and application of the general results to the special case when the interaction part of the Lagrangian is the Taylor expansion of the bounded, sufficiently smooth function. This example is instructive since one believes in the "good" large-momentum behavior of such theories,[8] but this feature is spoiled by the traditional expansion technique in powers of coupling constants. We argue that the $1/N$ expansion seems, hopefully, to be free of this problem. In particular, next-to-leading radiative corrections do not affect the boundedness of the effective potential. The restrictions on the validity of the technique used in the paper are discussed and a comment on the possibility of the extension of the method to include bounded nonlocal interactions is added.

II. THE EFFECTIVE POTENTIAL

Let us consider a field theory as defined by (1.1) and (1.2). Its bare Lagrangian is

$$L(\Phi^2/N) = \tfrac{1}{2}(\partial\Phi)^2 - N\sum_{n=1}^{\infty} \frac{1}{n!} V_0^{(n)}(0)(\Phi^2/N)^n . \qquad (2.1)$$

We assume $V_0(\Phi^2/N)$ to be an infinitely differentiable function of Φ^2/N, the nth derivative of which equals

$$V_0^{(n)}(\Phi^2/N) = \sum_{k=0}^{\infty} \frac{1}{k!} V_0^{(n+k)}(0)(\Phi^2/N)^k . \qquad (2.2)$$

For now we do not specify the space-time dimension or the particular form of the interaction part of (2.1).

The unrenormalized effective potential of our model is given by

$$V_{\text{eff}}(\varphi^2/N) = V_{(N)}(\varphi^2/N) + V_{(1)}(\varphi^2/N) + O(N^{-2}), \qquad (2.3)$$

where φ denotes the classical field and $V_{(N)}$ and $V_{(1)}$ stand for the leading and next-to-leading order parts of V_{eff},

$$V_{(N)}(\varphi^2/N) = N[V_0(\varphi^2/N + B_1) - B_1 V_0^{(1)}(\varphi^2/N + B_1)] + \tfrac{1}{2} N\hbar \int \frac{d^n p}{(2\pi)^n} \ln(p^2 + M^2) \qquad (2.4)$$

and

$$V_{(1)}(\varphi^2/N)=\frac{\hbar}{2}\int\frac{d^np}{(2\pi)^n}\ln\frac{(p^2+M^2)[1+4V_0^{(2)}(\varphi^2/N+B_1)B_2(p)]+4(\varphi^2/N)V^{(2)}(\varphi^2/N+B_1)}{p^2+M^2}. \tag{2.5}$$

The calculations leading to (2.4) and (2.5) are left to Appendix A, so for the definitions of B_1, B_2, and M^2 which are given by (A8), (A11), and (A7), respectively, expression (2.4) agrees with the result of Schnitzer.[5] The validity of (2.5) may be easily checked without recourse to lengthy calculations. The $1/N$ power counting shows that diagrams containing a vertex connected with other vertices by more than four lines are at most of order $1/N$. Therefore the relevant Feynman rules are simplified; all couplings present in (2.1) sum up to the "effective vierbein" vertex, as visualized in Fig. 1. The corresponding factor in the vertex is

$$\rho=V_0^{(2)}(\varphi^2/N+B_1). \tag{2.6}$$

One can then anticipate (2.4) and (2.5) from the formula (2.19) of Root's paper[2] devoted to the $1/N$ Φ^4 theory. To this end it is enough to replace his coupling $\lambda/4!$ by our infinite series $\rho/2!$. A similar observation was made by Townsend[7] while studying the $1/N(\Phi^6)_3$ model. In general it is more convenient to deal with the derivatives of the effective potential rather than with $V_{\rm eff}$ itself. Differentiating (2.4) and (2.5), we obtain

$$\frac{\partial V_{(N)}}{\partial\varphi^2/N}=V_0^{(1)}(\varphi^2/N+B_1) \tag{2.7}$$

and

$$\frac{\partial V_{(1)}}{\partial\varphi^2/N}=\frac{\hbar}{2}\int\frac{d^np}{(2\pi)^n}\frac{1}{(p^2+M^2)(1+4\rho B_2)+4\rho\varphi^2/N}\times\left\{\frac{4(p^2+M^2)\rho\partial B_2/\partial M^2-4\rho\varphi^2/N}{p^2+M^2}\frac{\partial M^2}{\partial\varphi^2/N}+4(p^2+M^2)B_2+\frac{4\varphi^2}{N}\frac{\partial\rho}{2\varphi^2/N}+4\rho\right\}. \tag{2.8}$$

In more than one dimension, subtractions are necessary to make (2.7) and (2.8) finite. In one dimension, integrations can be performed without regularization. In particular, for B_1 and B_2 we get

$$B_1=\hbar/2M, \tag{2.9}$$

$$B_2(k)=(\hbar/2M)(k^2+4M^2)^{-1}. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.7) and (2.5) and then integrating, we obtain

$$\frac{\partial V_{(N)}}{\partial\varphi^2/N}=NV_0^{(1)}(\varphi^2/N+\hbar/2M), \tag{2.11}$$

$$\frac{\partial V_{(1)}}{\partial\varphi^2/N}=\frac{\partial}{\partial\varphi^2/N}[(m_+{}^2)^{1/2}+(m_-{}^2)^{1/2}-3M], \tag{2.12}$$

where we have introduced

$$m_\pm{}^2=\tfrac{5}{2}M^2+\tfrac{1}{2}\rho\frac{\varphi^2}{N}+\hbar\rho M\pm\left[\left(\tfrac{5}{2}M+\tfrac{1}{2}\rho\frac{\varphi^2}{N}+\hbar\rho M\right)^2-4M^2-16M^2\rho\frac{\varphi^2}{N}-2\hbar M\right]^{1/2}. \tag{2.13}$$

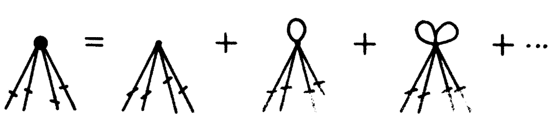

FIG. 1. The effective quartic vertex ρ.

The gap equation is

$$M^2=2V_0^{(1)}(\varphi^2/N+\hbar/2M). \tag{2.14}$$

It is apparent that for a reasonable choice of V_0 [$V_0(x)\to+\infty$ as $x\to\infty$] Eq. (2.14) has a positive, real solution for $\varphi^2/N=0$. Details will depend on the actual form of the interaction; however, this observation indicates that the symmetric minimum exists (at least in the large-N limit).

III. RENORMALIZATION IN TWO DIMENSIONS

In two dimensions, the scalar field theory is renormalizable to all orders. In the leading order all divergences are eliminated by normal ordering which is equivalent to shifting the arguments of the expansion coefficients $V_0^{(k)}$ by a divergent constant c,[5]

$$V_0^{(n)}(c)=V^{(n)}(0), \tag{3.1}$$

where

$$c=(\hbar/4)\ln(\Lambda^2/\mu^2) \tag{3.2}$$

is the divergent part of B_1,

$$B_1=\frac{\hbar}{4\pi}[\ln(\Lambda^2/\mu^2)-\ln(M^2/\mu^2)]. \tag{3.3}$$

We can then write

$$M^2=2V^{(1)}(X_2),\quad \rho=V^{(2)}(X_2), \tag{3.4}$$

where

$$X_2 = \varphi^2/N - (\hbar/4\pi)\ln(M^2/\mu^2)\,. \tag{3.5}$$

The substitution (3.1) makes the integrand in (2.5) finite, not so for the integral. Counterterms $(1/N)\delta^{(j)}$, countervailing the remaining divergences of (2.5), are of order $O(N^{-1})$, thus their relevant contribution to $V_{(1)}$ comes from graphs which are (topologically) of the leading order. Let us define

$$\bar{V}^{(j)}(0) = V^{(j)}(0) + \frac{1}{N}\delta^{(j)}\,. \tag{3.6}$$

Expanding $V^{(1)}(X_2)$ around its renormalized value $\bar{V}^{(1)}(X_2)$, we obtain

$$V^{(1)}(X_2) = \bar{V}^{(1)}(X_2) + \frac{1}{N}\sum_{j=1}^{\infty}\delta^{(j)}\frac{1}{(j-1)!}X_2^{\,j-1} \times \frac{1}{2\rho}\frac{\partial M^2}{\partial\varphi^2/N}\,. \tag{3.7}$$

We relegate the derivation of this formula to Appendix B. In order to determine $O(1/N)$ counterterms we have to separate the divergent part of (2.5) and then give it the shape of (3.7). To this end we rewrite (2.5) in a more tractable manner:

$$\frac{\partial V_{(1)}}{\partial\varphi^2/N} = \frac{\hbar}{2}\int\frac{d^2p}{(2\pi)^2}\Bigg\{\left[\frac{4B_2}{1+4\rho B_2} + \frac{4\varphi^2/N}{4\rho\varphi^2/N + (p^2+M^2)(1+4\rho B_2)}\right]\frac{\partial\rho}{\partial\varphi^2/N}$$
$$+\left[\frac{4}{1+4\rho B_2}\frac{\partial B_2}{\partial M^2}\frac{\partial M^2}{\partial\varphi^2/N} + \frac{4\rho}{4\rho\varphi^2/N+(p^2+M^2)(1+4\rho B_2)}\right]$$
$$-\frac{4\rho\varphi^2/N}{4\rho\varphi^2/N+(p^2+M^2)(1+4\rho B_2)}\left[\frac{1}{p^2+M^2} + \frac{4}{1+4\rho B_2}\,\rho\,\frac{\partial B_2}{\partial M^2}\frac{\partial M^2}{\partial\varphi^2/N} + B_2\frac{\partial\rho}{\partial\varphi^2/N}\right]\Bigg\}. \tag{3.8}$$

In two dimensions, B_2 is finite

$$B_2(k) = \frac{\hbar}{4\pi}\frac{1}{(k^4+4k^2M^2)^{1/2}}\ln\frac{[k+(k^2+4M^2)^{1/2}]^2}{4M^2}\,, \tag{3.9}$$

hence

$$\frac{\partial B_2}{\partial M^2} = -\frac{\hbar}{4\pi M^2}\frac{1}{k(k^2+4M^2)^{1/2}} + \cdots\,. \tag{3.10}$$

In (3.10) we have omitted terms which behave like k^{-4} or $k^{-4}\ln k$. Substituting (3.9) and (3.10) into (3.8), we obtain for the divergent part of (3.8)

$$\hbar\int\frac{d^2k}{(2\pi)^2}\left\{4\left[\frac{\hbar\ln(k^2/M^2)}{4\pi(k^4+4M^2k^2)^{1/2}} + \frac{\varphi^2}{N}\frac{1}{k^2+M^2}\right]\frac{\partial\rho}{\partial\varphi^2/N} + 2\rho\left[\frac{1}{k^2+M^2} - \frac{\hbar}{4\pi M^2(k^2+4M^2k^2)^{1/2}}\frac{\partial M^2}{\partial\varphi^2/N}\right]\right\}. \tag{3.11}$$

Introducing the cutoff, integrating, and then using the formulas (B5) and (B6), we obtain

$$\delta^{(k)} = -\frac{\hbar}{\pi}\ln(\Lambda^2/\mu^2)\{(k-1)V^{(k)}(0) + [\tfrac{1}{4}+\ln(\Lambda^2/\mu^2)]V^{(k+1)}(0)\}\,. \tag{3.12}$$

IV. RENORMALIZATION IN THREE DIMENSIONS

In three dimensions both logarithmic and power divergences are present in the next-to-leading order; however, the scalar field theory is still free of induced derivative couplings. Now the normal-ordering constant is

$$C = -\hbar\Lambda/2\pi^2\,. \tag{4.1}$$

In three dimensions B_2 is

$$B_2(p) = \frac{\hbar}{8\pi p}\arcsin\left(\frac{p^2}{p^2+4M^2}\right)^{1/2} = \frac{\hbar}{16}(p^2+M^2)^{-1/2}\left[1 - \frac{\hbar}{\pi}M(p^2+M^2)^{-1/2} + \cdots\right]; \tag{4.2}$$

hence,

$$\frac{\partial B_2}{\partial M^2} = -\frac{\hbar}{8\pi M}\frac{1}{p^2+M^2}\,. \tag{4.3}$$

Using (4.3) and the formulas of Appendix B [these formulas must be suitably modified according to

(B8)], we get

$$\frac{\partial V_{(1)}}{\partial\varphi^2/N}=\frac{\hbar}{2}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{(p^2+M^2)(1+4\rho B_2)+4\rho\varphi^2/N}\left(4\rho+4(p^2+M^2)B_2V^{(3)}(X_3)+\frac{4\varphi^2}{N}V^{(3)}(X_3)-\frac{8\rho^2\varphi^2/N}{p^2+M^2}\right)\frac{1}{2\rho}\,\frac{\partial M^2}{\partial\varphi^2/N}, \tag{4.4}$$

where

$$X_3=\varphi^2/N-\hbar M/4\pi \tag{4.5}$$

and

$$M^2=2V^{(1)}(X_3),\quad \rho=V^{(2)}(X_3). \tag{4.6}$$

Expanding B_2 according to (5.2), rejecting finite terms, and integrating (with cutoff) the divergent ones, we get from (4.4)

$$\frac{\partial V_{(1)}}{\partial\varphi^2/N}=\left\{V^{(3)}(X_3)C^2-2\rho C-2V^{(3)}(X_3)X_3C+\frac{\hbar^2}{32}V^{(3)}(X_3)C-\frac{\hbar}{4\pi}\ln(\Lambda^2/\mu^2)\left[\tfrac{1}{2}\hbar\rho^2+\hbar\rho V^{(3)}(X_3)X_3-2\hbar\rho^2\frac{\hbar^2}{16^2}V^{(3)}(X_3)\right]\right\}\frac{1}{2\rho}\,\frac{\partial M^2}{\partial\varphi^2/N}, \tag{4.7}$$

hence the $O(1/N)$ counterterms are

$$\delta^{(k)}=-\left\{2C^2V^{(k+2)}(0)-4kCV^{(k+1)}(0)+\frac{\hbar^2}{16}CV^{(k+2)}(0)-\frac{\hbar}{4\pi}\ln(\Lambda^2/\mu^2)[A_{k-1}+2(k-1)B_{k-2}-4(\hbar/16)^2D_{k-1}]\right\}, \tag{4.8}$$

where

$$A_k=\sum_{l=0}^{k}\binom{k}{l}V^{(l+2)}(0)\,V^{(k-l+2)}(0), \tag{4.9}$$

$$B_k=\sum_{l=0}^{k}\binom{k}{0}V^{(l+2)}(0)\,V^{(k-l+3)}(0), \tag{4.10}$$

$$D_k=\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{k}{l}\binom{l}{m}V^{(k-l+2)}(0)\,V^{(l-m+2)}(0)\,V^{(m+3)}(0). \tag{4.11}$$

V. APPLICATION TO BOUNDED INTERACTIONS AND CONCLUSIONS

Consistency of the renormalization procedure breaks down if the operator structure of divergences differs from that of the original Lagrangian. This drawback is absent from two-dimensional models of the type (2.1) but is unavoidable in three-or-more-dimensional space-time. However, in the large-N limit nonrenormalizable aspects of the theory may be suppressed by an appropriate power of $1/N$.[5] If one deals with the essentially infinite series in (2.1), then the corresponding factors are $1/N^2$ and $1/N$ for three and four dimensions, respectively. This has allowed us to derive the complete expressions for the next-to-leading counterterms in three dimensions.

Analogous expressions for the leading-order counterterms in four dimensions can be immediately anticipated from the formulas (5.14) and (5.15) of Ref. 5. Beyond these approximations, one needs to introduce derivative-coupling counterterms. The original Lagrangian does not involve such terms, hence one does not have appropriate normalization conditions available. The number of arbitrary constants increases unmanageably and the theory loses its predictive power. One could dodge this difficulty and admit nonlocal Lagrangians with all kinds of derivative couplings present from the beginning. An appropriate generalization of (1.1) would be an infinite polynomial both in fields and their derivatives. The case when the interaction part of the Lagrangian is an expansion of a bounded function of fields and their derivatives seems to be of particular interest because one believes in the "good" large-momentum behavior and unitarity of such models.[8]

Unfortunately progress in exploring this idea has been halted by the lack of a viable approximation technique allowing one to make use of the boundedness of the interaction.[8,9] Recently it has been recognized that the usual difficulties with nonrenormalizable interactions were caused by the invalid perturbation technique using the expansion in powers of coupling constants rather than by the fundamental failure of the models.[10] The $1/N$ expansion was recently applied with some success for studying various nonrenormalizable interactions[11,12] and seems to be suitable also for bounded interactions. The following argument may justify this hope:

Let the interaction Lagrangian be an infinitely differentiable function of fields and their derivatives. Moreover, let this function and all its derivatives vanish at infinity faster than any polynomial of fields and their derivatives. In the $1/N$ expansion, all diagrams contributing to the effective action are built of the "effective vertices" analogous to those of Fig. 1. The number of such vertices increases with the order of the approximation but they are all proportional to derivatives of the interaction Lagrangian. In the large-φ regime, factors from the vertices dominate, hence boundedness of Green's functions remains unaffected order by order in $1/N$.

The above argument requires a more formal treatment which shall not be attempted here. Instead, recourse will be made to calculations restricted to the specific three-dimensional case and to the next-to-leading order only. The result, however encouraging, must not be convincing.

Let us return to the Lagrangian (2.1) and let us make the following assumptions on V:

(i) for large x^2, let $V(x^2) \sim x^2$ (the mass term is included in V);

(ii) for $x^2 \to \infty$, let $V^{(1)}(x^2) \to \text{const} > 0$;

(iii) for $k^2 > 1$, let $V^{(k)}(x^2) \to 0$ as $x^2 \to \infty$.

From (4.5) and (4.6) we get

$$X_3(\varphi^2/N) = \varphi^2/N - (\hbar/4\pi)[2V^{(1)}(X_3)]^{1/2}. \tag{5.1}$$

The boundedness of $V^{(1)}$ combined with (5.1) implies

$$X_3(\varphi^2/N) \sim \varphi^2/N \text{ as } \varphi^2/N \to \infty. \tag{5.2}$$

The leading-order approximation to the renormalized effective potential gives[5]

$$V_{(N)}(\varphi^2/N) = N\{V(X_3) + (\hbar/24\pi)[2V^{(1)}(X_3)]^{3/2}\}. \tag{5.3}$$

Consulting (5.2), (5.3), and our assumptions for V, we find that $V_{(N)}(\varphi^2/N) \sim \varphi^2$ for $\varphi^2/N \to \infty$. In order to study the $O(N^0)$ contribution let us expand (2.5) such that

$$V_{(1)}(\varphi^2/N) = -\frac{\hbar}{2}\int \frac{d^3p}{(2\pi)^3} \times \sum_{j=1}^{\infty} \frac{1}{j}\left[-4V^{(3)}(X_3)B_2 - \frac{4V^{(3)}(X_3)\varphi^2/N}{p^2+M^2}\right]^j. \tag{5.4}$$

Neglecting the divergences which were just subtracted, we see that in the large-φ^2/N limit all terms of (5.4) are proportional to $V^{(3)}(X_3)\varphi^2/N$, but

$$\lim_{\varphi^2\to\infty} V^{(3)}(X_3)\frac{\varphi^2}{N} = \lim_{X_3\to\infty} V^{(2)}(X_3)X_3 = \lim_{X_3\to\infty} \frac{V^{(2)}(X_3)}{X_3^{-1}} = \lim_{X_3\to\infty} \frac{V^{(1)}(X_3)}{\ln X_3} = 0; \tag{5.5}$$

therefore,

$V_{(1)}(\varphi^2/N) \to 0$ as $\varphi^2/N \to \infty$.

In conclusion, in the large-φ^2 limit the effective potential grows as φ^2. Moreover, in this limit the $O(1)$ part of the potential is smaller than the $O(N)$ part, independent of the value of N.

However the above analysis may be destroyed by higher-order contributions. It, hopefully, gives strong encouragement to the ideas of bounded interactions and the $1/N$ expansion.

APPENDIX A

We shall derive the expressions (2.4) and (2.5) by the method of Cornwall, Jackiw, and Tomboulis.[13] The effective action $\Gamma(\varphi, G)$ equals

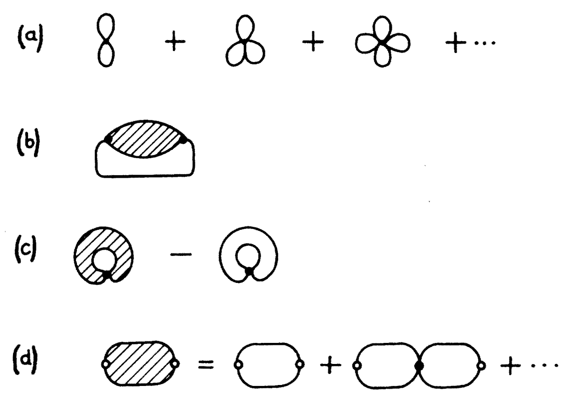

FIG. 2. Two-particle irreducible diagrams contibuting to Γ_2. (a) Graphs relevant in the leading and next-to-leading orders. (b) Contribution to the next-to-leading and higher orders. The shaded bubble represents the infinite chain of graphs presented on Fig. 2(d). The heavy dots stand for the effective three-legs vertex analogous to that of Fig. 1. The corresponding factors are $2\varphi_a\rho$. (c) Same as (b), except that the factor at the heavy dot is ρ. We reject the diagram stemming from the first member of the chain (d), because the corresponding contribution is that of Fig. 2(a). (d) The bubble chain of Figs. 2 (b) and 2 (c). Heavy dots represent ρ , factors at the ends are omitted.

$$\Gamma(\varphi, G) = W(\varphi) + \tfrac{1}{2} i\hbar \operatorname{tr} \ln G^{-1} + \tfrac{1}{2} i\hbar \operatorname{tr} D^{-1}(\varphi)G + \Gamma_2(\varphi, G), \tag{A1}$$

where $W(\varphi)$ is the classical action. The exact propagator G_{ab} is implicitly defined by

$$\delta\Gamma(\varphi, G)/\delta G_{ab}(x, y) = 0 . \tag{A2}$$

D^{-1} is the inverse free propagator which, in our case, equals

$$D_{ab}{}^{-1} = i[\Box + 2V_0^{(1)}(\varphi^2/N)(\delta_{ab} - \hat\varphi_a\hat\varphi_b)] + i[\Box + 2V_0^{(1)}(\varphi^2/N) + 4V_0^{(2)}(\varphi^2/N)/N]\hat\varphi_a\hat\varphi_b . \tag{A3}$$

In the previous formula the decomposition into transverse and longitudinal modes with respect to $\hat\varphi = \varphi/|\varphi|$ was performed. The analogous decomposition of G_{ab} yields

$$G_{ab} = g(\delta_{ab} - \hat\varphi_a\hat\varphi_b) + \tilde g\hat\varphi_a\hat\varphi_b . \tag{A4}$$

Finally, Γ_2 is the sum of all two-particle irreducible vacuum graphs built of the exact propagators and vertices generated by the Lagrangian written in terms of the shifted fields: $L(\Phi + \varphi)$. These graphs are visualized in Fig. 2. Diagrams of Fig. 2(a) yield

$$\Gamma_2^1 = -\int d^n x\, N\{V_0(\varphi^2/N + \hbar g) - V_0(\varphi^2/N) - \hbar g V_0^{(1)}(\varphi^2/N) + \hbar[\hbar g^2 V_0^{(2)}(\varphi^2/N + \hbar g) - (g - \tilde g)(V_0^{(1)}(\varphi^2/N - \hbar g) - V_0^{(1)}(\varphi^2/N)) + 2(\varphi^2/N) g (V_0^{(2)}(\varphi^2/N - \hbar g) - V_0^{(2)}(\varphi^2/N))]\} . \tag{A5}$$

The remaining graphs contribute only to the next-to-leading order, hence we are just in a position to evaluate g. (The leading-order result for Γ completely determines g as required for the next-to-leading-order calculations.[7]) In the (Minkowskian) momentum representation,

$$g = i/(k^2 - M^2), \tag{A6}$$

where

$$M^2 = 2V_0^{(1)}(\varphi^2/N + B_1) \tag{A7}$$

and

$$B_1 = i\hbar \int \frac{d^n p}{(2\pi)^n} \frac{1}{k^2 - M^2} . \tag{A8}$$

The contribution stemming from the graphs of Fig. 2(b) is

$$\Gamma_2^2 = -8\hbar \frac{\varphi^2}{N} \rho^2 \int \frac{d^n p}{(2\pi)^n} \frac{B_2(p)}{1 + 4\rho B_2(p)}, \tag{A9}$$

and from those of Fig. 2(c), the contribution is

$$\Gamma_2^3 = -\frac{\hbar}{2} \int \frac{d^n k}{(2\pi)^n} \ln[1 + 4\rho B_2(k)] + 2\hbar\rho \int \frac{d^n k}{(2\pi)^n} B_2(k), \tag{A10}$$

where

$$B_2(k) = -i\frac{\hbar}{2} \int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2 - M^2} \frac{1}{(k+p)^2 - M^2}, \quad \rho = V_0^{(2)}(\varphi^2/N + B_1) . \tag{A11}$$

Substituting all contributions into (A1), solving (A2) for g, then substituting the result back into (A1), finally changing for Euclidean momenta, we obtain the sum of (2.4) and (2.5).

APPENDIX B

Expanding $V^{(1)}(X_2)$ around $\overline{V}^{(1)}(X_2)$, as defined by (3.6), we get

$$V^{(1)}(X_2) = \overline{V}^{(1)}(X_2) + (1/N)\sum_k \delta^{(k)} \partial \overline{V}^{(1)}(X_2)/\partial \overline{V}^{(k)}(0) . \tag{B1}$$

Using (2.2), we obtain

$$\partial \overline{V}^{(1)}(X_2)/\partial \overline{V}^{(k)}(0) = \frac{1}{(j-1)!} X_2{}^{j-1} - \frac{\hbar}{4\pi M^2} \rho \frac{\partial M^2}{\partial \overline{V}^{(k)}(0)} + O(N^{-1}) . \tag{B2}$$

Using (3.4) we find

$$\partial \overline{V}^{(1)}(X_2)/\partial \overline{V}^{(k)}(0) = \frac{1}{(j-1)!} X_2 (1 + \hbar\rho/4\pi M^2)^{-1} . \tag{B3}$$

This can be written in a more convenient form. Let us notice that

$$\frac{\partial V^{(n)}(X_2)}{\partial \varphi^2/N} = V^{(n+1)}(X_2)\left(1 - \frac{\hbar}{4\pi M^2} \frac{\partial M^2}{\partial \varphi^2/N}\right) . \tag{B4}$$

In particular,

$$\frac{\partial M^2}{\partial \varphi^2/N} = 2\rho\left(1 - \frac{\hbar}{4\pi M^2} \frac{\partial M^2}{\partial \varphi^2/N}\right), \tag{B5}$$

thus

$$\frac{\partial V^{(n)}(X_2)}{\partial\varphi^2/N}=V^{(n+1)}(X_2)\frac{1}{2\rho}\frac{\partial M^2}{\partial\varphi^2/N} \tag{B6}$$

and, finally,

$$\frac{\partial V^{(1)}(X_2)}{\partial V^{(j)}(0)}=\frac{1}{(j-1)!}\left(\frac{\varphi^2}{N}-\frac{\hbar}{4\pi}\ln\frac{M^2}{\mu^2}\right)\times\frac{1}{2\rho}\frac{\partial M^2}{\partial\varphi^2/N}. \tag{B7}$$

Substituting (B7) into (B1) we get (3.7).

The above formulas remain valid also in three dimensions, but then we must make the following modifications:

$$(\hbar/4\pi)\ln(M^2/\mu^2),\ \text{replace by } \hbar M/4\pi;$$
$$\hbar/4\pi M^2,\ \text{replace by } \hbar/4\pi M. \tag{B8}$$

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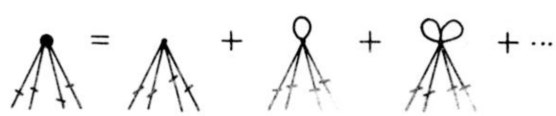

FIG. 1. The effective quartic vertex ρ .

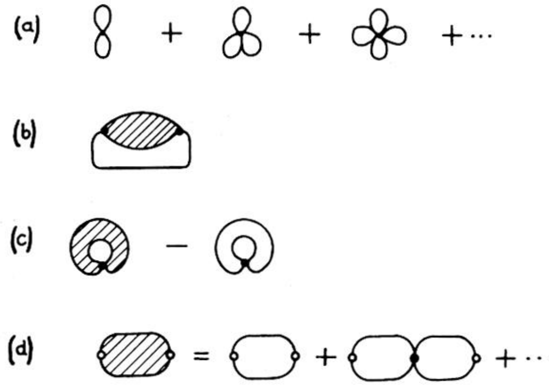

FIG. 2. Two-particle irreducible diagrams contibuting to Γ_2. (a) Graphs relevant in the leading and next-to-leading orders. (b) Contribution to the next-to-leading and higher orders. The shaded bubble represents the infinite chain of graphs presented on Fig. 2(d). The heavy dots stand for the effective three-legs vertex analogous to that of Fig. 1. The corresponding factors are $2\varphi_a \rho$. (c) Same as (b), except that the factor at the heavy dot is ρ. We reject the diagram stemming from the first member of the chain (d), because the corresponding contribution is that of Fig. 2(a). (d) The bubble chain of Figs. 2 (b) and 2 (c). Heavy dots represent ρ , factors at the ends are omitted.