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Anharmonic oscillator and the analytic theory of continued fractions

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We study anharmonic oscillators of the type $ax^2 + bx^4 + cx^6$ using the theory of continued fractions. Introducing a new set of coupling constants (depending on a, b, and c) in terms of which the associated difference equation simplifies, we write the Green's function of the theory in terms of an infinite continued fraction of the Stieltjes type, whose poles give the energy eigenvalues. We prove that this continued fraction converges where the corresponding perturbation series in the dominant coupling diverges. We obtain the analytic structure of the Green's function in the complex plane of this coupling constant. A scale transformation allows us to study the analyticity of the Green's function for $ax^2 + cx^6$ oscillators in the energy plane.

I. INTRODUCTION

The study of the quantum mechanics of the anharmonic oscillator is of considerable interest from both physical and mathematical points of view. The first detailed study of the properties of the perturbation series for the energy eigenvalues is due to Bender and Wu.[1] Simon[2] employed the theory of singular perturbations in Hilbert space to prove rigorously many of the properties conjectured by Bender and Wu and, generally, to present a rigorous study of the analyticity properties of the energy eigenvalues in the complex coupling-constant plane. One of his important results consisted of the proof that while the Rayleigh-Schrödinger perturbation series for the energy eigenvalues is divergent for all values of the coupling, arising from an essential singularity at the origin in the coupling-constant plane, it is asymptotic in an open domain around the origin. Numerical analyses starting from the perturbation series using Padé and Borel-Padé sums[3] confirmed the branch-point nature of the essential singularity. A nonperturbative method for the calculation of eigenvalues of λx^{2m} type oscillators using the infinite Hill determinant was developed by Biswas *et al.*[4] in a series of papers. The method, being nonperturbative, allowed the calculation of energy eigenvalues for arbitrary values of the coupling to confirm the asymptotic nature of the perturbation series and the singularity in the coupling constant at the origin. Following this approach, Hioe and Montroll[5] developed an iterative scheme for the computation of the energy of the nth excited level; the method is based on an expansion of the truncated Hill determinant arising from an orthogonal polynomial basis for the eigenfunctions. More recently, Graffi and Grecchi[6] have developed continued-fraction approximants to the Hill determinants arising in the study of λx^{2m} type oscillators.

It is our purpose here to study the properties of the oscillator potential of the type $ax^2 + bx^4 + cx^6$. As we have noted above, all the previous studies have considered potentials of the type λx^{2m}. For such potentials, studies which use the Hill determinant show that the difference equation from which the determinant arises, while only a three-term one, involves terms a_{n-3}, a_{n-1} and a_n. This feature presents considerable difficulties in the analytic study of such systems from the viewpoint of difference equations. In studying the $ax^2 + bx^4 + cx^6$ oscillator we find, however, that there exists for this problem a suitable three-term difference equation involving contiguous terms a_{n-1}, a_n, and a_{n+1} with appropriate coefficients. This allows, in this approach, an analytic study of the system in a fashion which was not possible in the case of λx^{2m} oscillators.

Our results may be summarized as follows: Introducing a new set of coupling constants α, β, and γ (which depend on a, b, and c) of which α is the coefficient of the dominant potential and in terms of which the difference equation simplifies, we show that the energy eigenvalues of this oscillator occur as poles in the energy plane of an infinite continued fraction which we may define as the Green's function for the problem, in analogy with standard results in quantum mechanics. We next show that this Green's function can be expressed as a Stieltjes fraction (known in the literature as an S fraction). This immediately allows us to establish that the Green's function $G(E)$ has, in the complex plane of the coupling constant α (β, γ, and the energy being held fixed), a domain of analyticity which consists of the entire plane, apart from the negative real axis. We further find that though this infinite continued fraction converges in the domain defined above, the perturbation series (in the same coupling constant) diverges, the divergence arising from a branch-point singularity at $\alpha=0$. An alternative representation for $G(E)$ results when use is made of the fact that the even part of an S fraction is a real J fraction. This representation allows us to show that $G(E)$ has a Lehmann-type spectral decomposition in the complex α plane.

From these results, we derive interesting properties for the ax^2+cx^6 type oscillator. We perform a scale transformation in the original problem and consider the limit $\beta\to 0$ which, for $c\neq 0$, requires $b\to 0$. The Schrödinger equation for this case then reduces to a three-term difference equation involving a_{n-1}, a_n, and a_{n+1} only, contrary to what was obtained earlier in discussions of λx^{2m} oscillators. The new feature which now appears is that the continued-fraction representation of $G(E)$ diverges by oscillation. However, the odd and even parts converge, and we conjecture that one of them represents the true Green's function for this problem since the $ax^2+bx^4+cx^6$ oscillator has a convergent continued-fraction representation for each fixed β, including $\beta=0$. The scale transformation also allows us, in this case, to study the analytic properties of $G(E)$ in the complex energy plane for each fixed α and obtain the well-known results that the poles of $G(E)$ are real and that $G(E)$ has a Lehmann spectral decomposition in this variable.

Finally, another of our results may be mentioned. The continued-fraction representation shows that for a particular set of values of the coupling parameters a, b, and c, it is possible to obtain exact solutions for a certain subset of the energy eigenvalues to which correspond polynomial eigenfunctions (weighted with the usual Gaussian). The remaining infinity of energy eigenvalues are obtainable from a reduced (infinite) continued fraction or, equivalently, from a reduced Hill determinant.

In Sec. II we formulate the difference equation, show that the energy eigenvalues are the zeros of an infinite continued fraction $G(E)$, and obtain the subset of exact solutions which result from particular values of the couplings. In Sec. III we use the analytic theory of continued fractions to study the analytic properties of $G(E)$ in the coupling constant α. The Stieltjes fraction form is obtained, its convergence and domain of analyticity established, and its relationship with the perturbation series examined. The J-fraction form and the Lehmann spectral representation in the coupling follows. In Sec. IV we study the ax^2+cx^6 oscillator and the analyticity of its Green's function in the energy plane using a scale transformation.

II. THE EIGENVALUE PROBLEM: STATEMENT AND SOLUTION

A. The difference equation and its solution

We start with the Schrödinger equation for the anharmonic oscillator in one dimension, viz.,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+(ax^2+bx^4+cx^6)\psi=E\psi. \quad (1)$$

Here $c>0$ and the sign of b is left free. a is the usual harmonic force constant and E the energy eigenvalue. We convert (1) to a difference equation and formulate the eigenvalue problem in terms of the zeros of an infinite continued fraction. We set

$$\psi=e^{f(x)}\phi(x), \quad (2)$$

where

$$f(x)=-\tfrac{1}{4}\alpha x^4+\tfrac{1}{2}\beta x^2, \quad \alpha>0. \quad (3)$$

α and β are constants as yet undetermined. Substituting Eq. (2) in (1) and choosing

$$\alpha=\left(\frac{2m}{\hbar^2}c\right)^{1/2}, \quad (4a)$$

$$\beta=-\frac{1}{2}\left(\frac{2m}{\hbar^2 c}\right)^{1/2} b, \quad (4b)$$

we obtain for $\phi(x)$ the following differential equation:

$$\frac{d^2\phi}{dx^2}+2(-\alpha x^3+\beta x)\frac{d\phi}{dx}+\left\{[\beta^2-3\alpha-(2m/\hbar^2)a]x^2+[(2mE/\hbar^2)+\beta]\right\}\phi=0. \quad (5)$$

We solve Eq. (5) by writing for $\phi(x)$ the power series

$$\phi(x)=\sum_{n=0}^{\infty}a_n x^{2n+v}, \quad (6)$$

where $v=0$ for the even-parity solution and $v=1$ for that of opposite parity. Substituting (6) in (5) we obtain the difference equation for the unknown coefficients, viz., the a_n's:

$$(2n+2+v)(2n+1+v)a_{n+1}+[\epsilon+\beta(4n+1+2v)]a_n+\alpha[\gamma-(4n-1+2v)]a_{n-1}=0, \quad (7)$$

where we have used

$$\epsilon=2mE/\hbar^2 \quad (8a)$$

and

$$\gamma=\left(\frac{2m}{\hbar^2}\right)^{1/2}\frac{1}{\sqrt{c}}\left(\frac{b^2}{4c}-a\right). \quad (8b)$$

From the difference equation it is easy to see that if $a_{-1}=0$, all a_{-k} for $k=2,3,\ldots$ vanish.

The difference equation (7) is an equivalent description of the original differential equation (1) for the eigenvalue problem. The necessary and sufficient condition that nontrivial a_n's (for $n=0,1,\ldots$) exist which solve (7) is that the following infinite determinant vanish:

$$0=\begin{bmatrix} \epsilon+\beta(2v+1) & (2+v)(1+v) & 0 & & & \cdots \\ \alpha(\gamma-3-2v) & \epsilon+\beta(5+2v) & (4+v)(3+v) & & & \cdots \\ & & \ddots & & & \\ \cdots & & \alpha\{\gamma-[4(n-1)+2v-1]\} & \epsilon+\beta[4(n-1)+2v+1] & [2(n-1)+v+2][2(n-1)+v+1] & \cdots \\ \cdots & & & \alpha[\gamma-(4n+2v-1)] & \epsilon+\beta(4n+2v+1) & \cdots \\ \cdots & & & & & \cdots \end{bmatrix}. \tag{9}$$

If D_{n+1} denotes the first $(n+1)\times(n+1)$ determinant, then

$$D_{n+1}=[\epsilon+\beta(4n+1+2v)]D_n-\alpha[\gamma-(4n+2v-1)][2(n-1)+v+2][2(n-1)+v+1]D_{n-1}. \tag{10}$$

The zeros of D_{n+1} in the energy parameter ϵ will determine the energy eigenvalues of the problem when $n\to\infty$. The numerical solution of this problem was discussed earlier in connection with the x^6 and x^4 anharmonic oscillators using these infinite determinants which are known as Hill determinants. In this paper we do not propose to proceed with such a numerical program. We merely note that the infinite number of the roots of D_n in the limit $n\to\infty$ gives rise to all the energy eigenvalues and the corresponding wave functions can be easily obtained from Eq. (6). Indeed, an exact expression for the coefficients a_n is given by

$$a_n=(-1)^n\frac{v!}{(2n+v)!}\begin{vmatrix} \epsilon+\beta(1+2v) & (2+v)(1+v) & & \cdots \\ \alpha(\gamma-3-2v) & \epsilon+\beta(5+2v) & (4+v)(3+v) & \cdots \\ & & \ddots & \\ \cdots & & \alpha[\gamma-(4(n-1)+2v-1)] & \epsilon+\beta(4n-3+2v) \end{vmatrix}, \tag{11}$$

where we have normalized a_0 to unity.

Thus the wave function

$$\psi=e^f\sum_n a_n x^{2n+v} \tag{12}$$

is completely known with a_n given by (11). In the limit when $n\to\infty$ the zeros of D_{n+1} give the eigenvalues and the corresponding a_n's are obtained from (11). It may be remarked that in the approximation procedure when one terminates the infinite determinants at the nth point and obtains the roots, the corresponding a_{n+1} vanishes as can be seen from (11), but all the other a_n's do not and these are finite. One also notes that the series (12) is convergent as $a_{n+1}/a_n\to O(1/\sqrt{n})$ for large n, and the wave functions are normalized whenever ϵ corresponds to an eigenvalue.

B. Continued-fraction representation and eigenvalues as zeros of the denominator

In this section we define a Green's function for the problem whose poles in the energy parameter occur as zeros of an infinite continued fraction. We first obtain the infinite-continued-fraction representation of the Green's function from the difference equation (7) and discuss the eigenvalue problem. It is easily verified from (7) that the ratio a_n/a_{n-1} is given by

$$\frac{a_n}{a_{n-1}}=\cfrac{-C_n}{B_n-\cfrac{A_nC_{n+1}}{B_{n+1}-\cfrac{A_{n+1}C_{n+2}}{B_{n+2}+\cdots}}} \tag{13}$$

where

$$A_n = (2n+1+v)(2n+2+v)\,,$$
$$B_n = \beta(4n+1+2v)+\epsilon\,, \qquad (14)$$
$$C_n = -\alpha(4n+2v-\gamma-1)\,.$$

In this notation our difference equation (7) reduces to

$$A_n a_{n+1} + B_n a_n + C_n a_{n-1} = 0\,. \qquad (15)$$

Noticing from (7) that

$$\frac{a_1}{a_0} = -\frac{\epsilon+\beta(1+2v)}{(v+1)(v+2)}\,,$$

we obtain from (13) and (14)

$$-\frac{\epsilon+\beta(1+2v)}{(v+1)(v+2)} = \cfrac{\alpha(3+2v-\gamma)}{\epsilon+\beta(5+2v)+\cfrac{\alpha(7+2v-\gamma)(v+3)(v+4)}{\epsilon+\beta(9+2v)+\cdots\;\epsilon+\beta(4n+1+2v)+\cfrac{\alpha(4n+2v+\gamma-1)(2n+2v+2)(2n+v+1)}{\epsilon+\beta(4n+5+2v)+\cdots}}}\,. \qquad (16)$$

Equation (16) gives the energy eigenvalues as the zeros of an infinite continued fraction, namely the equation

$$0 = \frac{\epsilon+\beta(1+2v)}{(v+1)(v+2)} + \cfrac{\alpha(3+2v-\gamma)(v+1)(v+2)}{\epsilon+\beta(5+2v)+\cdots\;\epsilon+\beta(4n+1+2v)+\cdots}\,. \qquad (17)$$

We define the right-hand side of this equation as the inverse of the equivalent Green's function whose zeros in the energy parameter are the energy eigenvalues.

C. Exact polynomial solutions of the problem

From (17) we note that the infinite continued fraction terminates as soon as we attribute some special values to γ which is an expression involving the harmonic and anharmonic potential strengths. If these are so arranged to ensure that

$$\gamma = 4n-1+2v\,, \quad n = 1, 2, \ldots \qquad (18)$$

the continued fraction terminates at the nth approximant, leading to an nth-order polynomial in ϵ. Thus we get a number n of energy eigenvalues and since the continued fraction terminates we obtain a similar number of (reduced) polynomial wave functions. This feature immediately raises the question as to what happens to the remaining eigenvalues as we must have for any anharmonic potential an infinity of solutions for all values of the harmonic and anharmonic coupling strengths. It is interesting to note that under condition (18) for which the continued fraction terminates, the equivalent Hill determinant as given by (9) factorizes into two determinants one of which is finite. Thus, for example, if we choose $\gamma = 7+2v$ (i.e., $n = 2$) the Hill determinant (9) breaks up as follows:

$$\begin{vmatrix} \epsilon+\beta(2v+1) & (2+v)(1+v) \\ \alpha(\gamma-3-2v) & \epsilon+\beta(5+2v) \end{vmatrix} \times \begin{vmatrix} \epsilon+\beta(9+2v) & (6+v)(5+v) & & \cdots \\ \alpha(\gamma-11-2v) & \epsilon+\beta(13+2v) & (8+v)(7+v) & \cdots \\ & \cdot & & \\ & \cdot & & \\ & \cdot & & \\ \cdots & \alpha\{\gamma-[4(n+2)+2v-1]\} & \epsilon+\beta[4(n+2)+2v+1] & \cdots \\ \cdots & & & \cdots \end{vmatrix} . \quad (19)$$

Here γ is $2v+7$. The finite number of roots will appear from the vanishing of the 2×2 determinant which appears at the top left-hand corner. The corresponding reduced eigenfunctions are polynomials. From Eq. (11) we easily note that in this case a_0 and a_1 are different from zero whereas all the a_n for $n \geq 2$ vanish. The remaining infinity of solutions are obtained from the zeros of the infinite determinant occurring on the lower right-hand side of Eq. (19). In other words these roots are the poles of the reduced Green's function

$$G^R(E) = \cfrac{1}{\epsilon+\beta(9+2v)+\cfrac{\alpha(11+2v-\gamma)(5+v)(6+v)}{\epsilon+\beta(13+2v)+\cdots}}, \quad (20)$$

where $\gamma = 2v+7$. The new coefficients a_n's for the eigenfunctions corresponding to the eigenvalues of the reduced infinite determinant are obtained from Eq. (11) and the requisite determinantal expression in (11) becomes

$$\begin{vmatrix} \epsilon+\beta(9+2v) & (6+v)(5+v) & \cdots \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ \cdots & \alpha\{\gamma-[4(n+1)+2v-1]\} & \epsilon+\beta[4(n+1)+2v+1] \end{vmatrix} . \quad (21)$$

The determinant (19) is being evaluated at $\gamma = 2v+7$. The generalization to the case $\gamma = 4n-1+2v$ is straightforward.

III. ANALYTIC PROPERTIES OF $G(E)$ USING THE THEORY OF CONTINUED FRACTIONS

A. The Stieltjes fraction form

Let us consider the difference equation (15). From this equation we have

$$\frac{a_1}{a_0} = \cfrac{-C_1}{B_1+\cfrac{(-A_1C_2)}{B_2+\cfrac{(-A_2C_3)}{B_3+ \cdots \; B_n+\cfrac{(-A_nC_{n+1})}{B_{n+1}+\cdots}}}} . \quad (22)$$

We now apply an equivalence transformation to this equation and reduce it to an S fraction. To this end we introduce a sequence of objects $\{\alpha_n\}$ such that

$$\alpha_{n+1}\alpha_n A_n C_{n+1} = -1 . \quad (23)$$

With the help of (23) and (15) we can rewrite (22) in the form

$$0 = B_0\alpha_0 + \cfrac{1}{B_1\alpha_1+\cfrac{1}{B_2\alpha_2+\cdots}} . \quad (24)$$

The right-hand side of (24) is $\alpha_0 A_0 G^{-1}(E)$ from our definition of the Green's function. Then we have

$$\alpha_0^{-1}A_0^{-1}G(E) = \cfrac{1}{B_0\alpha_0+\cfrac{1}{B_1\alpha_1+\cfrac{1}{B_2\alpha_2+\cdots}}} . \quad (25)$$

This is a Stieltjes continued fraction if

$$B_n\,\alpha_n > 0. \quad (26)$$

From (23) and (14) we easily obtain

$$\alpha_{2n+1}=\frac{1}{2^7\alpha}\frac{\Gamma\left(1+n+\frac{v-1}{4}\right)\Gamma\left(1+n+\frac{v}{4}\right)\Gamma\left(1+\frac{v-3}{4}\right)\Gamma\left(1+\frac{v-2}{4}\right)}{\Gamma\left(1+\frac{v-1}{4}\right)\Gamma\left(1+\frac{v}{4}\right)\Gamma\left(n+\frac{v+1}{4}+1\right)\Gamma\left(1+n+\frac{v+2}{4}\right)}\frac{\Gamma\left(1+n+\frac{2v-\gamma-1}{8}\right)\Gamma\left(1+\frac{2v-5-\gamma}{8}\right)}{\Gamma\left(1+\frac{2v-\gamma-1}{8}\right)\Gamma\left(1+n+\frac{3+2v-\gamma}{8}\right)}\frac{1}{\alpha_0}, \tag{27}$$

$$\alpha_{2n}=\frac{\Gamma\left(1+n+\frac{v-3}{4}\right)\Gamma\left(1+n+\frac{v-2}{4}\right)\Gamma\left(1+\frac{v-1}{4}\right)\Gamma\left(1+\frac{v}{4}\right)}{\Gamma\left(1+\frac{v-3}{4}\right)\Gamma\left(1+\frac{v-2}{4}\right)\Gamma\left(1+n+\frac{v-1}{4}\right)\Gamma\left(1+n+\frac{v}{4}\right)}\frac{\Gamma\left(1+n+\frac{2v-5-\gamma}{8}\right)\Gamma\left(1+\frac{2v-1-\gamma}{8}\right)}{\Gamma\left(1+\frac{2v-5-\gamma}{8}\right)\Gamma\left(1+n+\frac{2v-1-\gamma}{8}\right)}\alpha_0, \tag{28}$$

$$B_{2n+1}=\beta(8n+5+2v)+\epsilon, \tag{29}$$

and

$$B_{2n}=\beta(8n+1+2v)+\epsilon. \tag{30}$$

It is convenient to define a new set of quantities such that

$$B_{2n}\alpha_{2n}=k_{2n}, \tag{31}$$

$$B_{2n+1}\alpha_{2n+1}=k_{2n+1}(1/\alpha). \tag{32}$$

Thus $G(E)$ is given by the continued fraction

$$G(E)=\cfrac{1}{k_0+\cfrac{1}{k_1\tilde{\beta}+\cfrac{1}{k_2+\cfrac{1}{k_3\tilde{\beta}+\cdots}}}} \tag{33}$$

where $\tilde{\beta}=1/\alpha$.

From the expressions (27) to (30) for B_n and α_n we find that if α_0 is chosen to have the same sign as β and if the couplings a, b, and c are chosen so that $\gamma<2v+3$, all the k_n defined above are positive. $G(E)$ as given by (33) is then an S fraction. From the set of equations (27) to (30) and from the defining relations (31) and (32) we find, using Stirling's approximation, that $\alpha_n \sim n^{-3/2}$ for large n. Thus $k_n \sim n^{-1/2}$ in the same limit, causing $\sum k_n$ to diverge. We have thus proved the following theorem:[7]

Theorem I. If $\gamma<2v+3$, the function $G(E)$ as a function of $\tilde{\beta}$ $(=1/\alpha)$ for fixed β, γ, and ϵ is uniformly convergent over a finite closed domain of $1/\alpha$ whose distance from the negative half of the real axis is positive. Its value is an analytic function of the coupling constant α for all $1/\alpha$ not on the negative half of the real axis.

However, if the condition $\gamma<2v+3$ is relaxed, the choice $\operatorname{sgn}\alpha_0=\operatorname{sgn}\beta$ ensures only the positivity of all k_n for $n\geq m$, m being finite. In this case we have the same theorem for that infinite part of $G(E)$ which we denote by $\tilde{G}^R(E)$ and which is given by the following relation:

$$G(E)=\cfrac{1}{k_0+\cfrac{1}{k_1\tilde{\beta}+\cfrac{1}{k_2+\cfrac{1}{\ddots+\cfrac{1}{k_{n-1}\tilde{\beta}+\tilde{G}^R(E)}}}}} \tag{34}$$

(if, e.g., n is even). We also note that $G(E)$ may be obtained from $\tilde{G}^R(E)$ using the equation

$$G(E)=\frac{P_{n-1}\tilde{G}^R(E)+P_n}{Q_{n-1}\tilde{G}^R(E)+Q_n}, \tag{35}$$

where P_n, Q_n satisfy the recursive relations

$$P_{n+1}=b_{n+1}P_n+P_{n-1},\qquad Q_{n+1}=b_{n+1}Q_n+Q_{n-1}, \tag{36}$$

with

$$P_{-1}=1,\quad P_0=Q_{-1}=0,\quad Q_0=1,$$

$$b_{2n+1}=k_{2n},\quad b_{2n}=k_{2n-1}\tilde{\beta}.$$

Having established the domain of analyticity of the Green's function in the complex plane of the coupling constant α, we would like to establish the behavior of the power series obtained on expansion of $G(E)$ in powers of this coupling.

B. Relation of the perturbation expansion to the continued-fraction representation of $G(E)$

Equation (33) can also be written as

$$\alpha G(E)=\cfrac{1}{k_0\tilde{\beta}+\cfrac{1}{k_1+\cfrac{1}{k_2\tilde{\beta}+\cdots}}}\,. \tag{37}$$

We introduce new quantities d_n through the relations

$$d_n=1/k_{n-1}k_n\,. \tag{38}$$

With the help of (35) we can recast (34) as follows:

$$k_0G(E)=\cfrac{1}{1-\cfrac{d_1(-\alpha)}{1-\cfrac{d_2(-\alpha)}{1-\cfrac{d_3(-\alpha)}{\ddots\, 1-\cfrac{d_n(-\alpha)}{\ddots}}}}}\,. \tag{39}$$

This is another standard form of the Stieltjes S fraction. It is well known that this S fraction can be expanded in a power series in $(-\alpha)$ and that the series is unambiguously defined by the S fraction.[8] We write the power series as

$$k_0G(E)=C_0+C_1(-\alpha)+C_2(-\alpha)^2+\cdots\,. \tag{40}$$

The coefficients C_n are all positive. This series may be regarded as the perturbation expansion of the Green's function in powers of the coupling constant α for fixed β and γ. Regarding the nature of this perturbation, we have the following results due to Stieltjes[9,10]:

If $\sum_n k_n$ diverges, the continued-fraction representation (37) converges and we may distinguish between the following three cases:

(1) If $d_n\to 0$ for large n, then the perturbation expansion is a meromorphic function of α. The poles in α are on the positive real axis of α and the perturbation expansion converges for α sufficiently small.

(2) If for $n\to\infty$, $d_n\to d\neq 0$, a finite number, then $G(E)$ has at most polar singularities in α in the region exterior to the cut going from $(1/4d)$ to ∞ over the real axis. The perturbation series converges for d sufficiently small.

(3) If $d_n\to\infty$ when $n\to\infty$, then $G(E)$ has an essential singularity at $\alpha=0$ and the perturbation series (37) diverges.

In our case for $G(E)$ as given by (39) we find easily that $\sum_n k_n$ diverges and that $d_n\to\infty$ as $n\to\infty$. Hence we have the following theorem for our Green's function:

Theorem II. The continued-fraction representation (39) or (37) for $G(E)$ is convergent, but the corresponding perturbation series in powers of the coupling constant α for fixed β, γ, and ϵ is divergent and $G(E)$ has an essential singularity at $\alpha=0$ in the complex α plane.

We thus see that the continued-fraction representation is a good method of summing up the perturbation series even if the perturbation series diverges. The continued fraction is convergent and may be interpreted as an analytic continuation of the relevant perturbation series.

C. J-fraction representation of $G(E)$ and analyticity in the coupling constant: the Lehmann spectral representation

We have shown in Sec. III A that the Green's function $G(E)$ has a Stieltjes S-fraction form. It is well known[11] that the even part of a Stieltjes S fraction with $k_n>0$ is a real J fraction; in other words, the even approximants of the S fraction are the successive approximants of a real J fraction which also represents the function $G(E)$. Since the S fraction converges, both the even and odd approximants converge to the same limit which is the unique value of the infinite S fraction. Thus our $G(E)$ is also represented by the associated real J fraction. The J-fraction representation of our S fraction as given in (34) has the following form:

$$G(E)=\cfrac{1/k_0}{c_1+z-\cfrac{c_1c_2}{c_2+c_3+z-\cfrac{c_3c_4}{\ddots}}}\,, \tag{41}$$

where z stands for the coupling constant and the c_p are given by

$$c_p=1/k_pk_{p+1}\,,\quad p=0,1,\ldots\,.$$

For $k_p>0$ (41) is a real positive-definite J fraction. We thus obtain the following results for our $G(E)$[12]:

Theorem III. The zeros of the denominator of the J fraction (41) in the parameter z are all real.

Theorem IV. $G(E)$ converges uniformly over every finite closed region in the complex z plane whose distance from the real z axis is positive, and its value in each of the imaginary half planes is an analytic function of z in that half half plane.

Theorem V. If $\tilde{A}_p/\tilde{B}_p$ denotes the pth approximant[13] of (41) then this partial fraction has the further development

$$\sum_{r=1}^{p} \frac{L_r}{z+x_r},$$

where $L_r > 0$, $\sum L_r = 1/k_0$, and $0 < x_1 < x_2 < \cdots < x_p$. Thus from theorem V we obtain that the Green's function has a Lehmann-type spectral representation in the coupling-constant plane.

IV. SCALE TRANSFORMATION AND THE cx^6 ANHARMONIC OSCILLATOR

In Eq. (1) we make scale transformation

$$x = yE^{-1/2}$$

(where we have used $2m = \hbar = 1$). Then the difference equation (7) reduces to the form

$$(2n+2+v)(2n+1+v)b_{n+1} + [1+\beta'(4n+1+2v)]b_n + \alpha'[\gamma-(4n+2v-1)]b_{n-1} = 0, \quad (42)$$

where $\beta' = \beta/E$ and $\alpha' = \alpha/E^2$.

This result allows us to study the analytic behavior of the Green's function in the energy parameter also through α' $(=\alpha/E^2)$. We thus see from (42) and our earlier results of Sec. III that $G(\alpha')$ is analytic in the prescribed domain as shown in Sec. III for fixed β' and γ. In particular if we set $\beta=0$, then for fixed α, $G(\alpha'=\alpha/E^2)$ is a function of the energy parameter E. *Similar theorems as established in Secs. III A, III B and III C for the coupling constant are valid for the energy variable E, for a fixed value α.* These results in the case of $\beta=0$ are well known since in this limit our Schrödinger equation (1) becomes that for a pure x^6 anharmonic oscillator.[2,4] Nevertheless it is of some interest to note the following: In the limit when $\beta=0$, we easily obtain from Eqs. (27) to (30) that the S fraction (34) which represents the Green's function $G(E)$ has the property that the series $\sum k_n$ converges. Consequently, we have the result that[14] the continued fraction is divergent by oscillation for every value of α'. Thus the sequence of even and odd approximants of (37) converges uniformly over every closed region whose distance from the real α' axis is positive, to different distinct meromorphic functions $p(\alpha')/q(\alpha')$ and $p_1(\alpha')/q_1(\alpha')$, respectively, whose poles are all on the negative real axis. We thus obtain two different limiting Green's functions, and in such a case the perturbation series is always divergent. We thus conjecture that the even part of the S fraction, viz., the real positive-definite J fraction (which is convergent) is the true Green's function for the ax^2+cx^6 oscillator and its analytic continuation for $b\neq 0$ is the Green's function for the $ax^2+bx^4+cx^6$ oscillator which we have obtained earlier as a convergent S fraction. Further, since the even part of an S fraction is a real positive-definite J fraction, we conclude that $p(\alpha')/q(\alpha')$ has a Lehmann-type representation in E and that the zeros of the denominator function $q(\alpha'=\alpha/E^2)$ are all real. These will correspond to the real energy eigenvalues of the x^6 anharmonic oscillator.

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[1]C. M. Bender and T. T. Wu, Phys. Rev. Lett. 21, 406 (1968); Phys. Rev. 184, 1231 (1969).

[2]B. Simon, Ann. Phys. (N.Y.) 58, 76 (1969).

[3]J. J. Loeffel, A. Martin, B. Simon, and A. S. Wightman, Phys. Lett. 30B, 656 (1970); S. Graffi, V. Grecchi, and B. Simon, *ibid.* 32B, 631 (1970); B. Simon, Ann. Phys. (N.Y.) 58, 76 (1969); P. M. Mathews and T. R. Govindarajan, Pramana 8, 371 (1977).

[4]S. N. Biswas, K. Datta, R. P. Saxena, P. K. Srivastava, and V. S. Varma, Phys. Rev. D 4, 3617 (1971); J. Math. Phys. 14, 1190 (1973).

[5]F. T. Hioe and E. W. Montroll, J. Math. Phys. 16, 1945 (1975).

[6]S. Graffi and V. Grecchi, Lett. Nuovo Cimento 12, 425 (1975).

[7]H. S. Wall, *Analytic Theory of Continued Fractions* (Van Nostrand, New York, Toronto, London, 1948), Theorem 28.1, p. 120.

[8]H. S. Wall, Ref. 7, Theorem 52.1, p. 202.

[9]H. S. Wall, Ref. 7, Theorem 54.2, p. 209.

[10]T. J. Stieltjes, Recherches sur les fractions continués Oeuvres (unpublished), Vol. 2, p. 402; C. R. Acad. Sci. CXVIII, 1401 (1894).

[11]H. S. Wall, Ref. 7, Chap. VI, p. 119.

[12]H. S. Wall, Ref. 7, Theorem 27.1, p. 114 and Theorem 27.2, p. 114.

[13]The pth approximant $\tilde{A}_p/\tilde{B}_p$ to a continued fraction of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_n}{b_n + \cdots}}}$$

is given by the recurrence relations

$$\tilde{A}_{p+1} = b_{p+1}\tilde{A}_p + a_{p+1}\tilde{A}_{p-1},$$
$$\tilde{B}_{p+1} = b_{p+1}\tilde{B}_p + a_{p+1}\tilde{B}_{p-1},$$

with $\tilde{A}_{-1}=1$, $\tilde{B}_{-1}=0$, $\tilde{A}_0=b_0$, $\tilde{B}_0=1$.

[14]H. S. Wall, Ref. 7, Theorem 28.2, p. 120.