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# Constraint formalism of classical mechanics

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The constraint formalism for classical mechanics is developed with an eye toward facilitating a manifestly covariant relativistic quantization of classical mechanics. The very close relationship of this formalism to the more familiar Hamiltonian formalism has long obscured its independent status. In this paper, after defining the concept of a classical trajectory in a particularly natural and intrinsic way (that is, without reference to a special parameter such as the time), we prove that the trajectories satisfy the Hamilton equations of motion. In the following paper, the power of this formalism is demonstrated, when as an illustrative example it is employed to solve the nontrivial relativistic two-body problem, both classically and quantum mechanically.

## I. INTRODUCTION

The various formalisms of classical mechanics each have their own specialized advantages. Thus the Lagrangian formalism is particularly suited to exhibit the symmetry properties of a theory, in particular, its relativistic covariance, and the relationship of these symmetries to the constants of motion. The Hamiltonian formalism, while not readily adapted to manifest relativistic covariance due to the singling out of a preferred time variable, is ideally suited for the transition to quantum mechanics. The Hamilton-Jacobi formalism has the unique attribute of exhibiting the classical analog of the quantum state.

In an earlier paper[1] we discussed various formalisms of classical physics and showed to what extent they can be regarded as giving equivalent descriptions of nature, namely, that their respective algebras of observables are homomorphic. In that paper we described a less familiar classical formalism which we called the constraint formalism and developed some of its properties. In the present paper we shall develop the constraint formalism somewhat further. Our particular interest in this formalism stems from the fact that it combines many of the virtues of the Lagrangian, the Hamiltonian, and the Hamilton-Jacobi formalisms. That is, it facilitates a manifestly Poincaré-covariant transition to quantum theory. The problem of a nontrivial relativistic system of interacting particles is most naturally handled in this formalism. However, since this problem has long and intricate antecedents, we will present this important application of the constraint formalism in the following paper, rather than relegating it to the concluding sections of the present paper. We should note that our long-term interest in the constraint formalism derives from our desire to obtain a manifestly covariant general-relativistic quantum field theory. That generalization will not be treated in the present paper.

## II. KINEMATICS

The arena of the constraint formalism is a phase space, that is, an even-dimensional space with a symplectic form. If we adapt a set of coordinates to the symplectic form we may define a set of Poisson brackets in the usual manner. Thus we obtain

$$\{q^A, q^B\} \equiv \{p_A, p_B\} \equiv 0,$$
$$\{q^A, p_B\} \equiv -\{p_B, q^A\} \equiv \delta^A_B, \tag{2.1}$$

and more generally

$$\{A(q^M, p_N), B(q^M, p_N\} \equiv \frac{\partial A}{\partial q^M}\frac{\partial B}{\partial p_M} - \frac{\partial A}{\partial p_M}\frac{\partial B}{\partial q^M}, \tag{2.2}$$

where $A$ and $B$ are arbitrary scalar fields defined on phase space, and we employ a summation convention on repeated indices. (We employ the *identity* equality at this point to emphasize that such relations are to hold everywhere in the phase space, that is, including those regions which will subsequently prove to be nonphysical.) Since the machinery of phase space is quite familiar to us from the Hamiltonian fromalism, there is no need to repeat the proofs of well-known results. The reader need only refer to any standard text in classical mechanics.[2] In this spirit, if by a canonical mapping we mean a curvilinear coordinate transformation which preserves the given symplectic form, we note that any function of the coordinates of the phase space $A(q^M, p_N)$ may be regarded as the generator of an infinitesimal canonical transformation via the relations

$$\delta_A q^M \equiv \{q^M, A\},$$
$$\delta_A p_M \equiv \{p_M, A\}. \tag{2.3}$$

It is a consequence of Eqs. (2.2) and (2.3) that under an infinitesimal canonical transformation generated by $A$, any other function $B(q^M, p_N)$ defined on this phase space will transform accordingly:

$$\delta_A B \equiv \{A, B\}. \tag{2.4}$$

## III. DYNAMICS

The distinguishing feature of the constraint formalism is the manner in which the dynamics is stated. Let us be given a set of $k$ functions over the phase space $K_i(q^M, p_N)$, $i=1,\ldots,k$, which satisfy the set of relations

$$\{K_i, K_j\} \equiv \sum_{m=1}^{k} \lambda_{ij}^m K_m \tag{3.1}$$

(where $\lambda_{ij}^m$ are totally arbitrary, and need not be constants). That is, the functions $K_i$ form a first-class system in the sense of Dirac.[3] We define the set of $k$-constraint equations as follows:

$$K_i = 0, \quad i=1,\ldots,k. \tag{3.2}$$

These relations define a lower-dimensional surface in the phase space, the constraint hypersurface. It is an evident consequence of Eqs. (3.1) and (2.4) that this hypersurface is invariant under infinitesimal canonical transformation generated by functions $K_i$.

Thus if we start with any point lying within the constraint hypersurface of Eq. (3.2), and map it via iterating the infinitesimal canonical transformations generated by the functions $K_i$, we obtain an equivalence class of points lying entirely within the constraint hypersurface. *We define such an equivalence class to be a generalized dynamical tractory.* (For the case where $k=1$, it will be a trajectory in the ordinary sense, namely, a one dimensional path.) The virtue of this manner of definition of a trajectory is that it makes no explicit reference to a time parameter, and thus can be made in a manifestly relativistic covariant manner.

As a guide for the reader's intuition let us briefly exhibit two simple nonrelativistic examples:

(i) The three-dimensional simple harmonic oscillator. Let us consider an eight-dimensional phase space labeled by the canonical coordinates $q^0$, $\vec{q}$, $p^0$, $\vec{p}$. The constraint hypersurface for this physical system is seven dimensional, and is given by the single constraint equation

$$K_{\text{SHO}} \equiv p^0 - \left(\frac{\vec{p}^{\,2}}{2m} + \frac{k\vec{q}^{\,2}}{2}\right) = 0. \tag{3.3}$$

The usual six-dimensional phase space of the Hamiltonian formalism is obtained by factoring the constraint hypersurface modulo the one-dimensional equivalence class of trajectories generated by $K_{\text{SHO}}$.

(ii) Two particles interacting via a Kepler force. This physical system is particularly instructive for there is more than one natural way to treat it. Analogous to the previous example we can consider a 14-dimensonal phase space labeled by the canonical coordinates $q^0$, $\vec{q}_1$, $\vec{q}_2$, $p^0$, $\vec{p}_1, \vec{p}_2$. The constaint hypersurface for this system is the 13-dimensional surface given by the single constraint equation

$$K_k \equiv p^0 - \left(\frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} - \frac{k}{|\vec{q}_1 - \vec{q}_2|}\right) = 0. \tag{3.4}$$

As in the previous example, the usual 12-dimensional phase space is recovered by factoring the constraint hypersurface modulo the one-dimensional equivalence class of trajectories generated by $K_k$. However, a much more interesting, and in many ways, a much more powerful treatment of this dynamical system is obtained by considering a 16-dimensional phase space labeled by the canonical coordinates $q_1^0$, $\vec{q}_1$, $q_2^0$, $\vec{q}_2$, $p_1^0$, $\vec{p}_1$, $p_2^0$, $\vec{p}_2$. The constraint hypersurface for this system is now the 14-dimensional surface given by the following two *commuting* constraints:

$$K_1 = p_1^0 + p_2^0 - \left(\frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} - \frac{k}{|\vec{q}_1 - \vec{q}_2|}\right) = 0, \tag{3.5}$$

$$K_2 \equiv q_1^0 - q_2^0 = 0. \tag{3.6}$$

The equivalence class of generalized trajectories generated by $K_1$ and $K_2$ is now two dimensional. Thus by factoring the constraint hypersurface modulo the trajectories, we again recover the 12-dimensional phase space of the Hamiltonian formalism. (We leave it to the entertainment of the interested reader to verify that these two radically different ways of treating the Kepler problem are equivalent in the sense of Ref. 1.)

We see from this latter example that the constraint formalism is really not one formalism, but rather a family of related structures which can be adapted to give equivalent treatments of a given dynamical system. It is this freedom to modify the components of the structure (such as the dimensions of the phase space, or the number and functional form of the constraints) which gives this formalism its power. For we are now free to introduce other desiderata, in particular, manifest covariance and facility for quantization.

## IV. OBSERVABLES

In order to relate the present formalism to the more familiar descriptions of physical systems we introduce the concept of observables. We define an observable $Q(q^M, p_N)$ to be a function over the phase space which generates an infinitesimal canonical transformation which leaves invariant the constraint hypersurface given by Eq. (3.2). Thus, from Eq. (2.4), we see that a necessary and sufficient condition that a quantity $Q$ be an observable is

$$\{Q, K_i\}=0, \quad i=1, \ldots, k, \tag{4.1}$$

where the *ordinary* equality sign is employed throughout this paper whenever the relation need hold only modulo the constraint equations (3.2), and not necessarily everywhere in the phase space. With this notation, it is a consequence of Eq. (3.1) that

$$\{K_i, K_j\}=0, \quad i,j=1, \ldots, k. \tag{4.2}$$

It is evident from our definition, as well as from this relation, that the constraints themselves are observables. We distinguish between the constraints and the remaining observables in so far as we employ the constraints to define trajectories. The remaining observables then generate canonical mappings which preserve trajectories intact, merely permuting them within the constraint hypersurface. We should emphasize, however, that there is no requirement that the observables (other than the constraints) mutually commute.

As to the actual construction of observables, it can be accomplished in a number of ways. One particularly convenient procedure is to introduce as an additional set of constraints quantities which are canonically conjugate to the given constraints, Eq. (3.2). The combined set of relations then forms a second-class system in the sense of Dirac.[3] The observables can then be constructed via the Bergmann-Komar starring procedure.[4] We shall not be concerned with this construction in the present paper, since, in effect, it amounts to the reintroduction of preferred "timelike" parameters into the formalism thereby vitiating much of the impetus for its construction.

## V. HAMILTON-JACOBI THEORY

Let us for the moment consider a complete *commuting* set of observables. That is, consider a set $Q_M$, which in addition to satisfying Eq. (4.1), also satisfies

$$\{Q_M, Q_N\}=0. \tag{5.1}$$

The completeness condition is in this case understood to be given by the inequality

$$\left|\frac{\partial Q_M}{\partial p_N}\right| \neq 0. \tag{5.2}$$

Such a complete set of observables will in general include the constraints $K_i$. Let us assign a set of constant values to this set of observables such that

$$Q_R = C_R. \tag{5.3}$$

[Of course, those constants $C_R$ which are associated with constraint observables must be assigned to vanish, in view of Eq. (3.2).]

In view of the completeness condition, Eq. (5.2), we may regard Eq. (5.3) as defining the canonical momenta $p_N$ as implicit functions of the canonical coordinates $q^M$, and the constants $C_R$. Differentiating Eq. (5.3) with respect to $q^M$ we therefore obtain

$$\frac{\partial Q_R}{\partial q^M}+\frac{\partial Q_R}{\partial p_N}\frac{\partial p_N}{\partial q_M}=0. \tag{5.4}$$

If we multiply this equation by $\partial Q_S/\partial p^M$, antisymmetrize with respect to the indices $R$ and $S$, and refer to the definition of the Poisson brackets Eq. (2.2), we find that

$$\{Q_R, Q_S\}+\frac{\partial Q_R}{\partial p_M}\frac{\partial Q_S}{\partial p_N}\left(\frac{\partial p_M}{\partial q^N}-\frac{\partial p_N}{\partial q^M}\right)=0. \tag{5.5}$$

As a consequence of Eqs. (5.1) and (5.2) we obtain

$$\frac{\partial p_M}{\partial q^N}=\frac{\partial p_N}{\partial q^M}. \tag{5.6}$$

We therefore conclude that there exists a function $S(q^M)$, such that

$$p_M=\frac{\partial S}{\partial q^M}. \tag{5.7}$$

Since the set of constraints $K_i$ were assumed to be included among our complete commuting set of observables, we may substitute Eq. (5.7) into the constraint equations, (3.2), thereby obtaining the (generalized) set of Hamilton-Jacobi equations. We conclude therefore that the Hamilton-Jacobi function is determined uniquely up to an arbitrary additive constant by the values $C_R$, assigned to a complete commuting set of observables.

## VI. CONSTANTS OF THE MOTION

We shall now consider a phase space for $k$ particles, coordinated by the $4k$ canonical variables $q_i^\mu$, $p_{i\mu}$, $i=1, \ldots, k$, $\mu=0, \ldots, 3$. Even though we are for the most part confining our considerations to nonrelativistic mechanics, it is convenient to assume a Minkowski signature $(+,-,-,-)$ for raising and lowering Greek indices. For heuristic reasons it will be convenient in the sequel to separate the space and time variables. The expression for the Poisson brackets, Eq. (2.2), may now be written as

$$\{A,B\}\equiv\frac{\partial A}{\partial q_i^0}\frac{\partial B}{\partial p_i^0}-\frac{\partial A}{\partial p_i^0}\frac{\partial B}{\partial q_i^0}-[A,B], \qquad (6.1)$$

where the square brackets are the standard Poisson bracket expression of the Hamiltonian formalism.[2] (The negative sign is merely due to our utilization of the Minkowski signature in raising the spatial subscripts of the momenta.)

As we have indicated in the second example of Sec. III, there are many ways in which we can introduce the constraints. We now find it instructive to consider the $k$ constraints as follows:

$$K_i\equiv p_i^0-H_i(q_j^0,\vec{q}_j,\vec{p}_i)=0,\quad i,j=1,\ldots,k. \qquad (6.2)$$

While, at present, the reader may view this as an alternative form for treating a system of many particles, in fact, this form of the constraints is in part motivated by our intention ultimately to extend the formalism to field theories, in which case several free parameters are needed to label points of trajectory. The expressions for $H_i$, the generalized Hamiltonians, are not arbitrary, but as a consequence of Eq. (4.2) must satisfy

$$\frac{\partial H_j}{\partial q_i^0}-\frac{\partial H_i}{\partial q_j^0}-[H_i,H_j]=0. \qquad (6.3)$$

In our present notation the condition for an observable, Eq. (4.1), now becomes

$$\frac{\partial Q}{\partial q_j^0}+\frac{\partial Q}{\partial p_m^0}\frac{\partial H_j}{\partial q_m^0}+[Q,H_j]=0. \qquad (6.4)$$

Let us now define $Q^*$, the restriction of $Q$ to the constraint hypersurface:

$$Q^*(q_j^0,\vec{q}_j,\vec{p}_j)=Q(q_j^\mu,p_j^\mu)\big|_{p_j^0=H_j}. \qquad (6.5)$$

It therefore follows that

$$\frac{\partial Q^*}{\partial q_j^0}=\frac{\partial Q}{\partial q_j^0}+\frac{\partial Q}{\partial p_m^0}\frac{\partial H_m}{\partial q_j^0} \qquad (6.6)$$

and

$$[Q^*,H_j]=[Q,H_j]+\frac{\partial Q}{\partial p_m^0}[H_m,H_j]. \qquad (6.7)$$

Substituting these last two equations into Eq. (6.4), and employing Eq. (6.3), we obtain after some manipulation the following:

$$\frac{\partial Q^*}{\partial q_j^0}+[Q^*,H_j]=0. \qquad (6.8)$$

It was evident from our original definition of observables as quantities which commute with the constraints, and from our definition of trajectories as the orbits generated by the constraints, that observables are constant along a trajectory. The purpose of the demonstration of Eq. (6.8) is to show how the more familiar expression for the constant of motion in the Hamiltonian formalism can be recovered. [We chose to accomplish this in a slightly more general scheme than the usual single-constraint Hamiltonian formalism in order to illustrate how one can handle multidimensional trajectories. If the reader prefers, he may instead refer to our first treatment of the Kepler problem in Sec. III, with the single constraint given by Eq. (3.4).]

We note that the same set of relations and manipulations which led to the derivation of Eq. (6.8), can also be employed to demonstrate that

$$\{Q_A,Q_B\}=-[Q_A^*,Q_B^*]. \qquad (6.9)$$

Thus in order to determine whether observables commute on the constraint hypersurface it is sufficient to consider the usual Poisson brackets of their respective restrictions.

## VII. GENERALIZED HAMILTON EQUATIONS OF MOTION

We have shown that observables are constant along each trajectory. If we have sufficiently many observables at our disposal we can employ them to specify the trajectory uniquely. The trajectory specified will satisfy dynamical equations of motion. With this in mind, let us define a set of observables to be complete if it satisfies the inequality

$$\left|\frac{\partial Q_i^*}{\partial \vec{q}_j},\frac{\partial Q_i^*}{\partial \vec{p}_j}\right|\neq 0\quad (i=1,\ldots,6k;\ j=1,\ldots,k). \qquad (7.1)$$

(Note that in the present case we cannot employ the constraint observables as their restriction vanishes identically.) Consider the trajectory determined by assigning constant values $C_i$ to such a complete set:

$$Q_i^*=C_i,\quad i=1,\ldots,6k. \qquad (7.2)$$

The canonical variables $\vec{q}_i$, $\vec{p}_i$, thereby become implicit functions of the $q_j^0$. Since the definition of the trajectory is not at our disposal, the equations of motion of these canonical variables are fully determined. Differentiating Eq. (7.2) with respect to $q_j^0$ we obtain

$$\frac{\partial Q_i^*}{\partial q_j^0}+\frac{\partial Q_i^*}{\partial \vec{q}_m}\frac{\partial \vec{q}_m}{\partial q_j^0}+\frac{\partial Q_i^*}{\partial \vec{p}_m}\frac{\partial \vec{p}_m}{\partial q_j^0}=0. \qquad (7.3)$$

Since each of the observables $Q_i^*$ must satisfy Eq. (6.8), we find by comparing with the above equation that

$$\frac{\partial Q_i^*}{\partial \vec{q}_m}\cdot\frac{\partial \vec{q}_m}{\partial q_j^0}+\frac{\partial Q_i^*}{\partial \vec{p}_m}\cdot\frac{\partial \vec{p}_m}{\partial q_i^0}=\frac{\partial Q_i^*}{\partial \vec{q}_m}\cdot\frac{\partial H_j}{\partial \vec{p}_m}-\frac{\partial Q_i^*}{\partial \vec{p}_m}\cdot\frac{\partial H_i}{\partial \vec{q}_m}. \qquad (7.4)$$

By virtue of Eq. (7.1) we can now conclude that

$$\frac{\partial \vec{q}_i}{\partial q_j^0} = \frac{\partial H_j}{\partial \vec{p}_i}, \qquad \frac{\partial \vec{p}_i}{\partial q_j^0} = -\frac{\partial H_j}{\partial \vec{q}_i}. \tag{7.5}$$

We have thus succeeded in *deriving* the generalized set of Hamilton equations of motion. (Again if the reader prefers to confine himself to a single constraint treatment he readily recovers the standard Hamilton equations of motion.)

In this process of having established the full equivalence of the constraint formalism with that of Hamiltonian mechanics, we learn the manner by which a functional dependence is established between what were initially independent canonical coordinates of the phase space, namely, $q_j^0$, and the remaining canonical variables. The power of the constraint formalism, however, lies in our never having to impose such a dependence; the definition of the trajectory is determined by the constraints alone. In fact, when we look ahead to the quantization of such theories we see that the ability to parametrize the individual points of a classical trajectory is neither very useful nor indeed particularly meaningful. It will be seen that the critical feature of the mechanical system is the algebraic form of the constraints. In fact, the essential virtue of the constraint formalism, which makes it particularly convenient for quantization, is that it is totally algebraic—not even "time" derivatives need enter into its statement of the dynamical laws.

## VIII. CONCLUSION

In this paper we have established the constraint formalism as an independent formalism of classical mechanics, although it is frequently confused with the superficially similar Hamiltonian formalism. We have also shown how to relate it to the more familiar Hamiltonian and Hamilton-Jacobi formalisms. In view of the fact that most of the results are quite familiar and certainly none of the physics, up to this point, is new, the reader may well question the point of this exercise. The value of a new approach ultimately rests on the increased insight it may provide toward the resolution of outstanding questions. We shall briefly allude to several such areas.

From the perspective of the constraint formalism, the natural definition of a symmetry group is a Lie group generated by a closed algebra of observables. Thus, to take the trivial case of a free particle, defined (in an eight-dimensional phase space represented by the canonical variables $q^\mu$, $p_\mu$) by the constraint

$$K \equiv p^0 - \frac{\vec{p}^{\,2}}{2m} = 0, \tag{8.1}$$

it is easy to see that a complete nonredundant algebra of observables is given by the ten quantities $p^0, \vec{p}, \vec{q}\times\vec{p}, m\vec{q} - q^0\vec{p}$. It is also easily confirmed that the algebra of these ten quantities is that of the Galilean group. We can thus assert that the Galilean group is the symmetry group of the free particle. Although this seems quite natural and even obvious, the reader should be cautioned that it is *not* the existing practice in the Hamiltonian formalism. In that formalism, the Hamiltonian $\vec{p}^{\,2}/2m$ does *not* commute with the Galilean boosts $m\vec{q} - q^0\vec{p}$. The Galilean group is thus not regarded as an invariance group of the theory and is relegated to the puzzling category of being a noninvariance group.[5] Even in this trivial example we see a new perspective being made available for the understanding of so-called noninvariance groups and their associated spectrum-generating algebras.

The constraint formalism provides a unified perspective for understanding symmetries of a physical system in another respect. For mechanical systems, as well as for most field theories, the space-time symmetries, such as the Galilean group, or the Poincaré group, are generated by the nontrivial observables of the theory, such as the energy, the momentum, the angular momentum, etc. For gauge theories, such as general relativity, the principal symmetries, frequently including the space-time symmetries, are generated by the constraints themselves. It thus becomes difficult to identify the familiar observables in such theories, and in view of the fact that their values are constrained to vanish, they are of no value in identifying trajectories. The constraint formalism provides a unified view point for identifying symmetries: Whether or not the associated observables are constraints, their hallmark is that they leave invariant the constraint hypersurface. That is, the algebra of the constraints forms a normal subalgebra of the total symmetry group generated by the observables. In order to identify trajectories by means of observables, we are required to examine and identify the elements of the factor algebra.

The most important application of the constraint formalism to date will be presented in the following paper. Within the context of the Hamiltonian formalism there is a "no interaction" theorem[6] which asserts that it is impossible to have a system of particles consistent with the principles of special relativity whose laws of motion may be described by means of the Hamilton equations of motion, and whose trajectories transform correctly under Lorentz transformations, except for the sole case of a collection of free particles. It is this theorem which has impeded the development of a relativistic quantum theory of interacting particles. We shall demonstrate in the following paper that the constraint formalism does not suffer from such a pathology, and in fact provides the natural forum for quantization. If there were any lingering doubt that the constraint formalism differed in essential ways from the Hamiltonian formalism this example should lay it it rest.

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