Trace anomaly for gravitons

Raymond Critchley
Department of Theoretical Physics, The University of Manchester, England
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The trace anomaly for gravitons in an arbitrary background space-time is considered for four and six dimensions. The integrated energy-momentum tensor is also computed in six dimensions which allows the single-loop divergences to be calculated.

I. INTRODUCTION

In view of the large number of recent works[1] on the so-called conformal trace anomaly it would seem justifiable to complete the set of known results by considering the graviton anomaly. Since the graviton Lagrangian is not conformally invariant we will have to differentiate between the trace of the full stress tensor and the anomalous part.[2]

A similar situation occurs for the minimally coupled scaled field, and it may be enlightening to consider this situation before moving on to the graviton case.

Starting from the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\phi_{;\mu}\phi^{;\mu} - \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{2}\zeta R\phi^2 \tag{1}$$

(where ζ is a constant) we can take the variational derivative to compute the trace of the stress tensor $T_\mu{}^\mu$. We find

$$T_\mu{}^\mu = \frac{2}{(-g)^{1/2}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int (-g)^{1/2} \mathcal{L}\, dx^4 \tag{2}$$

$$= -(1-6\zeta)(\phi_{;\mu}\phi^{;\mu} + m^2\phi^2 - \zeta R\phi^2) + 6\zeta\phi(\phi_{;\sigma}{}^{\sigma} - m^2 + \zeta R\phi) - m^2\phi^2 . \tag{3}$$

In order to define the vacuum average we will follow the standard procedure (see, for example, Christensen[3]) and take the arguments of the ϕ to be at x and x'; later we will take the coincidence limit $x\to x'$. We find that $T_\mu{}^\mu$ has the vacuum average

$$\langle T_\mu{}^\mu\rangle = \lim_{x\to x'} [i(1-6\zeta)(\nabla_\mu\nabla^{\mu'} + m^2 - \zeta R)G(x,x') - i6\zeta(\Box - m^2 + \zeta R)G(x,x') + im^2G(x,x')] , \tag{4}$$

where the Green's function is defined by

$$G(x,x') = \frac{i\langle \text{out} \,|\, T[\phi(x),\phi(x')]\,|\text{in}\rangle}{\langle \text{out}\,|\,\text{in}\rangle} \tag{5}$$

and satisfies the equation

$$(\Box - m^2 + \zeta R)G(x,x') = -\frac{\delta^4(x-x')}{(-g)^{1/2}} . \tag{6}$$

Not knowing how to define the coincidence limit of a δ function, we prefer to avoid this issue and equate it to zero. This will mean that we will drop the second term in Eq. (4) since by Eq. (6) it is zero modulo a δ-function singularity. In the massless limit (4) becomes

$$\langle T_\mu{}^\mu\rangle = \lim_{x\to x'} i(1-6\zeta)(\nabla_\mu\nabla^{\mu'} - \zeta R)G(x,x') \tag{7}$$

and is formally zero only for the value $\zeta=\frac{1}{6}$. For the minimally coupled scalar field $\langle T_\mu{}^\mu\rangle$ is not formally zero and we might expect the regularized $\langle T_\mu{}^\mu\rangle$ to contain a nonzero term, even if there were no anomalous part. Such an expectation is borne out by the work of Bunch and Davies.[4] In the massless (minimally coupled case) they find a $\Box R \ln R$ term as well as the standard anomaly. For the conformal scalar field ($\zeta=\frac{1}{6}$) the extra term vanishes. This is circumstantial evidence for the $\Box R \ln R$ term arising from the term in Eq. (7) rather than, say, in the δ-function term in Eq. (4).

The anomalous part of $\langle T_\mu{}^\mu\rangle$, denoted in the text by $\langle \bar{T}_\mu{}^\mu\rangle_{\text{ren}}$ can be found in several ways. The method of Tsao[5] is to take the variational derivative of the one-loop counterterms, whereas Gibbons, Hawking, and Perry[6] use the ζ-function method of regularizing. In the method of this paper we will allow the massless field to have, initially, a small mass. For the scalar field having the Lagrangian (1) we can compute $\langle T_\mu{}^\mu\rangle$ [Eq. (4)]. If we consider only the contribution due to the third term in Eq. (4) denoted by $\langle \bar{T}_\mu{}^\mu\rangle$ then we have

$$\langle T_\mu{}^\mu\rangle \sim \langle \bar{T}_\mu{}^\mu\rangle = \lim_{x\to x'} im^2G(x,x') , \tag{8}$$

which is formally zero in the massless limit. The description of the trace anomaly that we will take is that of Ref. 7. We expand the Green's function in terms of $1/m^2$, the coefficients being linear in the Minakshisundaram coefficients, a_n, defined by the asymptotic expansion of the quantum-mechanical propagator

$$\langle x'(s)|x''(0)\rangle = \frac{-i}{16\pi^2 s^2} e^{-i\sigma/2s} \sum_{n=0}^{\infty} a_n(x,x')(is)^n . \quad (9)$$

The trace of the stress-tensor contribution can similarly be expanded by Eq. (8). Absorption of terms up to a_2 into the renormalization of the cosmological constant, Newton's constant, and of the Weyl terms can be done and should leave a finite renormalized trace $\langle \tilde{T}_\mu{}^\mu \rangle$ which contains no term proportional to a_2. However, it might be thought that, since an asymptotic expansion in terms of $1/m^2$ was made, the calculation cannot be applied to the massless theory. The opinion of the authors of Ref. 7, and, recently, Bunch and Davies,[8] is that the a_2 term should be removed even in the strictly massless theory. If we do this we find the trace anomaly

$$\langle T_\mu{}^\mu \rangle_{\text{ren}} \sim \langle \tilde{T}_\mu{}^\mu \rangle_{\text{ren}} = \frac{-1}{16\pi^2} a_2(x,x) . \quad (10)$$

For $\xi = \frac{1}{6}$ we expect this to be exact, but for other values we might expect other contributions to the trace due to the first terms in Eq. (4).

In Sec. II, as a preliminary to our calculation of the anomaly, we will consider the existing literature on one-loop graviton processes in an arbitrary background metric. We will write down the Green's function satisfied by the gravitons and ghost particles.

II. THE METHOD OF THE BACKGROUND FIELD

In this section we make use of the work of 't Hooft and Veltman,[9] Deser and van Nieuwenhuizen,[10] and Deser, Tsao, and van Nieuwenhuizen.[11] These authors consider an expansion of the Einstein-Hilbert Lagrangian about a fixed background metric and use the quadratic part in the fields to compute the form of the single-loop divergences via an algorithm. We will need the quadratic part of the Lagrangian in order to compute the Green's function equations for the fields and so our calculations will diverge from those of the previous authors after Eq. (20).

For a pure graviton field we take the graviton Lagrangian

$$\mathcal{L} = -(-\bar{g})^{1/2} R(\bar{g}_{\mu\nu}) . \quad (11)$$

The standard procedure is to split the metric tensor into two parts, i.e., a fixed background $g_{\mu\nu}$ and a field part $h_{\mu\nu}$ which satisfies the linearized graviton field equations. We write, using, the conventions of Ref. 10,

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + K h_{\mu\nu} . \quad (12)$$

Equation (11) is then expanded in terms of the field $h_{\mu\nu}$. According to Ref. 10, for example, the quadratic part of the Lagrangian is given by

$$\mathcal{L}_2 = (-g)^{1/2}\left[-\tfrac{1}{2}(D_\nu h_{\alpha\beta})P^{\alpha\beta\rho\sigma}(D_\nu h_{\rho\sigma}) + \tfrac{1}{2}(X_g)^{\alpha\beta\rho\sigma} h_{\alpha\beta} h_{\rho\sigma} + \tfrac{1}{2}(h_\mu - \tfrac{1}{2}D_\mu h)^2\right] . \quad (13)$$

In Eq. (13) D_ν is the covariant derivative, $h_\mu = D^\nu h_{\mu\nu}$, and all indices are to be raised and lowered by the background metric tensor $g_{\mu\nu}$. $P^{\alpha\beta\rho\sigma}$ and $(X_g)^{\alpha\beta\rho\sigma}$ are defined later in the text. Standard theory states that it is not possible to derive the graviton propagator directly since the Lagrangian is singular. To obtain the propagator we have to add a gauge part and a compensating ghost part, the ghost term being related to the gauge-breaking term by

$$\mathcal{L}_{\text{ghost}} = \sum_{a,b=1}^{4} \phi^{*a} \frac{\partial}{\partial \eta_b} C_a \phi^b , \quad (14)$$

with the gauge term being given by

$$-\tfrac{1}{2} C_a C^a . \quad (15)$$

Here ϕ^b are the gauge fields and η_b is a variable introduced by a coordinate transformation. According to Ref. 10, if we make the transformation

$$\bar{g}_{\mu\nu}(x') = \bar{g}_{\alpha\beta}(x) \left(\frac{\partial x^\alpha}{\partial x'^\mu}\right)\left(\frac{\partial x^\beta}{\partial x'^\nu}\right) , \quad (16)$$

then the field $h_{\mu\nu}$ transforms as

$$h_{\mu\nu} \to h_{\mu\nu} + (g_{\mu\alpha} + K h_{\mu\alpha}) h^\alpha{}_{,\nu} + (g_{\alpha\nu} + K h_{\alpha\nu}) h^\alpha{}_{,\mu} + \eta^\alpha (g_{\mu\nu} + K h_{\mu\nu})_{,\alpha} , \quad (17)$$

where

$$x^\alpha - x'^\alpha = K\eta^\alpha ,$$

and the comma stands for an ordinary derivative. These last few equations allow us to determine the form of the ghost part for any gauge-breaking term. The most convenient choice of gauge is

$$-\tfrac{1}{2} C_\mu C^\mu = -\tfrac{1}{2}(-g)^{1/2}(h_\mu - \tfrac{1}{2}D_\mu h)^2 , \quad (18)$$

which corresponds to having

$$C_\mu = +(-g)^{1/4} e^a{}_\mu (h_\mu - \tfrac{1}{2}D_\mu h) , \quad g^{\mu\nu} = e^{a\mu} e_a{}^\nu . \quad (19)$$

Substituting into (14) gives

$$\mathcal{L}_{\text{ghost}} = (-g)^{1/2}\left[(D_\mu \phi^*_\nu)(D^\mu \phi^\nu) + \phi^*_\mu R^{\mu\nu} \phi_\nu\right] . \quad (20)$$

Thus the Lagrangian becomes (with the addition of mass terms)

$$\frac{1}{(-g)^{1/2}}\mathcal{L}_{\text{total}} = -\tfrac{1}{2}(D_\nu h_{\alpha\beta})P^{\alpha\beta rs}(D^\nu h_{rs}) + \tfrac{1}{2}(h_{\alpha\beta}X_g^{\alpha\beta rs}h_{rs}) - \tfrac{1}{4}m^2 h_{\alpha\beta}h^{\alpha\beta} + \tfrac{1}{8}m^2h^2 + m^2\phi^{*\alpha}\phi^\alpha + (D_\mu\phi_\nu^*)(D^\mu\phi^\nu) + \phi_\mu^* R^{\mu\nu}\phi_\nu . \tag{21}$$

The exact form of the mass terms is chosen so that the equation satisfied by the massless Green's function becomes modified by the replacement

$$\Box \to \Box - m^2 . \tag{22}$$

If we wish, we can put Eq. (21) entirely in real form by writing

$$\phi_\nu = \frac{1}{\sqrt{2}}(a_\nu + i\bar{a}_\nu); \quad \phi_\nu^* = \frac{1}{\sqrt{2}}(a_\nu - i\bar{a}_\nu), \qquad \phi_\nu^*\phi^\nu = \tfrac{1}{2}(a_\nu a^\nu + \bar{a}_\nu\bar{a}^\nu). \tag{23}$$

Thus the ghost term becomes equivalent to two real ghosts and this is the form we will use in our calculations.

Before proceeding further we can put Eq. (21) into the form

$$\delta\hat{S} = \tfrac{1}{2}\delta\left(\frac{\delta^2 S}{\delta h_{\alpha\beta}\delta h_{\mu'\nu'}}\right)\hat{h}_{\alpha\beta}\hat{h}_{\mu'\nu'} + \tfrac{1}{2}\delta\left(\frac{\delta^2 S}{\delta a_\beta \delta a_\gamma}\right)\hat{a}_\beta\hat{a}_\gamma$$

with

$$S = \int \mathcal{L}_{\text{total}}\, dx . \tag{24}$$

Taking the vacuum average gives

$$\langle\delta\hat{S}\rangle = -\frac{i}{2}\delta\left(\frac{\delta^2 S}{\delta h_{\alpha\beta}\delta h_{\mu'\nu'}}\right)(\bar{G}_E)_{\alpha\beta\mu'\nu'} - \frac{i}{2}\delta\left(\frac{\delta^2 S}{\delta a_\beta\delta a_{\gamma'}}\right)(G_V)_{\beta\gamma'} . \tag{25}$$

Here the Green's functions are defined by

$$(\bar{G}_E)_{\alpha\beta\mu'\nu'} = i\langle\hat{h}_{\alpha\beta}|\hat{h}_{\mu'\nu'}\rangle , \qquad (G_V)_{\beta\gamma'} = i\langle\hat{a}_\beta|\hat{a}_{\gamma'}\rangle . \tag{26}$$

The second derivatives are easily found to be

$$\frac{\delta^2 S}{\delta h_{\mu\nu}\delta h_{\alpha''\beta''}} = \tfrac{1}{2}g^{\mu\alpha}g^{\nu\beta}\Box''\delta(x,x'') - \tfrac{1}{4}g^{\alpha\beta}g^{\mu\nu}\Box''\delta(x,x'') + X_g^{\alpha\beta\mu\nu}\delta(x,x'') - \tfrac{1}{2}m^2 g^{\mu\alpha}g^{\nu\beta}\delta(x,x'') - \tfrac{1}{4}m^2 g^{\alpha\beta}g^{\mu\nu}\delta(x,x'') \tag{27}$$

(where $\Box''$ means that the derivatives are to be taken with respect to x'') and

$$\frac{\delta^2 S}{\delta a_\beta\delta a_{\gamma'}} = -2[g^{\beta\gamma}\Box'\delta(x,x') - R^{\beta\gamma}\delta(x,x') - m^2 g^{\beta\gamma}\delta(x,x')] . \tag{28}$$

The next step is to obtain the inverse of Eqs. (27) and (28). We define the inverse by the equation

$$\frac{\delta^2 S}{\delta h_{\alpha\beta}\delta h_{\mu''\nu''}}(\bar{G}_E)_{\mu''\nu''r's'} = -I^{\alpha\beta}{}_{r's'}\delta(x,x') , \quad \text{where} \quad I^{\alpha\beta}{}_{r's'} = \tfrac{1}{2}(g^\alpha{}_{r'}g^\beta{}_{s'} + g^\alpha{}_{s'}g^\beta{}_{r'}) . \tag{29}$$

Substituting Eq. (27) we obtain

$$\int dx''(\bar{G}_E)_{\alpha''\beta''r's'}[\tfrac{1}{2}g^{\mu\alpha}g^{\nu\beta}(\Box'' - m^2) - \tfrac{1}{4}g^{\alpha\beta}g^{\mu\nu}(\Box'' - m^2) + X_g^{\alpha\beta\mu\nu}]\delta(x,x'') = -I^{\mu\nu}{}_{r's'}\delta(x,x') . \tag{30}$$

Integration by parts gives

$$(\Box - m^2)(\bar{G}_E)^{\mu\nu}{}_{r's'} - \tfrac{1}{2}g^{\mu\nu}(\Box - m^2)(\bar{G}_E)^\alpha{}_{\alpha r's'} + 2(X_g)^{\alpha\beta\mu\nu}(\bar{G}_E)_{\alpha\beta r's'} = -2I^{\mu\nu}{}_{r's'}\delta(x,x'). \tag{31}$$

By taking the trace of Eq. (31) and substituting the resulting expression back into the equation we find, in d dimensions,

$$\Box(\bar{G}_E)^{\mu\nu}{}_{r's'} + 2(X_g)^{\alpha\beta\mu\nu}(\bar{G}_E)_{\alpha\beta r's'} - m^2(G_E)^{\mu\nu}{}_{r's'} - \left(\frac{d-4}{d-2}\right)G^{\alpha\beta}g^{\mu\nu}(\bar{G}_E)_{\alpha\beta r's'} = -4(P^{-1})^{\mu\nu}{}_{rs}\delta(x,x') , \tag{32}$$

with

$$(P^{-1})^{\mu\nu}{}_{rs} = g^\mu{}_r g^\nu{}_s + g^\mu{}_s g^\nu{}_r - \frac{2}{d-2}g^{\mu\nu}g_{rs}$$

and

$$G^{\alpha\beta} = R^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}R .$$

Here, $(\bar{G}_E)^{\mu\nu}{}_{r's'}$ is the graviton Green's function and is the same function as appears in Eq. (25).

The calculation for the vector ghost is more straightforward and leads to

$$\frac{\delta^2 S}{\delta a_\beta\delta a_{\gamma'}} = -2[(G_V)^{-1}]^{\beta\gamma'} \tag{33}$$

where G_V satisfies

$$\Box(G_V)^\mu{}_\nu - R^{\mu\alpha}(G_V)_{\alpha\nu} - m^2(G_V)^\mu_\nu = -g^\mu_\nu\delta(x,x') . \quad (34)$$

Thus,

$$\langle\delta\hat{S}\rangle = -\frac{i}{2}\mathrm{tr}\,\delta(G_B)^{-1}G_B + i\,\mathrm{tr}\,\delta(G_V)^{-1}G_V$$
$$= -\frac{i}{2}\mathrm{tr}\,\delta\ln(G_B) + i\,\mathrm{tr}\,\delta\ln(G_V)$$
$$= \delta W^{(1)} , \quad (35)$$

where we have written everything in matrix notation. In Eq. (35), tr denotes the trace over both space-time and continuous indices. $W^{(1)}$ is the effective action. If required ln G can be interpreted by using the Schwinger-DeWitt formalism.[12] If this was done then Eq. (35) could be used to define a finite regularized effective action. We will not consider this problem but will deal directly with the stress tensor.

III. THE ENERGY-MOMENTUM TENSOR

Our first priority is to find that contribution to $\langle T_\mu{}^\mu\rangle$ analogous to the $\langle\tilde{T}_\mu{}^\mu\rangle$ introduced in the introduction. We can find this by using the well-known result

$$T_\mu{}^\mu = \frac{2}{(-g)^{1/2}} g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\int \mathcal{L}\,dx , \quad (36)$$

and substituting the result (21). We note that the variation is to be performed with respect to the background metric only. All of the results that we need have been given by DeWitt.[12] They are

$$\delta(-g)^{1/2} = \tfrac{1}{2}(-g)^{1/2}g^{\mu\nu}\delta g_{\mu\nu} ,$$
$$\delta R_{\alpha\tau\beta}{}^s = \tfrac{1}{2}g^{s\rho}[(\delta g_{\rho\tau})_{;\beta\alpha} + (\delta g_{\rho\beta})_{;\tau\alpha} - (\delta g_{\tau\beta})_{;\rho\alpha} - (\alpha \to \tau)] ,$$
$$\delta R_{\mu\nu} = \delta R_{\mu\tau\nu}{}^\tau = \tfrac{1}{2}g^{\sigma\tau}[(\delta g_{\mu\nu})_{;\sigma\tau} + (\delta g_{\sigma\tau})_{;\mu\nu} - (\nu \to \tau)] ,$$
$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}g^{\sigma\tau}[(\delta g_{\mu\nu})_{;\sigma\tau} - (\delta g_{\mu\sigma})_{;\nu\tau}] . \quad (37)$$

The semicolon stands for a covariant derivative. The contribution, for example, from the $R_{\alpha\tau\beta}{}^s$ term in Eq. (21) is given by

$$\frac{1}{(-g)^{1/2}}g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}R_{\alpha\tau\beta}{}^\tau g_{\tau s}h^{\alpha\beta}h^{rs} = 2h^{\alpha\beta}h^{rs}R_{\alpha r\beta s} - 3h^{\alpha\beta}h^{rs}R_{\alpha r\beta s} + \tfrac{1}{2}\{[h^{\alpha\beta}h]_{;\alpha\beta} - [h^{\alpha\beta}h^r{}_\alpha]_{;r\beta}\} . \quad (38)$$

The bracketed term is a total derivative and will not contribute to the anomaly. Equation (36) becomes

$$T_\mu{}^\mu = -\frac{\mathcal{L}_{\mathrm{total}}}{(-g)^{1/2}} - \tfrac{1}{2}m^2h_{\alpha\beta}h^{\alpha\beta} + \tfrac{1}{4}m^2h^2 + 2m^2a^\alpha a_\alpha + 2[(D_\mu a_\nu)(D^\mu a^\nu) + a_\mu R^{\mu\nu}a_\nu + m^2a^\alpha a_\alpha] . \quad (39)$$

We note that, even with the neglect of the total divergences, $T_\mu{}^\mu$ is not formally zero. This is only to be expected since the linearized graviton field is not conformally invariant. However, we expect that the anomalous part of the trace can be found by neglecting the $\mathcal{L}_{\mathrm{total}}$ term and proceeding analogously to the minimally coupled case as discussed in the Introduction.

On taking the vacuum average of (39) and with the neglect of $\mathcal{L}_{\mathrm{total}}$ we find

$$\langle T_\mu{}^\mu\rangle \sim \langle\tilde{T}_\mu{}^\mu\rangle = \tfrac{1}{2}im^2[(\bar{G}_B)_{\alpha\beta}{}^{\alpha'\beta'} - \tfrac{1}{2}(\bar{G}_B)_\alpha{}^\alpha{}_{\beta'}{}^{\beta'} - 4(G_V)_\alpha{}^\alpha]$$
$$= \tfrac{1}{4}im^2\,\mathrm{tr}P^{-1}\bar{G}_B - 2im^2\,\mathrm{tr}G_V . \quad (40)$$

Equation (40) is now formally zero as $m \to 0$. The equations satisfied by $\tfrac{1}{4}P^{-1}G_B = G_B$ and G_V are

$$\Box(G_B)^{\mu\nu}{}_{r's'} + 2(X_g)^{\alpha\beta\mu\nu}(G_B)_{\alpha\beta r's'} - m^2(G_B)^{\mu\nu}{}_{r's'} - \left(\frac{d-4}{d-2}\right)G^{\mu\nu}g_{rs}(\bar{G}_B)_{\alpha\beta r's'} = -I^{\mu\nu}{}_{r's'}\delta(x,x') , \quad (41)$$
$$\Box(G_V)^\mu{}_\nu - R^{\mu\alpha}(G_V)_{\alpha\nu} - m^2(G_V)^\mu{}_\nu = -g^\mu{}_\nu\delta(x,x') .$$

Equations (40) and (41) are sufficient for us to obtain the trace anomaly. We note that Eqs. (41) are of the form (in matrix notation)

$$(\Box + E - m^2)G(x,x') = -I\delta(x,x') . \quad (42)$$

A propagator $\langle x'(s)|x''(0)\rangle$ is easily defined by

$$G(x',x'') = i\int ds\, e^{-im^2s}\langle x|(s)|x''(0)\rangle \quad (43)$$

and has the coincidence limit

$$\langle x'(s)|x''(0)\rangle = i\sum_{n=0}^{\infty}E_{2n}(is)^{n-d/2} . \quad (44)$$

Gilkey has obtained very general expressions for the expansion coefficients of (44) for any second-order operator of the form which acts on the Green's function in Eq. (42). Of most interest for our work is the coefficient E_4, which Gilkey[13,14] shows to be given by

$$E_4 = \frac{1}{16\pi^2}[I(-\tfrac{1}{30}\Box R + \tfrac{1}{72}R^2 - \tfrac{1}{180}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{180}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}) - \tfrac{1}{6}RE + \tfrac{1}{2}E^2 + \tfrac{1}{12}W_{ij}{}^2 + \tfrac{1}{6}\Box E] . \quad (45)$$

E_{2n} is what we would normally call $(a_{n/2})/16\pi^2$ where $(a_{n/2})$ is the Minakshisundaram coefficient.

To compute our final result for the anomaly we note that Eqs. (40), (42), and (44) bear a strong resemblance to the analogous scalar equations (8), (6), and (9), respectively, except that discrete matrix elements are present in the graviton case. The calculation of the anomaly is therefore virtually identical to the scalar case and we can merely quote the result. We find

$$\langle \tilde{T}_\mu{}^\mu \rangle_{\rm ren} = \{[(E_g)d]_{\alpha\beta}{}^{\alpha\beta} - 2[(E_V)d]_\alpha{}^\alpha\}. \tag{46}$$

The only coefficients computed at the present time are E_0, E_2, and E_4, and recently E_6. Thus only the anomaly in two, four and six dimensions can be computed at the present time.

An alternative procedure for computing the anomaly has been communicated to be by Stuart Dowker. This method, which avoids the massless limit and therefore might be preferable, is best illustrated for the scalar case. Instead of (8) we have

$$\langle T_\mu{}^\mu \rangle = i \lim_{x\to x'} \Box G(x,x'), \tag{47}$$

which formally diverges. ζ-function regularization replaces G by the complex power $G^s = \zeta(s)$ so that, if $\Box G^s = -G^{s-1}$ is used,

$$\begin{aligned}\langle T_\mu{}^\mu \rangle_{\rm ren} &= -i \lim_{x\to x'} G^{s-1}(x,x')\\ &= -i\,\zeta(0)\\ &= \frac{a_{d/2}}{(4\pi)^{d/2}}.\end{aligned} \tag{48}$$

This method is easily adapted to the present case. The only differences are that there are two contributions corresponding to the two terms in (46) and that matrix traces must be taken. The result (46) follows from the theory described by Gilkey (Refs. 13 and 14 and the references therein).

IV. RESULTS FOR FOUR-DIMENSIONAL SPACE-TIMES

The only remaining part of the calculation is to substitute (45) into (46). To evaluate (45) we need to know W_{ij} and the values of E. We have from Eq. (41)

$$(E_g) = 2X_g,$$

where X_g is given by

$$\begin{aligned}(X_g)^{\alpha\beta\mu\nu} = {}& \tfrac{1}{2}(R^{\alpha\mu\beta\nu} + R^{\beta\mu\alpha\nu}) - \tfrac{1}{4}(g^{\alpha\mu}R^{\beta\nu} + g^{\beta\mu}R^{\alpha\nu} + g^{\beta\nu}R^{\alpha\mu} + g^{\alpha\nu}R^{\mu\beta})\\ &+ \tfrac{1}{2}(g^{\alpha\beta}R^{\mu\nu} + g^{\mu\nu}R^{\alpha\beta}) + \tfrac{1}{4}(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu})R\end{aligned} \tag{49}$$

(this result is valid in any dimension d). Direct calculation shows that for $d=4$

$$\begin{aligned}&(E_g)^{\alpha\beta}{}_{rs}(E_g)^{rs}{}_{\alpha\beta} = 3R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 6R_{\mu\nu}R^{\mu\nu} + 5R^2,\\ &(E_g)^{\alpha\beta}{}_{\alpha\beta} = 6R;\quad I^{\alpha\beta}{}_{\alpha\beta} = 10.\end{aligned} \tag{50}$$

The spin curvature W_{ij} can either be calculated from $W_{ij} = -\frac{1}{4}R_{ij\alpha\beta}J^{\alpha\beta}$, where $J^{\alpha\beta}$ are the generators of the Lorentz group,[15] or we can expand the covariant derivatives of the fields as, for example,[16]

$$\begin{aligned}&\Box h_{\alpha\beta} = \Box_0 h_{\alpha\beta} + 2(n_\mu)_{\alpha\beta}{}^{\rho\sigma}h_{\rho\sigma}{}^{,\mu} + m_{\alpha\beta}{}^{\rho\sigma}h_{\rho\sigma},\\ &\Box_0 = g^{\alpha\beta}\partial_\alpha\partial_\beta.\end{aligned} \tag{51}$$

W_{ij} is then defined by

$$W_{ij} = \partial_i n_j - \partial_j n_i + n_i n_j - n_j n_i. \tag{52}$$

We obtain

$$\begin{aligned}(W_{ij})^{\alpha\beta\rho\sigma} = -\tfrac{1}{2}(&g^{\alpha\rho}R^{\sigma\beta}{}_{ij} + g^{\beta\rho}R^{\sigma\alpha}{}_{ij}\\ &+ g^{\alpha\sigma}R^{\rho\beta}{}_{ij} + g^{\beta\sigma}R^{\rho\alpha}{}_{ij})\end{aligned} \tag{53}$$

(valid for any dimension).

The corresponding quantities for the vector ghost are, in four dimensions,

$$\begin{aligned}&(E_V)_{\mu\nu} = -R_{\mu\nu},\\ &(E_V)_\mu{}^\mu = -R,\quad I_\alpha{}^\alpha = 4,\\ &(E_V)^\mu{}_\alpha(E_V)^\alpha{}_\mu = R_{\mu\nu}R^{\mu\nu},\end{aligned}$$

and, in d dimensions,

$$(W_{ij})^{\mu\nu} = R^{\mu\nu}{}_{ij}. \tag{54}$$

With the help of Eqs. (40)–(54) we can derive the E_n. We find [remember that Eq. (45) contains the unit matrix I]

$$\begin{aligned}[(E_g)_4]^{\alpha\beta}{}_{\alpha\beta} &= \frac{1}{576\pi^2}(24\Box R + 38R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + 42R_{\mu\nu}R^{\mu\nu} + \tfrac{101}{3}R^2),\\ [(E_V)_4]^\alpha{}_\alpha &= \frac{1}{16\pi^2}[-\tfrac{3}{10}\Box R + \tfrac{2}{9}R^2 + \tfrac{43}{90}R_{\mu\nu}R^{\mu\nu} - \tfrac{11}{180}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}].\end{aligned} \tag{55}$$

Substituting into (50) gives us our final result,

$$\langle \tilde{T}_\mu{}^\mu \rangle_{\rm ren} = \frac{1}{2880\pi^2}[212C^2 - 298(R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}R^2) + 12\Box R + 45R^2], \tag{56}$$

where C^2 is the squared Weyl tensor defined as

$$C^2 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + 2R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}R^2.$$

The result (56) agrees with Tsao[5] (he has $\Box R = 0$) and also Gibbons, Hawking, and Perry,[6] who set $R_{\mu\nu} = 0$.

We should point out that only the first term in Eq. (56) is gauge independent. Furthermore, when $R_{\mu\nu} = 0$ then there is a nonzero contribution, linear in the fields in the expansion of the graviton Lagrangian, Eq. (11). Inclusion of this term would alter the coefficients of the last three terms in Eq. (56). For a further elaboration of this point see the article by Duff.[17]

In Sec. V we briefly describe the result for six dimensions.

V. THE SIX-DIMENSIONAL CASE

The coefficients (E_E) and $E_V)_6$ are very complicated and contain 46 possible different terms. However a considerable simplification is possible if we evaluate all terms on the mass shell $R_{\mu\nu} = 0$. With the definitions

$$x = R_{\mu\nu\rho\sigma} R^{\rho\sigma\alpha\beta} R_{\alpha\beta}{}^{\mu\nu}, \quad y = R^{\mu\nu\rho\sigma} R_{\mu\alpha\rho\beta} R_{\nu}{}^{\alpha}{}_{\sigma}{}^{\beta},$$
$$\bar{x} = R_{\mu\alpha\beta\nu} R^{\mu\beta}{}_{\rho\sigma} R^{\alpha\nu\beta\rho}, \quad \bar{y} = R_{\mu\nu\rho\sigma} R^{\mu\alpha\rho\beta} R^{\nu}{}_{\alpha}{}^{\sigma}{}_{\beta},$$
$$|\nabla R|^2 = R_{\mu\nu\rho\sigma;\alpha} R^{\mu\nu\rho\sigma;\alpha}, \quad |R \Box R| = R_{\mu\nu\rho\sigma} \Box R^{\mu\nu\rho\sigma}, \tag{57}$$

and the internal relations (derivable from the cyclic relation $R^{\mu[\nu\alpha\beta]} = 0$)

$$\bar{y} = y - \tfrac{1}{4}x, \quad \bar{x} = \tfrac{1}{2}x, \tag{58}$$

we find from Gilkey's tables, with the help of Eqs. (53) and (54)

$$\mathrm{tr}(E_E)_6 = \frac{1}{(4\pi)^3} \frac{1}{2160} (237|\nabla R|^2 + 2384y - 64x + 612|R \Box R|)$$

and

$$\mathrm{tr}(E_V)_6 = \frac{1}{(4\pi)^3} \frac{1}{840} (-\tfrac{29}{3}|\nabla R|^2 - \tfrac{82}{9}x - \tfrac{172}{9}y - 16|R \Box R|). \tag{59}$$

Hence, in six dimensions,

$$\langle \tilde{T}_{\mu}{}^{\mu} \rangle_{\mathrm{ren}} = \frac{1}{(4\pi)^3 15120} (2007|\nabla R|^2 + 4860|R \Box R| + 17376y - 120x). \tag{60}$$

A spin-off from Eq. (60) is that it allows us to compute the form of the single-loop divergences in six dimensions and hence find the value of the coefficient α_6 of van Nieuwenhuizen and Wu.[18] These authors show that

$$\int \mathfrak{L}^{(1)}_{\mathrm{total}}\, dx^6 = \frac{-\alpha_6}{d-6} \int dx^6 (-g)^{1/2} x. \tag{61}$$

Analogous to Tsao[5] we can put the anomalous part of the trace proportional to the divergent part of the effective Lagrangian. We write

$$\int dx^6 (-g)^{1/2} \langle \tilde{T}_{\mu}{}^{\mu} \rangle_{\mathrm{ren}} = \frac{-\alpha_6}{d-6} \int dx^6 (-g)^{1/2} X. \tag{62}$$

Use of the relations (see Gilkey[13] or van Nieuwenhuizen and Wu[18])

$$\int Y(-g)^{1/2} dx^6 = \tfrac{1}{4}\int |\nabla R|^2 (-g)^{1/2} dx^6 - \tfrac{1}{4}\int x(-g)^{1/2} dx^6,$$

$$\int |\nabla R|^2 (-g)^{1/2} dx^6 = -\int |R \Box R| (-g)^{1/2} dx^6, \tag{63}$$

$$\int (Y - \tfrac{1}{2}x)(-g)^{1/2} dx^6 = 0$$

implies

$$\int dx^6 (-g)^{1/2} \langle T_{\mu}{}^{\mu} \rangle_{\mathrm{ren}} = \tfrac{9}{15120} \int dx^6 (-g)^{1/2} X, \tag{64}$$

so that

$$\alpha_6 = \tfrac{9}{15120}. \tag{65}$$

This value differs from the value given in Ref. 16. It is possible to compare van Nieuwenhuizen's calculation to ours step by step, and the only difference is one factor of 2 which is missing in his Eq. (41), but is needed when going from real to complex fields. Multiplying therefore the first five terms in his Eq. (63) by 2, and the first four as well as the seventh term in his Eq. (80) by 2, we get complete agreement. Apart from this minor change the conclusions of Ref. 16 are correct.

It is interesting to note that the integrated trace anomaly for scalar, vector, and graviton fields has the form

$$\mathfrak{L}^{(1)}_j = \frac{(j+1)^2}{4\pi^3 15120(n-6)} \int X\sqrt{-g}\, dx^6. \tag{66}$$

Here j is the spin of the particle. The values given above for the scalar and vector particles correct values given by Dowker.[19]

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