Stress tensor of massless conformal quantum fields in hyperbolic universes

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The vacuum expectation value of the renormalized stress tensor for a conformally coupled massless scalar field propagating in an arbitrary hyperbolic Robertson-Walker spacetime is calculated. The vacuum state used is that obtained by conformal transformation to the static hyperbolic universe. The result differs from calculations performed using the vacuum defined by conformal transformation to Minkowski spacetime, and is also different from earlier results obtained using the conformally static vacuum which are shown to be incorrect.

I. INTRODUCTION

In the semiclassical theory of gravity, in which all fields apart from the gravitational field are quantized, the calculation of expectation values, $\langle T_{\mu\nu}\rangle$, of the stress tensor of a quantum field is of considerable importance. In the Robertson-Walker spacetimes, which are conformally flat, it is possible to obtain exactly the vacuum expectation value of the stress tensor for conformally invariant fields.[1-4] This can be done either by the direct application of regularization[2] or by using arguments based on conformal invariance and the existence of trace anomalies.[1,3,4] However, the result obtained by the methods used in Refs. 1 and 2 is not entirely correct. While it is correct for the spatially flat and closed universes, it requires modification for the open (hyperbolic) universes. In this paper, the correct expression for $\langle T_{\mu\nu}\rangle$ in the open Robertson-Walker universe, which replaces that of Refs. 1 and 2, is derived. The expression obtained differs from those of Refs. 3 and 4 and this is shown to be the result of a different choice of vacuum state. There is no disagreement in the results.

In Sec. II, the vacuum expectation value of the renormalized stress tensor of a conformally coupled massless scalar field in the static open universe is shown to be zero. The vacuum state used is the natural one obtained by decomposing solutions of the covariant wave equation into positive- and negative-frequency solutions with respect to the timelike Killing vector field which is globally defined in this spacetime. In Sec. III, $\langle T_{\mu\nu}\rangle$ is calculated for all open universes by using the trace anomaly coefficients and the result derived in Sec. II: the vacuum state used is formally the same as that used in Sec. II and is obtained by applying a conformal transformation to the positive- and negative-frequency solutions defined with respect to the aforementioned Killing vector field.

A short discussion of the relationship between the results obtained in this paper and those of Refs. 3 and 4 is given in Sec. IV. The sign conventions used are the same as in Ref. 6.

II. THE OPEN STATIC UNIVERSE

It is well known that the vacuum expectation value of the quantum stress tensor is in general a divergent quantity and must be regularized to give a finite, physical result. The method which will be employed in this section is covariant point separation[5-10] and the renormalization ansatz is that of subtracting from the regularized $\langle T_{\mu\nu}\rangle$ those terms obtained by Christensen[7] which contain no more than four derivatives of the metric. This renormalization ansatz has been discussed in some detail in Refs. 8–11.

The metric of the open static universe is

$$ds^2 = dt^2 - a^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] . \tag{1}$$

The general solution of the conformally coupled massless scalar wave equation is

$$\psi(t,\chi,\theta,\phi) = a^{-1}\int_0^\infty dq \sum_{l,m} (2q)^{-1/2}[A_{\vec{k}}\mathcal{Y}_{\vec{k}}(\vec{x})e^{-iqt/a} + A_{\vec{k}}^\dagger\mathcal{Y}_{\vec{k}}^*(\vec{x})e^{iqt/a}] , \tag{2}$$

where $\vec{k}=(q,l,m)$; $0\le q<\infty$; $l=0,1,2,\ldots$; $m=-l, -l+1,\ldots,l-1,l$; and $\mathcal{Y}_{\vec{k}}(\vec{x}) = Z_{ql}(\chi)Y_{lm}(\theta,\phi)$. $Y_{lm}(\theta,\phi)$ denotes a spherical harmonic and $Z_{ql}(\chi)$ is a solution of

$$\left(\sinh^2\chi\frac{d^2}{d\cosh^2\chi} + 3\cosh\chi\frac{d}{d\cosh\chi} - \frac{l(l+1)}{\sinh^2\chi} + q^2 + 1\right)Z_{ql}(\chi) = 0 . \tag{3}$$

The operators $A_{\vec{k}}$, $A_{\vec{k}}^\dagger$ are annihilation and creation operators satisfying the usual commutation relations and giving rise to a vacuum state, $|\,\rangle$, which is annihilated by all the $A_{\vec{k}}$. Consistency with the canonical commutation relations is ensured if the basis functions $Z_{ql}(\chi)Y_{lm}(\theta,\phi)$ are correctly normalized: This is guaranteed by choosing the completeness relation to have the precise form given by Eq. (6) below. The properties of $Z_{ql}(\chi)$ are discussed in Ref. 12 in which a wave equation of arbitrary dimensionality is considered. The functions $Z_{ql}(\chi)$ correspond to $Z^{(f)}_{N,\alpha}(\theta)$ in Ref. 12 for the special case $f=3$.

The first step towards obtaining $\langle T_{\mu\nu}\rangle$ is the calculation of the two-point function $\langle\psi(x')\psi(x'')\rangle$, where x' and x'' are taken to be two points on a geodesic through x each a parameter distance ϵ from x. The tangent vector to the geodesic will be denoted t^μ and will be unspecified except that it is required to be non-null and is normalized to unity. However, because of the homogeneity and isotropy of the spatial sections, the coordinate axes can without loss of generality be oriented so that the geodesic lies in a plane of constant θ and ϕ. Then $t^\mu=(t^0,t^1,0,0)$ and the normalization condition is

$$g_{\mu\nu}t^\mu t^\nu \equiv (t^0)^2 - a^2(t^1)^2 = \Sigma, \tag{4}$$

where

$$\Sigma=\begin{cases}+1 \text{ if } t^\mu \text{ is timelike}\\ -1 \text{ if } t^\mu \text{ is spacelike}.\end{cases}$$

Using the expansion (2), one obtains

$$\langle\psi(x')\psi(x'')\rangle = \int_0^\infty \frac{dq}{2qa^2}\left[\exp\left(\frac{-iq\Delta t}{a}\right)\sum_{l,m}\left[Z_{ql}(\chi')Z^*_{ql}(\chi'')|Y_{lm}(\theta,\phi)|^2\right]\right], \tag{5}$$

where $\Delta t=t'-t''$ and the points x' and x'' have coordinates (t',χ',θ,ϕ) and (t'',χ'',θ,ϕ), respectively. From Eq. (21) in Ref. 12 one obtains

$$\sum_{l,m} Z_{ql}(\chi')Z^*_{ql}(\chi'')|Y_{lm}(\theta,\phi)|^2=\frac{q}{2\pi}\frac{\sin(q\Delta\chi)}{\sinh\Delta\chi}, \tag{6}$$

where $\Delta\chi=\chi'-\chi''$. Substituting (6) in (5) gives

$$\langle\psi(x')\psi(x'')\rangle = \frac{1}{4\pi^2a^2}\int_0^\infty \frac{\exp(-iq\Delta t/a)\sin(q\Delta\chi)dq}{\sinh\Delta\chi}$$

and so

$$\langle\psi^2\rangle \equiv \langle\psi(x')\psi(x'')\rangle = \frac{\Delta\chi}{4\pi^2\sinh\Delta\chi\,(a^2\Delta\chi^2-\Delta t^2)}. \tag{7}$$

The geodesic equation has the simple solution,

$$t''=t(\epsilon)=t+\epsilon t^0,\qquad \chi''=\chi(\epsilon)=\chi+\epsilon t^1, \tag{8}$$

where t^0 and t^1 are constant. The point $x'=(t',\chi',\theta,\phi)$ is given by

$$t'=t(-\epsilon)=t-\epsilon t^0,\qquad \chi'=\chi(-\epsilon)=\chi-\epsilon t^1. \tag{9}$$

Substituting (4), (8), and (9) in (7) and expanding $(\sinh\Delta\chi)^{-1}$ as a power series in ϵ gives

$$\langle\psi^2\rangle = -(16\pi^2\epsilon^2\Sigma)^{-1} + (24\pi^2\Sigma)(t^1)^2 - (360\pi^2\Sigma)^{-1}7\epsilon^2(t^1)^4. \tag{10}$$

The Ricci tensor $R_{\mu\nu}$ has components $R_{tt}=0$ and $R_{\chi\chi}=2$ so that (10) may be written

$$\langle\psi^2\rangle = -\frac{1}{16\pi^2\epsilon^2\Sigma}+\frac{1}{48\pi^2}R_{\alpha\beta}\frac{t^\alpha t^\beta}{\Sigma} - \frac{7}{1440\pi^2}R_{\alpha\beta}R_{\gamma\delta}\frac{t^\alpha t^\beta t^\gamma t^\delta}{\Sigma^2}. \tag{11}$$

Equation (11) is the unique geometrical expression for $\langle\psi^2\rangle$. The renormalized stress tensor $\langle T_{\mu\nu}\rangle_{\text{ren}}$ can now be obtained from (11) by first differentiating $\langle\psi^2\rangle$ according to the method given in Appendix D of Ref. 6 to obtain the ϵ- and t^μ-dependent stress tensor $\langle T_{\mu\nu}\rangle$, and then by subtracting the terms in Christensen's expression[7] for $\langle T_{\mu\nu}\rangle$, which contain no more than four derivatives of the metric. This procedure may be expressed as

$$\langle T_{\mu\nu}\rangle_{\text{ren}}=\langle T_{\mu\nu}(x',x'')\rangle - \langle T^{(c)}_{\mu\nu}(x',x'')\rangle,$$

where $\langle T_{\mu\nu}(x',x'')\rangle = D_{\mu\nu}(x',x'')\langle\psi(x')\psi(x'')\rangle$ for some second-order differential operator $D_{\mu\nu}$ acting at both x' and x'', and $\langle T^{(c)}_{\mu\nu}(x',x'')\rangle$ denotes Christensen's expression for $\langle T_{\mu\nu}\rangle$ calculated up to fourth order in derivatives of the metric. The ϵ and t^μ dependence of $\langle T_{\mu\nu}(x',x'')\rangle$ and $\langle T^{(c)}_{\mu\nu}(x',x'')\rangle$ are the same, so that $\langle T_{\mu\nu}\rangle_{\text{ren}}$ is independent of these quantities: It is a function only of the point x. Christensen's expression is for a massive field and is obtained by differentiating those terms of a two-point function which he calls $\frac{1}{2}G^{(1)}(x',x'')$ which give rise to no more than four derivatives of the metric in $\langle T^{(c)}_{\mu\nu}(x',x'')\rangle$:

$$\langle T^{(c)}_{\mu\nu}(x',x'')\rangle = D_{\mu\nu}(x',x'')\tfrac{1}{2}G^{(1)}(x',x'') + \tfrac{1}{2}\alpha m^2 G^{(1)}(x',x'')g_{\mu\nu},$$

where $D_{\mu\nu}$ is the same differential operator as above, and α is a constant. There is some arbitrariness in the choice of $D_{\mu\nu}$ since the scalar field satisfies a second-order differential equation. However, it is usual to fix $D_{\mu\nu}$ for the conformal scalar field by the requirement that it be formally traceless: Then the constant $\alpha=\frac{1}{4}$. Hence,

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = D_{\mu\nu}(x',x'')[\langle\psi(x')\psi(x'')\rangle - \tfrac{1}{2}G^{(1)}(x',x'')] - \tfrac{1}{8}m^2G^{(1)}(x',x''),$$

where it is understood that the limit $m\to 0$ is to be taken at the end. Define

$$\langle \psi^2 \rangle_{\rm ren} = \langle \psi(x')\psi(x'') \rangle - \tfrac{1}{2}G^{(1)}(x',x'') .$$

Then

$$\langle T_{\mu\nu} \rangle_{\rm ren} = \lim_{m\to 0} D_{\mu\nu}(x',x'')\langle \psi^2 \rangle_{\rm ren} - \tfrac{1}{8}\lim_{m\to 0} m^2 G^{(1)}(x',x'')g_{\mu\nu} .$$

This expression provides a simpler method of obtaining $\langle T_{\mu\nu}\rangle_{\rm ren}$: First renormalize $\langle \psi^2\rangle$ by subtracting the terms in $\frac{1}{2}G^{(1)}(x',x'')$ which on differentiation give rise to no more than four derivatives of the metric, and then differentiate $\langle \psi^2\rangle_{\rm ren}$, a much simpler task than differentiating the unrenormalized $\langle \psi^2\rangle$. Finally one must add a term,

$$-\tfrac{1}{8}\lim_{m\to 0} m^2 G^{(1)}(x',x'')g_{\mu\nu} ,$$

which is nonzero since $G^{(1)}(x',x'')$ contains a term involving four derivatives of the metric which is proportional to m^{-2}. This term gives rise to the anomalous trace in $\langle T_{\mu\nu}\rangle_{\rm ren}$. In the special case of the open static universe the terms to be subtracted to renormalize $\langle \psi^2\rangle$ are identical to those in (11); hence,

$$\langle \psi^2 \rangle_{\rm ren} = 0. \tag{12}$$

Performing the differentiation and adding in the conformal anomaly terms (which are zero in this case) yields the result

$$\langle T_{\mu\nu} \rangle_{\rm ren} = 0. \tag{13}$$

This result differs from that obtained in Refs. 1 and 2 where there appeared a nonzero result numerically equal to that found by Ford[13] in the closed static universe. The calculation of $\langle T_{\mu\nu}\rangle_{\rm ren}$ given in Ref. 2 is based on an expression for $\langle \psi(x')\psi(x'')\rangle$ in the Einstein universe obtained by Critchley.[14] It was assumed in Ref. 2 that this expression is also valid for the open static universe but comparison of Eq. (7) above with Eq. (2.4) of Ref. 2 shows that this assumption is false. The differences arises because $\langle \psi^2\rangle$ in the Einstein universe is obtained by summing products of a discrete set of modes whereas in the open static universe, $\langle \psi^2\rangle$ is obtained by integrating products of a continuous set of modes. The transition from an integral to a sum gives rise to a nonzero renormalized stress tensor in the Einstein universe: This is shown very clearly in Ford's calculation.[13]

III. EXTENSION TO EXPANDING HYPERBOLIC UNIVERSES

The metric of the expanding hyperbolic universes will be taken to be

$$ds^2 = C(\eta)\,[d\eta^2 - \rho(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)] , \tag{14}$$

where $\rho(r) = (1-kr^2)^{-1}$ and, for the hyperbolic universes, $k = -a^{-2}$. Taking the metric in form (14) enables the spatially flat and closed universes to be considered as well, by taking $k=0$ or $k=a^{-2}$ respectively. Because of the symmetry of the Robertson-Walker universes, the stress tensor will have only two independent components, the $\eta\eta$ component, $T_{\eta\eta}$, and the rr component, T_{rr}. The other components are then given by

$$T_{\theta\theta} = r^2\rho^{-1}T_{rr}, \quad T_{\phi\phi} = \sin^2\theta T_{\theta\theta} ,$$

and

$$T_{ab} = 0, \quad a \neq b .$$

For any such tensor, $T_{\mu\nu}$, the conservation condition $T_\mu{}^\nu{}_{;\nu} = 0$ may be written

$$T_{\eta\eta,\eta} + \tfrac{1}{2}DT_{\eta\eta} + \tfrac{3}{2}D\rho^{-1}T_{rr} = 0 , \tag{15}$$

where

$$D = C'/C \quad\text{and}\quad C' = dC/d\eta .$$

But,

$$T_\alpha{}^\alpha \equiv C^{-1}(T_{\eta\eta} - 3\rho^{-1}T_{rr}) . \tag{16}$$

Therefore,

$$T_{\eta\eta,\eta} + DT_{\eta\eta} = \tfrac{1}{2}DCT^\alpha{}_\alpha . \tag{17}$$

For a conformally coupled massless scalar field, it is well known that[15-17]

$$T_\alpha{}^\alpha = (2880\pi^2)^{-1}(-C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{3}R^2 + \Box R) , \tag{18}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor. For the metric (14) one obtains

$$C_{\alpha\beta\gamma\delta} = 0,$$

$$R^{\alpha\beta}R_{\alpha\beta} = C^{-2}(3D'^2 + \tfrac{3}{2}D'D^2 + \tfrac{3}{4}D^4 + 6D'k + 6D^2k + 12k^2) ,$$

$$R = C^{-1}(3D' + \tfrac{3}{2}D^2 + 6k) ,$$

$$\Box R = C^{-2}(3D''' - \tfrac{9}{2}D'D^2 - 6D'k) .$$

Hence,

$$T_\alpha{}^\alpha = (2880\pi^2 C)^{-1}(3D''' - 3D'D^2) . \tag{19}$$

Substituting in (17) gives

$$\frac{d}{d\eta}(CT_{\eta\eta}) = (2880\pi^2)^{-1}(\tfrac{3}{2}D'''D - \tfrac{3}{2}D'D^3) ,$$

and so

$$T_{\eta\eta} = (2880\pi^2 C)^{-1}(\tfrac{3}{2}D''D - \tfrac{3}{4}D'^2 - \tfrac{3}{8}D^4 + A) . \tag{20}$$

A is a constant of integration which will be fixed by using knowledge of $\langle T_{\eta\eta}\rangle_{\rm ren}$ in the static universe. Using (16), (19), and (20) one finds

$$T_{rr} = (2880\pi^2 C)^{-1}\rho(-D''' + \tfrac{1}{2}D''D - \tfrac{1}{4}D'^2 + D'D^2 - \tfrac{1}{8}D^4 + \tfrac{1}{3}A) . \tag{21}$$

Note that since equations (15) and (18) do not depend on the choice of quantum state the results (20) and (21) are valid for any state having all the symmetries of the spacetime. The constant A is now fixed by considering the static universe, for which $D=0$. Consistency with (13) requires that

$$A=0\,.$$

Now since (13) is state dependent, the final expression for $\langle T_{\mu\nu}\rangle_{\rm ren}$ will be. The state is the vacuum state defined in the conformally-related static hyperbolic spacetime by the decomposition (2). In this state, the expectation value of the stress tensor is

$$\langle T_{\eta\eta}\rangle_{\rm ren}=(2880\pi^2C)^{-1}(\tfrac{3}{2}D''D-\tfrac{3}{4}D'^2-\tfrac{3}{8}D^4)\,, \quad (22)$$

$$\langle T_{rr}\rangle_{\rm ren}=(2880\pi^2C)^{-1}\rho(-D'''+\tfrac{1}{2}D''D-\tfrac{1}{4}D'^2+D'D^2-\tfrac{1}{8}D^4)\,. \quad (23)$$

These expressions are identical to those obtained in the $k=0$ universe.[6] In the closed universes, $k=a^{-2}$, the constant A is[13]

$$A=6a^{-4}$$

and expressions (22) and (23) must be modified accordingly. The tensor defined by (22) and (23) can be written as

$$\langle T_{\mu\nu}\rangle_{\rm ren}=(2880\pi^2)^{-1}(-\tfrac{1}{6}{}^{(1)}H_{\mu\nu}+{}^{(3)}H_{\mu\nu}-6\,{}^{(6)}H_{\mu\nu})\,, \quad (24)$$

where the conserved geometrical tensors ${}^{(1)}H_{\mu\nu}$ and ${}^{(3)}H_{\mu\nu}$ are

$${}^{(1)}H_{\mu\nu}=2R_{;\mu\nu}-2\Box R g_{\mu\nu}+2(RR_{\mu\nu}-\tfrac{1}{4}R^2g_{\mu\nu}),$$

$${}^{(3)}H_{\mu\nu}=-R^{\alpha\beta}R_{\alpha\mu\beta\nu}+\tfrac{1}{12}R^2g_{\mu\nu}\,,$$

and the conserved nongeometrical tensor ${}^{(6)}H_{\mu\nu}$ has components

$${}^{(6)}H_{\eta\eta}=a^{-4},\quad {}^{(6)}H_{rr}=\tfrac{1}{3}a^{-4}\,.$$

(For the origin of the notation ${}^{(6)}H_{\mu\nu}$, see Ref. 10.) Note that if, as in Ref. 4, one assumes that the coefficient of $\Box R$ in (18) is zero, then one finds that the tensor ${}^{(1)}H_{\mu\nu}$ does not appear in the corresponding expression to (24).

Equation (24) is the expression which in the hyperbolic universes replaces that obtained in Refs. 1 and 2, namely,

$$\langle T_{\mu\nu}\rangle_{\rm ren}=(2880\pi^2)^{-1}(-\tfrac{1}{6}{}^{(1)}H_{\mu\nu}+{}^{(3)}H_{\mu\nu})\,. \quad (25)$$

Equation (25) is valid for spatially flat (in which ${}^{(6)}H_{\mu\nu}=0$) and closed universes. The Casimir term in the closed Robertson-Walker universes is precisely the nongeometrical tensor $(480\pi^2)^{-1}\,{}^{(6)}H_{\mu\nu}$. Adding this to (24) yields (25).

The result (24) can be extended to massless fields of spin $\frac{1}{2}$ and 1 if one assumes that the renormalized stress tensor in the open static universe is zero for these fields as well. Expression (18) must be replaced by

$$T_\alpha{}^\alpha=(2880\pi^2)^{-1}[-\tfrac{7}{4}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}-\tfrac{11}{2}(R^{\alpha\beta}R_{\alpha\beta}-\tfrac{1}{3}R^2)+3\Box R] \quad (\text{spin }\tfrac{1}{2})\,,$$

$$T_\alpha{}^\alpha=(2880\pi^2)^{-1}[13C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}-62(R^{\alpha\beta}R_{\alpha\beta}-\tfrac{1}{3}R^2)-18\Box R] \quad (\text{spin }1)\,.$$

The result is

$$\langle T_{\mu\nu}\rangle_{\rm ren}=(2880\pi^2)^{-1}[-\tfrac{1}{2}{}^{(1)}H_{\mu\nu}+\tfrac{11}{2}{}^{(3)}H_{\mu\nu}-\tfrac{51}{2}{}^{(6)}H_{\mu\nu}] \quad (\text{spin }\tfrac{1}{2}), \quad (26)$$

$$\langle T_{\mu\nu}\rangle_{\rm ren}=(2880\pi^2)^{-1}[3\,{}^{(1)}H_{\mu\nu}+62\,{}^{(3)}H_{\mu\nu}-132\,{}^{(6)}H_{\mu\nu}] \quad (\text{spin }1). \quad (27)$$

A particular hyperbolic universe of interest is the Milne universe for which $C(\eta)=e^{2\eta}$. In this spacetime the Riemann tensor vanishes so that it is merely a patch of Minkowski spacetime. As a result the usual construction of a Minkowski vacuum can be performed and the renormalized stress tensor with respect to this state will be zero. The tensor (24) is not, however, zero because of the presence of the nongeometrical tensor ${}^{(6)}H_{\mu\nu}$. Thus the state with respect to which (24) is calculated cannot be the usual Minkowski vacuum. A similar result arises in two dimensions[18] and it can be shown[10,11] that the nonzero energy density $\langle T_{\eta\eta}\rangle_{\rm ren}$ can be obtained as a Planckian integral with a negative coefficient. An analogous calculation for the four-dimensional Milne universe yields

$$\langle T_{\eta\eta}\rangle_{\rm ren}=-\frac{1}{2\pi^2C}\int_0^\infty\frac{k^3dk}{e^{2\pi k}-1}=-\frac{1}{480\pi^2C}=-\frac{{}^{(6)}H_{\eta\eta}}{480\pi^2}\,. \quad (28)$$

This is more conveniently expressed in terms of the coordinate $t=e^\eta$,

$$\langle T_{tt}\rangle_{\rm ren}=-\frac{1}{2\pi^2}\int_0^\infty\frac{q^3dq}{e^{2\pi qt}-1}=-\frac{1}{480\pi^2t^4}\,. \quad (29)$$

This shows that the "temperature" of the Planckian distribution is $T=1/2\pi t$.

IV. DISCUSSION

The result (24) for the hyperbolic Robertson-Walker universes differs from that obtained in

Refs. 1–4, where the expression (25) was obtained for all Robertson-Walker universes, although in Ref. 4 the tensor ${}^{(1)}H_{\mu\nu}$ was omitted since that author considered the appearance of fourth derivatives of the metric unphysical. As explained at the end of Sec. II, the calculation of Ref. 2 is not valid for hyperbolic universes. The derivation given in Ref. 1 is based on the assumption that $\langle T_{\mu\nu}\rangle_{\text{ren}}$ is a geometrical object in Robertson-Walker universes so it is not surprising that the tensor ${}^{(6)}H_{\mu\nu}$ did not appear in that treatment. This treatment was extended in Ref. 10 to allow for the appearance of ${}^{(6)}H_{\mu\nu}$ in $\langle T_{\mu\nu}\rangle_{\text{ren}}$, but there it was incorrectly assumed that $\langle T_{\mu\nu}\rangle_{\text{ren}}$ was nonzero in the open static universe and the result (25) was obtained. All that remains to be resolved is the discrepancy between (24) and the expressions obtained in Refs. 3 and 4. In fact the results in Refs. 3 and 4 are quite correct (modulo the ambiguity about whether ${}^{(1)}H_{\mu\nu}$ can appear in $\langle T_{\mu\nu}\rangle_{\text{ren}}$)—the difference arises because a different vacuum state is being used. The vacuum used in this paper is defined by making a conformal transformation to the static hyperbolic universe with metric (1) and defining a vacuum in that spacetime by decomposing the field ψ into positive and negative frequencies with respect to the timelike Killing vector field $\partial/\partial t$, as in (2). The vacuum used in Refs. 3 and 4 is defined by making a conformal transformation to Minkowski space and making use of the usual Minkowski vacuum. That the two vacuum states are indeed different can be seen by considering the static hyperbolic universe with metric (1): This can be case into the conformally flat form

$$ds^2=\Omega(\tau,r)[d\tau^2-dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2)]\ , \qquad (30)$$

for some function $\Omega(\tau,r)$ whose precise form is not important to what follows. The solution to the wave equation are then taken to be

$$\psi(\tau,\vec{x})=(2\pi)^{-3/2}\Omega^{-1/2}(\tau,r)\int(2k)^{-1/2}[a_{\vec{k}}e^{i(\vec{k}\cdot\vec{x}-k\tau)}+a^{\dagger}_{\vec{k}}e^{-i(\vec{k}\cdot\vec{x}-k\tau)}]d^3k\ , \qquad (31)$$

where $\vec{x}=(x_1,x_2,x_3)$ are Cartesian coordinates and $k=|\vec{k}|$. The decomposition (31) is clearly very different from (2) and so the corresponding vacuums are also different. If one denotes the vacuum state defined by (31) by $|0\rangle$, so that $a_{\vec{k}}|0\rangle=0$ for all $\vec{k}$, then the result of Ref. 3 when expressed in the coordinate system defined by (1) is

$$\langle 0|T_{tt}|0\rangle=(480\pi^2a^4)^{-1}\ , \qquad (32)$$

$$\langle 0|T_{xx}|0\rangle=(1440\pi^2a^4)^{-1}\ . \qquad (33)$$

Using the expression for $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$ obtained in Ref. 4 gives precisely half of these results. The physical significance of the state $|0\rangle$ in the static hyperbolic universe seems rather obscure. It is certainly not as natural a state as that defined in Sec. II. However in the expanding hyperbolic universes it is much less clear which state is the more physical. If the spacetime is initially static but then undergoes an expansion, the conformally static vacuum would seem to be preferred; however, in the Milne universe, the Minkowski vacuum has a clear physical interpretation.

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[1]T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London A356, 569 (1977).
[2]T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London A357, 381 (1977).
[3]L. S. Brown and J. P. Cassidy, Phys. Rev. D 16, 1712 (1977).
[4]R. Wald, Ann. Phys. (N.Y.) 110, 472 (1978).
[5]P. C. W. Davies, S. A. Fulling, and W. G. Unruh, Phys. Rev. D 13, 2720 (1976).
[6]P. C. W. Davies, S. A. Fulling, S. M. Christensen, and T. S. Bunch, Ann. Phys. (N.Y.) 109, 108 (1977).
[7]S. M. Christensen, Phys. Rev. D 14, 2490 (1976).
[8]T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London A360, 117 (1978).
[9]T. S. Bunch and P. C. W. Davies, J. Phys. A 11, 1315 (1978).
[10]T. S. Bunch, Ph.D. thesis [University of London (King's College), 1977] (unpublished).
[11]T. S. Bunch, S. M. Christensen, and S. A. Fulling (unpublished).
[12]M. Bander and C. Itzkyson, Rev. Mod. Phys. 38, 346 (1966).
[13]L. H. Ford, Phys. Rev. D 11, 3370 (1975).
[14]R. Critchley, Ph.D. thesis (University of Manchester, 1976) (unpublished).
[15]S. Deser, M. J. Duff, and C. J. Isham, Nucl. Phys. B111, 45 (1976).
[16]S. M. Christensen and S. A. Fulling, Phys. Rev. D 15, 2088 (1977).
[17]L. S. Brown, Phys. Rev. D 15, 1469 (1977).
[18]P. C. W. Davies and S. A. Fulling, Proc. R. Soc. London A354, 59 (1977).